Generating solutions of the Einstein-Maxwell equations with prescribed physical properties

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(Received 31 May 1991)

A linear transformation for generating electrovacuum solutions is presented. The multipole moments of the new solutions can be expressed explicitly in terms of those of the seed solution. This provides a method of generating solutions with physical properties determined *a priori*. We discuss the specific example of a solution representing the gravitational field of a deformed rotating source with zero charge and a vanishing magnetic-monopole moment but nonvanishing higher electric and magnetic multipoles.

PACS number(s): 04.20.Jb, 95.30.Sf

The active development of solution-generating techniques [1] during the past two decades has provided researchers in general relativity with a powerful tool for generating exact solutions of Einstein's equations. At present, the derivation of new solutions does not present major difficulties, especially when using algebraic packages operated by computers. More important and difficult, however, is the task of determining the physical relevance of the generated solutions. Here we are interested in solutions that can be used for the description of the gravitational field of astrophysical bodies. In this context, the investigation of the asymptotic properties of the solution and its singularities are very useful. Nevertheless, the most important properties of a solution are determined by the physical significance of the independent parameters entering the solution. If we limit ourselves to stationary gravitational fields, then the physical interpretation of the parameters reduces, in principle, to the problem of calculating the corresponding coordinate-invariant multipole moments. To do this, one may use any one of the known equivalent definitions originally due to Geroch and Hansen [2].

To derive a new solution, one need only select any one of the known solutions and apply one of the solution-generating techniques according to well-established procedures. If one is interested in deriving physically meaningful solutions, one can only hope that the seed solution has been chosen appropriately. This might be considered a trial and error approach. In this paper, we present a method that permits one to specify *a priori* the physical properties of the solution to be generated. The method is based upon a linear transformation that acts in the set of stationary axisymmetric solutions of the Einstein-Maxwell equations and can be used to generate new solutions. The linearity of the transformation allows one to represent easily *all* multipole moments of the new solution in terms of those of the known one, so that the new multipoles can be fixed so as to correspond to a realistic source.

The field equations for stationary axisymmetric fields can be written in the Ernst representation as [3]

$$(\xi\xi^* - qq^* - 1)\Delta\xi = 2(\xi^*\nabla\xi - q^*\nabla q)\nabla\xi \,, \tag{1}$$

$$(\xi\xi^* - qq^* - 1)\Delta q = 2(\xi^*\nabla\xi - q^*\nabla q)\nabla q \,, \tag{2}$$

where ξ is the gravitational and q the electromagnetic complex potential as defined by Ernst. The symbol ∇ represents the two-dimensional gradient operator defined by the two nonignorable coordinates (for instance, ρ and z in Weyl canonical coordinates t,ρ,z,ϕ), and $\Delta = \nabla^2$. Moreover, the asterisk denotes complex conjugation. Since ξ and q satisfy the same type of differential equation, one is then tempted to make the ansatz $q = e\xi$, where e is a complex constant, in general. Then it turns out that Eqs. (1) and (2) are identically satisfied if one chooses $\xi = \eta\xi_0$, where $\eta = (1 - ee^*)^{-1/2}$ and ξ_0 is a vacuum solution. This method of generating electrovacuum solutions from vacuum ones is known as the Harrison transformation [4] and has been used recently to obtain a generalization of Kerr-Newman spacetime [5].

Attempting to use more effectively the symmetric structure of Eqs. (1) and (2), we consider the vector $\chi = (\begin{smallmatrix}\xi \\ q\end{smallmatrix})$, where ξ and q represent a known electrovacuum solution, and define a *linear transformation* of χ as

$$\chi' = A\chi \quad \text{with} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,, \tag{3}$$

where a, b, c, and d are complex constants. Introducing this ansatz into Eqs. (1) and (2), one sees that χ represents a new solution of the same equations if the following conditions are satisfied: $ab^* = cd^*$, $aa^* = 1 + cc^* = dd^*$. It follows then that the set of linear transformations (3) generates the group U(1,1), whose general element can be expressed as ($\beta^2 = cc^*$)

$$\begin{aligned} a &= \sqrt{1+\beta^2}\exp(i\tau_a), \quad b = \beta\exp(i\tau_b), \\ c &= \beta\exp(i\tau_c), \quad d = \sqrt{1+\beta^2}\exp(\tau_d) \,, \end{aligned} \tag{4}$$

where β, τ_a, τ_b, and τ_c are real constants, and $\tau_a + \tau_d = \tau_c + \tau_b$. Therefore, a linear transformation generates a four-parameter family of electrovacuum solutions. The Harrison transformation is contained in this transformation with $\beta = |e|(1 - |e|^2)^{-1/2}$, $\tau_a = \tau_d = 0$, and $\tau_b = -\tau_c$ such that $e = |e|\exp(i\tau_c)$. It can be shown that other known transformations [6], which generate electro-

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vacuum solutions, are contained in the linear transformation (3), as special cases [7]. Defining the norm of the vector χ as $\chi^2 \equiv \chi^\dagger g \chi$, where $g = \mathrm{diag}(+1,-1)$ and the dagger denotes Hermitian conjugation, it is interesting to note that this norm is invariant under the linear transformation (3), i.e., $\chi'^2 = \chi^2$.

To investigate the physical significance of the solutions generated by a linear transformation, we calculate the corresponding multipole moments by using the invariant definition proposed by Geroch and Hansen (see Ref. [2]) for vacuum fields and recently generalized by Hoenselaers and Perjés [8] to include electrovacuum fields. According to these results, all information about the multipole moments is contained in the potentials ξ and q, which are related to the gravitational (ϕ_M and ϕ_J) and electromagnetic (ϕ_E and ϕ_H) potentials by

$$\xi = \phi_M + i\phi_J , \quad q = \phi_E + i\phi_H . \tag{5}$$

Then the gravitoelectric (M_n), gravitomagnetic (J_n), electric (E_n), and magnetic (H_n), $n = 0,1,2,\ldots$, multipoles are determined in an invariant way by the asymptotic behavior of ξ and q [9]. Let the seed solution χ be characterized by the set of multipole moments M_n, J_n, E_n, and H_n. Then the linear transformation (3) generates a new solution χ' whose multipole moments are given by

$$\begin{aligned} M_n' &= a_R M_n - a_I J_n + b_R E_n - b_I H_n , \\ J_n' &= a_I M_n + a_R J_n + b_I E_n + b_R H_n , \\ E_n' &= c_R M_n - c_I J_n + d_R E_n - d_I H_n , \\ H_n' &= c_I M_n + c_R J_n + d_I E_n + d_R H_n , \end{aligned} \tag{6}$$

where $a_R = \mathrm{Re}(a)$, $a_I = \mathrm{Im}(a)$, etc. Equation (6) represents an algebraic relationship between the seed multipoles and the new ones. It is easy to see that the determinant of this algebraic system is nonsingular and is equal to 1. Hence Eq. (6) can be solved with respect to the seed multipoles for any given M_n', J_n', E_n', and H_n'. That is, one can specify *a priori* the physical properties of the new solution by fixing its multipole moments. The following important example illustrates our approach.

Astrophysical objects, such as stars, planets, etc., are characterized by aspherical deformations, rotation, and magnetic fields *with vanishing magnetic monopole*. In a first-order approximation we can consider the aspherical deformations as axially symmetric. Astrophysical observations indicate that the *net charge* of astrophysical bodies may be vanishing small, although on the average, higher electric multipoles might exist [10]. By a straightforward application of the known solution-generating techniques, attempts to derive a solution describing this specific kind of gravitational field were unsuccessful. It turns out that it is not possible to get rid of the net charge without introducing undesirable parameters such as the gravitomagnetic or magnetic monopoles. The method presented here avoids this difficulty easily. According to the descriptions given above, the desired solution should have the following monopoles: $M_0' = m$ (total mass), $J_0' = 0$ (asymptotic flatness), $E_0' = 0$ (no charge), $H_0' = 0$ (no magnetic monopole); the dipoles should be $M_1' = 0$ (center of mass coincides with the origin of coordinates), $J_1' = J$ (angular momentum), $H_1' = \mu$ (magnetic dipole), $E_1' = p$ (electric dipole) [11], etc. Higher multipole moments can be fixed to correspond to any specific configuration of the mass distribution. Putting these values into Eq. (6) and solving with respect to the seed multipoles, we obtain

$$\begin{aligned} & M_0 = m\sqrt{1+\beta^2}\cos\tau_a , \quad J_0 = -m\sqrt{1+\beta^2}\sin\tau_a , \\ & E_0 = -m\beta\cos\tau_b , \quad H_0 = m\beta\sin\tau_b , \\ & M_1 = J\sqrt{1+\beta^2}\sin\tau_a - p\beta\cos\tau_c - \mu\beta\sin\tau_c , \\ & J_1 = J\sqrt{1+\beta^2}\cos\tau_a + p\beta\sin\tau_c - \mu\beta\cos\tau_c , \\ & E_1 = \mu\sqrt{1+\beta^2}(\sin\tau_d + p\cos\tau_d) - J\beta\sin\tau_b , \\ & H_1 = \mu\sqrt{1+\beta^2}(\cos\tau_d - p\sin\tau_d) - J\beta\cos\tau_b , \ \text{etc.} , \end{aligned} \tag{7}$$

where we have used the representation (4), and $\tau_d = \tau_b + \tau_c - \tau_a$. Higher multipole moments can be calculated in the same way. Equation (7) shows that the multipole moments of the seed solution must satisfy certain conditions, for instance, $J_0 = -M_0 \tan\tau_a$, $H_0 = -E_0 \tan\tau_b$, etc. Therefore the physical significance of the seed solution becomes rather unclear. However, this is not a disadvantage of our approach since it is the very physical meaning of the new solution that really matters, and this has been established *a priori* according to the specific gravitational source under consideration. Moreover, we do not need to worry about the physical significance of the parameters introduced by the linear transformation (4), because these are now entering only the seed solution.

The question arises whether there is a seed solution with multipole moments satisfying Eq. (7). In a series of publications, Quevedo and Mashhoon [12] presented a stationary vacuum solution with an infinite set of arbitrary *parameters*. This work was recently extended to the electrovacuum case [5] by using a real Harrison transformation and can easily be generalized to include electromagnetic fields by means of a complex Harrison transformation. Another class of solutions with arbitrary parameters has been investigated by Manko *et al.* [13]. Although the solutions presented in Refs. [12] and [13] are equipped with sets of arbitrary parameters, say m_n, $n = 0,1,2,\ldots$, the corresponding relativistic multipole moments are not arbitrary. Indeed, the gravitoelectric multipole moments of these solutions can be written as

$$M_n = N_n + R_n , \tag{8}$$

where $N_n \propto m_n$ are the Newtonian multipoles and $R_n = R_n(N_{n-1}, N_{n-2}, \ldots, N_0)$ are the relativistic contributions. The Newtonian multipoles are arbitrary since they are uniquely determined by the arbitrary parameters m_n. Using Eq. (8), one may write the relativistic contributions as $R_n = R_n(M_{n-1}, M_{n-2}, \ldots, M_0)$. That is, Eq. (8) represents a relationship between $M_n, M_{n-1}, \ldots, M_0$. Consequently, the *relativistic* multipole moments are not arbitrary.

Arbitrary multipoles should have the form $M_n \propto s_n$, where all s_n are linear independent. To generate solutions with such sets of multipoles, one may use linear transformations. Indeed, using the solutions presented in Refs. [12] and [13] as a seed solution and applying a linear transformation, one can always use the arbitrariness of the parameters in order to satisfy Eq. (7). Furthermore, one can also use any other seed solution with nonarbitrary parameters and still satisfy Eq. (7) (at least for some finite number of multipoles) by exploiting the freedom of the parameters β, τ_a, τ_b, and τ_c introduced by the linear transformation.

Although the potentials ξ and q contain very useful information about the metric, one would like to have the metric coefficients explicitly in order to investigate other properties of the solution. Let f', ω', and γ' be the metric coefficients of the new stationary axisymmetric electrovacuum solution, which are related to the potential ξ and q by certain algebraic and differential equations (see, for instance, Ref. [12] for notation and details). Then the function f' can be calculated algebraically by means of

$$f'=(1-\chi^2)|1+a\xi+bq|^{-2}, \quad (9)$$

with $\chi^2=\xi\xi^*-qq^*$, and the function ω' is determined by the differential equations

$$(1-\chi^2)^2\nabla_+\omega'=2\rho\,\mathrm{Im}\{(1+\xi')[(1+\xi')\nabla_-\xi'^* - q'(1+\xi'^*)\nabla_- q'^* + q'q'^*\nabla_-\xi'^*]\}, \quad (10)$$

where $\nabla_+=(\partial_\rho,\partial_z)$, $\nabla_-=(\partial_z,-\partial_\rho)$, and ρ and z are the nonignorable cylindrical coordinates. Finally, the function γ' can be calculated by quadratures once f' and ω' are known. The explicit integration of the differential equations (10) for *arbitrary* ξ and q becomes rather cumbersome and requires detailed study. This work will be presented elsewhere.

The importance of the method presented here lies in the possibility of generating solutions with well-defined and *a priori* fixed physical properties. The specific example discussed above describes, for the first time, an exact solution of the Einstein-Maxwell equations representing the gravitational field of a source with physical characteristics closely related to those of realistic astrophysical objects. A more complete investigation of this solution is necessary in order to clarify the possible consequences for astrophysics. It turns out that the specific relationship between the gravitational and electromagnetic potentials as expressed by the linear transformation (3) has deep physical significance related to the classical theories of gravity and electromagnetism, and leads to unexpected physical consequences [14].

I would like to thank Professor F. W. Hehl for encouragement and support, and Professor B. Mashhoon for illuminating comments and invaluable discussions. I am grateful to Dr. R. Hecht for interesting comments regarding the linear transformation. Thanks are due to the Department of Physics and Astronomy at UMC for excellent hospitality while part of this work was done. This work was supported by the Deutsche Forschungsgemeinschaft, Bonn, Germany.

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