# Traveling waves on a magnetic universe

David Garfinkle
*Department of Physics, University of California, Santa Barbara, California 93106*
*and Department of Physics, Oakland University, Rochester, Michigan 48309**

M. A. Melvin
*1300 Orchid Drive, Santa Barbara, California 93111*
(Received 4 October 1991)

We find solutions of the Einstein-Maxwell equations representing gravitational and electromagnetic waves traveling along the axis direction in a cylindrical magnetic universe. The waves are strongly gravitating and can have an arbitrary profile.

PACS number(s): 04.20.Jb, 04.40.+c

## I. INTRODUCTION

A traveling wave is a wave that propagates without any change in amplitude or shape. The wave solution can be regarded as a modified space-time solution, i.e., the wave viewed as a perturbation on a background spacetime where the perturbation need not be small. Electromagnetic plane waves in a flat background spacetime are examples of traveling waves. In addition both certain vacuum Einstein and Einstein-Maxwell background spacetimes admit traveling waves as modifications. In this paper we find solutions of the Einstein-Maxwell equations representing traveling waves whose background is the cylindrically symmetric magnetic universe [1]. While small disturbances on the magnetic universe have been previously studied [2], our treatment allows the disturbances to have arbitrary strength. The solution is found by applying a variant of the generalized Kerr-Schild ansatz to the background. Our variant imposes more stringent symmetry conditions on the null vector field defining the ansatz. These are satisfied for the magnetic universe. In Sec. II we review some properties of the magnetic universe and then use the ansatz to reduce the Einstein-Maxwell equations to a single ordinary differential equation. Section III contains our treatment of this equation and shows how, by an iterative procedure, one can obtain the solution to arbitrary accuracy. Section IV is a discussion of the properties of the traveling waves.

## II. EINSTEIN-MAXWELL EQUATIONS

The cylindrical magnetic universe is a solution of the Einstein-Maxwell equations. The solution is for an antisymmetric tensor $F_{ab}$ and a metric $g_{ab}$ satisfying

$$\nabla_{[a}F_{bc]}=0 , \tag{2.1}$$

$$\nabla_a F^{ab}=0 , \tag{2.2}$$

$$R_{ab}=2F_{ac}F_b{}^c-\tfrac{1}{2}g_{ab}F_{cd}F^{cd} , \tag{2.3}$$

with the metric and Maxwell fields given by

$$ds^2=H(2du\,dv+d\rho^2)+H^{-1}\rho^2 d\theta^2 , \tag{2.4}$$

$$F_{\mu\nu}=2H^{-1}B\rho\delta^\rho{}_{[\mu}\delta^\theta{}_{\nu]} . \tag{2.5}$$

Here $B$ is a constant for any given total flux. The magnitude of the physical magnetic field, $B_{\text{phys}}=F_{\rho\theta}/\rho$, on the axis is the constant $B$ and the total magnetic flux is $4\pi/B$. The function $H$ is given by

$$H\equiv(1+\tfrac{1}{4}B^2\rho^2)^2 \tag{2.6}$$

and $u\equiv(z-t)/\sqrt{2}$, $v\equiv(z+t)/\sqrt{2}$ where $t$, $z$, $\rho$, $\theta$ are the conventional time and cylindrical space coordinates. The magnetic universe possesses a Killing vector $k^a$ that is null and hypersurface orthogonal. The vector $k^a$ is given by

$$k^a=\left[\frac{\partial}{\partial v}\right]^a . \tag{2.7}$$

The derivative of $k^a$ is given by

$$\nabla_a k_b=H^{-1}k_{[b}\nabla_{a]}H . \tag{2.8}$$

The traveling-wave solutions that we will find will be of the form $(\tilde{g}_{ab},F_{ab})$ where $F_{ab}$ is the same as in the magnetic universe and $\tilde{g}_{ab}$ has the form

$$\tilde{g}_{ab}=g_{ab}+H^{-1}\Psi k_a k_b \tag{2.9}$$

for some scalar $\Psi$. This ansatz is called the generalized Kerr-Schild ansatz [3] when the vector $k^a$ is null, geodetic, and shear-free. In this case we have imposed the stronger condition that $k^a$ be a null, hypersurface orthogonal Killing vector (for null vectors Killing implies geodetic and shear-free, but not conversely). As shown in Ref. [4] the metric $\tilde{g}_{ab}$ with the Maxwell field $F_{ab}$ satisfies the Einstein-Maxwell equations provided that the scalar $\Psi$ satisfies

$$k^a\nabla_a\Psi=0 , \tag{2.10}$$

*Present address.

 

$$\nabla_a \nabla^a \Psi = 0 \ . \tag{2.11}$$

One can derive this result as follows. First impose Eq. (2.10). This ensures that the vector $k^a$ is also a null Killing vector for the metric $\tilde{g}_{ab}$. It is this fact that makes $\tilde{g}_{ab}$ a traveling wave. Since $\tilde{g}_{ab}$ is invariant under a null symmetry, any disturbances must propagate at the speed of light without changing their amplitude or shape. Next use Eqs. (2.8) and (2.9) to compute $\tilde{R}^a{}_b$ the Ricci tensor of $\tilde{g}_{ab}$. The result is that $\tilde{R}^a{}_b - R^a{}_b$ is proportional to $\nabla_a \nabla^a \Psi$. Thus imposing Eq. (2.11) ensures that the Ricci tensor is unchanged when $\tilde{g}_{ab}$ is substituted for $g_{ab}$. Using the fact that $k^a F_{ab} = 0$ we then find that the Einstein equation (2.3) is satisfied by $\tilde{g}_{ab}$. As for Maxwell's equations (2.1) and (2.2) the first is independent of the metric while the second can be written as $\nabla_{[a} {}^*F_{bc]} = 0$ where ${}^*F_{ab}$ is the dual of $F_{ab}$. Now from Eq. (2.9) and $k^a F_{ab} = 0$ it follows that ${}^*F_{ab}$ is the same in the metric $\tilde{g}_{ab}$ as in $g_{ab}$. So Maxwell's equations are still satisfied in the new metric. Thus to find a new solution of the Einstein-Maxwell equations we have only to solve Eqs. (2.10) and (2.11) in the magnetic universe.

In the magnetic universe equation (2.10) becomes

$$\frac{\partial \Psi}{\partial v} = 0 \ , \tag{2.12}$$

so the scalar $\Psi$ is independent of the coordinate $v$, depending only on $u$, $\rho$, $\theta$. The wave equation (2.11) then becomes

$$\rho \frac{\partial}{\partial \rho} \left[ \rho \frac{\partial \Psi}{\partial \rho} \right] + H^2 \frac{\partial^2 \Psi}{\partial \theta^2} = 0 \ . \tag{2.13}$$

One can separate variables in this equation. We first introduce the coordinate $x$ by $x \equiv B^2 \rho^2 / 4$. Then the separated solutions for $\Psi$ take the form

$$\Psi = P(x) f(u) \cos m(\theta - \theta_0) \ . \tag{2.14}$$

Here $f$ is any smooth function; $\theta_0$ is a constant; $m$ is an integer and $P$ satisfies the equation

$$x \frac{d}{dx} \left[ x \frac{dP}{dx} \right] - \tfrac{1}{4} m^2 (1+x)^4 P = 0 \ . \tag{2.15}$$

A solution of this equation yields a traveling-wave solution of the Einstein-Maxwell equations. The function $f$ is the profile of the wave.

## III. SMALL- AND LARGE-$x$ APPROXIMATIONS TO THE EXACT EQUATION FOR $P$

Equation (2.15) can be solved in closed form [5] for the case $m = 0$. The solution is $P = \ln x$. The general case $m \neq 0$ is more complicated. We first define the function $Q$ by $Q \equiv x^{1/2} P$. Then $Q$ satisfies the equation

$$Q'' + \frac{1}{4x^2} [1 - m^2 (1+x)^4] Q = 0 \ . \tag{3.1}$$

Here a prime denotes a derivative with respect to $x$.

For $x$ very small, $Q \to q$ where Eq. (3.1) takes the form

$$q'' + \frac{1-m^2}{4x^2} q = 0 \tag{3.2}$$

which has as its general solution the linear combination

$$q = x^{1/2} (C_1 x^{m/2} + C_2 x^{-m/2}) \quad (C_1, C_2 = \text{real constants}) \ . \tag{3.3}$$

Thus, at $x = 0$, $P$ has a zero of order $m/2$ or a pole of order $m/2$ according to which special solution we adopt.

In contrast, for $x$ very large, the dominant behavior of $Q$ is given by $\tilde{Q}$ where Eq. (3.1) takes the form

$$\tilde{Q}'' - \frac{m^2}{4} x^2 \tilde{Q} = 0 \tag{3.4}$$

for which the general real solution is the linear combination

$$\tilde{Q} = x^{1/2} \left[ C_1 I_{1/4} \left[ \frac{m x^2}{4} \right] + C_2 K_{1/4} \left[ \frac{m x^2}{4} \right] \right] \ . \tag{3.5}$$

Here $I_{1/4}$ and $K_{1/4}$ are Bessel functions of imaginary argument. Asymptotically, for large $x$, the solution becomes [6]

$$\tilde{Q} \to \left[ \frac{2}{mx} \right]^{1/2} \left[ \frac{C_1}{\sqrt{\pi}} \exp \left[ \frac{m x^2}{4} \right] + \sqrt{\pi} C_2 \exp \left[ -\frac{m x^2}{4} \right] \right] \ . \tag{3.6}$$

Though possibly meromorphic elsewhere, we see that as $x \to \infty$ $P$ has an essential singularity, either of the zero type or of the infinite type according to which solution we adopt.

We may conjecture that the exact solution of the exact $P$ equation that becomes zero at $x = 0$ is the one that becomes infinite at $x = \infty$ and, conversely, the solution that becomes infinite at $x = 0$ becomes zero at $x = \infty$. We will discuss the verification of this conjecture subsequently.

## IV. SOLUTION OF THE EXACT EQUATION FOR $P$

It appears that there is no general solution of Eq. (3.1) in finite elementary terms [7]. However this equation can be treated by the iterative method of Langer [8] which allows us to compute the solution to arbitrary accuracy. The Langer method provides not only a more powerful approximation for the solution of any ordinary differential equation such as our Eq. (3.1), but also provides a convergent iterative solution of the exact equation to any desired accuracy. The key point is to construct a "related equation" to the given equation for $Q$. The related equation is such that (a) the solution $Y(x)$ is known (by construction) and (b) $Y(x)$ can serve as a first-order ground function upon which a convergent iterative solution to an integral for $Q/Y$ can be found.

Equation (3.1) can be written as

$$Q'' + [\mu^2 \phi^2(x) - \chi(x)] Q = 0 \ . \tag{4.1}$$

In our case the parameter $\mu$ and the functions $\phi(x)$ and $\chi(x)$ are given by $\mu = im/2$, $\phi = (1+x)^2/x$, and

$\chi=-1/(4x^2)$. The related equation has the form

$$Y''+[\mu^2\phi^2(x)-\omega(x)]Y=0 \;. \tag{4.2}$$

Here the function $\omega(x)$ will be chosen so that Eq. (4.2) has a known solution. Let $\xi$ be the integral of $\mu\phi$. In our case

$$\xi=\frac{im}{2}[\ln x+\tfrac{1}{2}x(x+4)] \;. \tag{4.3}$$

Writing $\Upsilon=d\ln\xi/dx$ for the logarithmic derivative of $\xi$, the function $\omega(x)$ has the form

$$\omega(x)=\frac{1}{4}\left[-\frac{2\Upsilon''}{\Upsilon}+\frac{3(\Upsilon')^2}{\Upsilon^2}+\Upsilon^2\right] . \tag{4.4}$$

(In our case $\omega$ comes out to be given by $\omega-\chi=(x-1)/[x(1+x)^2]$.) In general, in cases such as our second-order linear differential equation, where $\phi$ has no zero between $x=0$ and $x=\infty$, each of the two solutions of the related equation has the form of an order-$\frac{1}{2}$ Hankel function, $H^{(1)}_{1/2}(\xi)$ or $H^{(2)}_{1/2}(\xi)$, divided by the square root of $\Upsilon$. We have that

$$H^{(1)}_{1/2}(iz)=-i\left[i\frac{\pi}{2}z\right]^{-1/2}e^{-z} \;, \tag{4.5}$$

$$H^{(2)}_{1/2}(iz)=i\left[i\frac{\pi}{2}z\right]^{-1/2}e^{z} \;. \tag{4.6}$$

The related equation associated with Eq. (3.1) having the form

$$Y''+\frac{1}{4x^2}\left[1-m^2(1+x)^4+\frac{4x(1-x)}{(1+x)^2}\right]Y=0 \tag{4.7}$$

has the solutions

$$Y^{\pm}=(1+x)^{-1}x^{(1-m)/2}\exp\left[-\frac{m}{4}x(x+4)\right] \tag{4.8}$$

with positive and negative values assigned to a given value of $|m|$ giving two linearly independent solutions. Since the equations for $Q$ and $Y$ are similar, one might expect that $Q$ is well approximated by $Y$. Choose the sign of $m$ to be positive and define the function $U$ by $U\equiv Q/Y^{\mp}$. Then $U$ satisfies the integral equation

$$U(x)=1+\frac{1}{m}\int_x^\infty C(t,x)U(t)dt \;. \tag{4.9}$$

In general the kernel $C(t,x)$ is calculated from $\chi$, $\omega$ and the known solutions of the related equation and is given by

$$\frac{1}{m}C(t,x)=\frac{\chi(t)-\omega(t)}{W}\left[Y^+(t)Y^-(t)-[Y^+(t)]^2\frac{Y^-(x)}{Y^+(x)}\right] . \tag{4.10}$$

Here the constant $W$ is the Wronskian of $Y^+$ and $Y^-$. That is

$$W\equiv Y^+(Y^-)'-Y^-(Y^+)' \;.$$

In our particular case the function $C(t,x)$ is given by

$$C(t,x)\equiv\frac{1-t}{(1+t)^4}\left[1-(x/t)^m\exp\left[\frac{m}{2}[x(x+4)-t(t+4)]\right]\right] . \tag{4.11}$$

Equation (4.9) is solved by iteration. Define $U_0$ by $U_0\equiv1$. Then define $U_n$ by

$$U_{n+1}(x)\equiv1+\frac{1}{m}\int_x^\infty C(t,x)U_n(t)dt \;. \tag{4.12}$$

Then $U_n$ converges uniformly to a solution of Eq. (4.9) as $n\to\infty$. One proves the convergence as follows. First note that since $t\geq x>0$ for the range of integration it follows that $|C(t,x)|\leq|1-t|/(1+t)^4$. Then for any $x$ it follows that

$$\int_x^\infty|C(t,x)|dt\leq\int_0^\infty\frac{|1-t|}{(1+t)^4}dt=\frac{1}{4} \;. \tag{4.13}$$

Now define $k_n$ by $k_n\equiv\sup|U_{n+1}-U_n|$. Then from Eq. (4.12) we have

$$k_{n+1}=\sup\left|\frac{1}{m}\int_x^\infty C(t,x)[U_{n+1}(t)-U_n(t)]dt\right|\leq\frac{1}{m}(\sup|U_{n+1}-U_n|)\left[\sup\int_x^\infty|C(t,x)|dt\right]\leq\frac{1}{4m}k_n \;. \tag{4.14}$$

Similarly for $k_0$ we have

$$k_0\leq\sup\left[\frac{1}{m}\int_x^\infty|C(t,x)|dt\right]\leq\frac{1}{4m} \;. \tag{4.15}$$

It then follows by induction that $k_n\leq(4m)^{-(n+1)}$. Now for any positive integers $n,j$ with $n>j$ we have

$$\sup|U_n-U_j|\leq\sup\sum_{r=0}^{n-(j+1)}|U_{r+j+1}-U_{r+j}|\leq\sum_{r=0}^{n-(j+1)}k_{r+j}<\left[\frac{1}{4m-1}\right]\left[\frac{1}{4m}\right]^j . \tag{4.16}$$

Thus for any $\epsilon>0$ there exists an integer $M$ such that $\sup|U_n-U_j|<\epsilon$ for all $n,j>M$. Thus the functions $U_n(x)$ are a Cauchy sequence in the sup norm. The $U_n$ therefore converge uniformly to a function $U(x)$ in the limit as $n\rightarrow\infty$.

To summarize, the function $P$ is given by

$$P\equiv\frac{U}{1+x}x^{-m/2}\exp\left[-\frac{m}{4}x(x+4)\right]. \tag{4.17}$$

The function $U$ is a solution of Eq. (4.9) and can be computed to arbitrary accuracy by iterating Eq. (4.12). Thus we obtain a traveling-wave solution of the Einstein-Maxwell equations to arbitrary accuracy.

## V. PROPERTIES OF THE TRAVELING-WAVE METRIC

The metric of the traveling wave is given by

$$ds^2=H(2du\,dv+d\rho^2)+H^{-1}\rho^2d\theta^2 + Hf(u)\cos m(\theta-\theta_0)P(B^2\rho^2/4)du^2 . \tag{5.1}$$

The function $f$, which can be freely specified, is the profile of the wave. In regions where $f$ vanishes the metric is just that of the magnetic universe. Thus the traveling wave can be regarded as a disturbance propagating in the magnetic universe.

We now consider the properties of the metric in the regions where $f$ does not vanish. First consider the case where $m>0$. Then the function $P$ is singular at $\rho=0$; and $P$ approaches zero extremely rapidly as $\rho\rightarrow\infty$. The traveling-wave metric then rapidly approaches the magnetic-universe metric for large $\rho$. However, there is a singularity at $\rho=0$ and the curvature diverges there. The case $m=0$ is also a metric singular at $\rho=0$. The case $m<0$ is somewhat different. Here the function $P$ and the metric are well behaved at $\rho=0$. However, $P$ diverges rapidly as $\rho\rightarrow\infty$. This is reflected in a divergence of the curvature as $\rho\rightarrow\infty$. Furthermore, this is an actual curvature singularity; since some geodesics now approach $\rho=\infty$ in a finite proper time.

Thus for all the choices of $m$ the metric is singular. However, the singularity only exists in regions where $f\neq0$. Thus if $f$ has compact support then the spacetime is a singular traveling wave with a singularity of finite width that propagates along with the wave.

## ACKNOWLEDGMENTS

This work was supported in part by NSF Grant No. PHY-9008502 to The University of California.

[1] M. A. Melvin, Phys. Lett. **8**, 65 (1964).

[2] M. A. Melvin, Phys. Rev. **139**, B225 (1965), gives solutions and establishes stability for small radial perturbations; K. S. Thorne, *ibid.* **139**, B244 (1965), confirms stability for large radial perturbations but does not give explicit solutions; see also D. F. Cattell and M. A. Melvin, J. Math. Phys. **20**, 1911 (1979).

[3] A. Taub, Ann. Phys. (N.Y.) **134**, 326 (1981).

[4] D. Garfinkle and T. Vachaspati, Phys. Rev. D **42**, 1960 (1990).

[5] D. Garfinkle, in *Nonlinear Problems in Relativity and Cosmology,* edited by J. R. Buchler, S. L. Detweiler, and J. R. Ipser (New York Academy of Sciences, New York, 1991).

[6] G. N. Watson, *Theory of Bessel Functions,* 2nd ed. (Cambridge University Press, New York, 1945).

[7] This phrase is used in the sense of Joseph Liouville [J. Liouville, J. Math. Pures Appl. **IV**, 423 (1839); **V**, 1 (1841)]. See also J. F. Ritt, *Integration in Finite Terms* (Columbia, New York, 1948), and G. N. Watson, *Theory of Bessel Functions,* 2nd ed. (Cambridge University Press, New York, 1945), pp. 87 and 111–123. The concept of solutions in finite terms has been made more general and precise in recent modern literature on differential field extensions. It is clearly less ambiguous than the frequently used phrase "in closed form."

[8] R. E. Langer, Trans. Am. Math. Soc. **33**, 23 (1932); R. E. Langer, *ibid.* **34**, 447 (1932).