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Trapped surfaces in the Schwarzschild geometry and cosmic censorship

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We prove that in extended Schwarzschild spacetime there exists a family of Cauchy surfaces which come arbitrarily close to the black-hole singularity at $r=0$ but are such that there do not exist any trapped surfaces lying within the past of any of these Cauchy surfaces. We argue that, in any spherically symmetric spacetime describing gravitational collapse to a Schwarzschild black hole, it should be possible to choose a spacelike slicing by Cauchy surfaces which terminates when a singularity (outside the matter) is reached, such that the region of the spacetime covered by the slicing contains no outer trapped surfaces. Thus, an important feature of the numerical collapse examples of Shapiro and Teukolsky, cited by them as evidence against cosmic censorship, does not appear to be qualitatively different from features which occur (for an appropriate choice of slicing) in the standard examples of collapse to a black hole.

Recently, Shapiro and Teukolsky [1,2] have reported results of numerical simulations involving the gravitational collapse of a prolate gas spheroid in general relativity. They numerically evolved the Einstein-Vlasov equations using maximal (i.e., $K=0$, where K denotes the trace of extrinsic curvature) slices until a singularity was reached. In some of their models, this singularity appeared in the vacuum region just outside the "tips" of the spindlelike configuration of gas. They then searched for the presence of an outer marginally trapped surface on their slices and found that none occurred, thus showing that no outer trapped surfaces lie within any of their time slices. As they noted, no general theorems require the presence of trapped surfaces in the collapse to a black hole. (The event horizon of the black hole must "settle down" to an outer marginally trapped surface at late times, but will normally have a positive expansion at any finite time.) Nevertheless, the usual physical arguments concerning why black holes rather than naked singularities should be formed by collapse strongly suggests that outer trapped surfaces always should accompany black-hole formation. Thus, Shapiro and Teukolsky concluded that their examples provided strong evidence that cosmic censorship can be violated, i.e., that in their examples the singularity which forms is "naked."

The purpose of this paper is to point out that, in fact, phenomena qualitatively very similar to those of their examples should occur for appropriately chosen (highly nonspherical) slicings of spacetimes describing standard examples of spherical gravitational collapse of a body to a Schwarzschild black hole. Hence, it does not appear that the phenomena they find should be viewed as evidence against cosmic censorship. In order to determine whether a black hole or naked singularity occurs in the examples of Shapiro and Teukolsky, it will be necessary to continue the evolution of their spacetimes considerably further into the future.

To explain how a choice of slicing of a spherical-collapse spacetime can be made so as to produce features similar to the Shapiro-Teukolsky examples, we shall first consider the pure vacuum, extended Schwarzschild (i.e., Kruskal) spacetime. We shall prove that one can choose Cauchy surfaces for a Schwarzschild spacetime which come arbitrarily close to hitting the black-hole singularity at $r=0$ such that no trapped surfaces can be found in the past of any of these Cauchy surfaces. The modifications needed to obtain similar results for spherical-collapse spacetimes then will be described at the end of this paper.

The key fact upon which our construction is based is that the Schwarzschild singularity at $r=0$ possesses "angular horizons" of much the same nature as the well-studied "cosmological horizons" occurring in Robertson-Walker models. Just as two observers emerging from the "big-bang" singularity cannot attain causal contact until a finite time after the big bang, even though $a=0$ at the big bang so that they start out "zero distance" apart, two observers radially falling into a Schwarzschild black hole from different angles will lose causal contact a finite time before reaching the singularity, even though $r=0$ at the singularity. The angular horizons are easily computed by integrating the "orbit equation"

$$\frac{d\theta}{dr}=\frac{1}{[E^2r^4/L^2+r(2m-r)]^{1/2}} \tag{1}$$

for a null geodesic in Schwarzschild spacetime [see Eq. (6.3.35) of [3]]. This equation is commonly used to calculate the bending of light rays outside the black hole. However, it also is valid inside the black hole, where r plays the natural role of a time coordinate:

$$ds^2=-\frac{r}{2m-r}dr^2+\frac{2m-r}{r}dt^2 +r^2(d\theta^2+\sin^2\theta d\phi^2)\,. \tag{2}$$

From Eq. (1), we see that if an observer at the north pole ($\theta=0$) of a two-sphere inside the black hole at "time" r emits a light ray, the total change in the angular coordinate of the light ray before it is absorbed by the singularity at $r=0$ is

$$\Delta\theta=\int_0^r\frac{dr}{[E^2r^4/L^2+r(2m-r)]^{1/2}} \le\int_0^r\frac{dr}{(2mr-r^2)^{1/2}} =2\arcsin\sqrt{r/2m}\,. \tag{3}$$

Thus, an observer at radius $r<2m$ and $\theta=0$ cannot send a signal to any event whatsoever in the spacetime which is located at an angular coordinate greater than

$$\theta_0(r)=2\arcsin\sqrt{r/2m}\,. \tag{4}$$

Note that $\theta_0(r)<\pi$ for all $r<2m$. Thus, an observer who enters the black hole at the north pole ($\theta=0$) cannot causally communicate with any observer who falls into the black hole along the south pole.

The basic idea of our construction may now be explained. The above calculation shows that the north pole of any sphere within the black hole is spacelike related to the south pole of any other two-sphere within the black hole. Hence, it should be possible to choose a spacelike Cauchy surface $\mathcal{C}$ of extended Schwarzschild spacetime, which interpolates between trajectories of the north and south poles similar to those illustrated in Fig. 1. The causal past $J^-(\mathcal{C})$ of such a Cauchy surface would then contain no "complete two-sphere" (i.e., a two-sphere which is homotopically nontrivial) within the black hole.

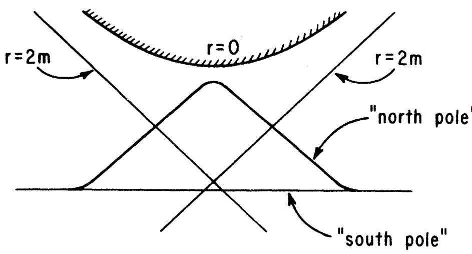

FIG. 1. A spacelike hypersurface in extended Schwarzschild spacetime, determined by the equation $F(r,t,\theta)=0$. Depicted are the trajectories of the north and south poles, i.e., the curves determined by $F(r,t,0)=0$ and $F(r,t,\pi)=0$. (The trajectories at intermediate angles interpolate between these two.) The north pole comes close to the black-hole singularity, but the south pole remains outside the black hole.

Hence $J^-(\mathcal{C})$ would be an excellent candidate for a suitable portion of extended Schwarzschild spacetime which comes arbitrarily close to the singularity and yet contains no trapped surfaces.

We now shall establish that in extended Schwarzschild spacetime, Cauchy surfaces of this sort do indeed exist. Specifically, we prove the following theorem.

Theorem. Given $\varepsilon>0$, there exists a Cauchy surface $\mathcal{C}$ of extended Schwarzschild spacetime having the properties that (i) there exists a point $x\in\mathcal{C}$ which lies inside the black hole at coordinate radius $r\le\varepsilon$ and (ii) $J^-(\mathcal{C})$ contains no trapped surfaces.

Proof. To begin the proof of this theorem, let γ be any future-directed timelike curve along which $\theta=0$, such that γ terminates at the singularity at $r=0$. (In other words, γ is the world line of an observer who enters the black hole at the north pole and undergoes no angular motion as he falls into the black-hole singularity.) Consider the chronological past of γ, denoted $I^-(\gamma)$. [$I^-(\gamma)$ defines a terminal indecomposable past set (TIP), i.e., a point of the causal boundary of Schwarzschild spacetime; see, e.g., [4].] We can characterize $I^-(\gamma)$ most usefully by specifying the portion of each two-sphere in a Kruskal diagram which lies in $I^-(\gamma)$. This is done in Fig. 2. Note that for any $r<2m$ the maximum value of θ over all two-spheres on the hypersurface Σ_r consisting of all events

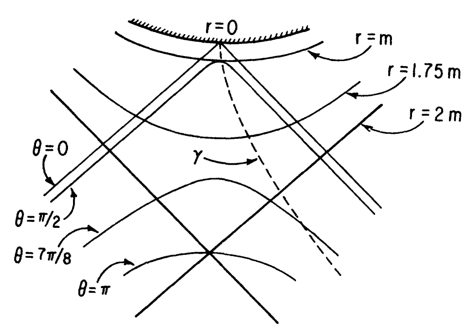

FIG. 2. A Kruskal diagram, with the lines of constant θ showing the portion of each two-sphere which lies in $I^-(\gamma)$. [Below the line $\theta=\pi$, the entire two-sphere lies in $I^-(\gamma)$.]

with radial coordinate r is given by Eq. (4). In Fig. 3, we also show the intersection of $I^-(\gamma)$ with the time slice Σ_r with $r=m$.

The proof of the theorem will be based upon the fact that $I^-(\gamma)$ contains no outer trapped surfaces lying within $r \leq 2m$. To demonstrate this, we first show that for any $r<2m$, $I^-(\gamma)$ contains no outer trapped surfaces which lie on the hypersurface Σ_r. To prove this, suppose that $T \subset \Sigma_r \cap I^-(\gamma)$ were outer trapped; i.e., we suppose that T is a two-dimensional, compact (without boundary) surface lying within Σ_r such that the convergence ρ of the outgoing null geodesics orthogonal to T is everywhere positive, $\rho>0$. Now, the outgoing null normal k^a can be written as

$$k^a = n^a + s^a \,, \tag{5}$$

where n^a is the unit (timelike) normal to Σ_r and s^a is the (spacelike) outgoing unit normal to T in Σ_r. Hence, we have

$$\rho = -q^{ab}\nabla_a k_b = -q^{ab}(\nabla_a n_b + \nabla_a s_b) = -q^{ab}K_{ab} - P \,. \tag{6}$$

Here q_{ab} denotes the induced metric on T, P denotes the trace of the extrinsic curvature of T in Σ_r, and K_{ab} denotes the extrinsic curvature of Σ_r in Schwarzschild spacetime, which is easily computed to be

$$K_{ab} = \left(\frac{2m}{r} - 1\right)^{1/2} \left[\frac{m}{r^2}(dt)_a(dt)_b - r[(d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b]\right] . \tag{7}$$

As Eq. (7) shows, the Σ_r hypersurfaces are contracting in the θ and ϕ directions but expanding in the t direction as one moves "forward in time," i.e., as r decreases. Hence, insofar as the first term in Eq. (6) is concerned, the most difficult place on T to maintain positive convergence is a point where the surface is tangent to the $(\partial/\partial t)$ direction. This suggests that we focus attention on a neighborhood of the point $p \in T$ at which the angular coordinate θ attains its maximum value, $\theta_{\max}$. Now, in the neighborhood of p, the surface T on Σ_r can be described locally by the equation $\theta = f(t,\phi)$. In terms of f, the convergence ρ at p is given from Eqs. (6) and (7) by

$$\rho = \frac{m/r - 1}{(2mr - r^2)^{1/2}} - \frac{\cot\theta}{r} + \frac{r^2}{2m - r}\frac{\partial^2 f}{\partial t^2} + \frac{1}{r\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \,. \tag{8}$$

Since θ is maximized at p, we have $\partial^2 f/\partial t^2 \leq 0$, $\partial^2 f/\partial\phi^2 \leq 0$, and, hence, at p,

$$\rho \leq \frac{m/r - 1}{(2mr - r^2)^{1/2}} - \frac{\cot\theta_{\max}}{r} \,. \tag{9}$$

Furthermore, since T lies in $I^-(\gamma)$, we have $\theta_{\max} < \theta_0(r)$, where θ_0 is given by Eq. (4). Remarkably, we obtain

$$\frac{\cot\theta_0}{r} = \frac{m/r - 1}{(2mr - r^2)^{1/2}} \,, \tag{10}$$

and, hence, at $p \in T$ we find that

$$\rho < 0 \,, \tag{11}$$

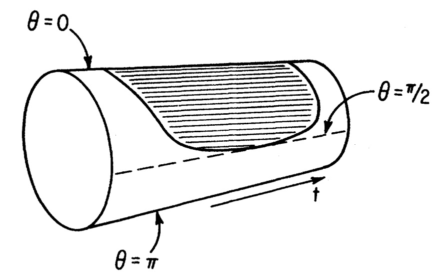

FIG. 3. The hypersurface $r=m$, which has the topology (and geometry) of $S^2 \times \mathbb{R}$. (The ϕ coordinate is suppressed.) The shaded region shows the portion of this hypersurface lying in $I^-(\gamma)$.

which shows that T cannot be outer trapped, as we desired to show.

To prove that no trapped surfaces exist in $I^-(\gamma)$ we assume that $T' \subset I^-(\gamma)$ is trapped. We note first that by a standard theorem (see, e.g., [3,4]) T' must be entirely contained within the black hole ($r \leq 2m$), so we may restrict attention to that region of extended Schwarzschild spacetime. We let $p' \in T'$ be the point which maximizes the function $g(r,\theta,\phi) = \theta + \theta_0(r)$ on T', where θ_0 is defined by Eq. (4). In a neighborhood of p', consider the congruence of the outgoing null geodesics normal to T'. Let $\delta > 0$ and consider the intersection of this congruence with the hypersurface $\Sigma_{r-\delta}$, where r denotes the radial coordinate of p'. (Thus $\Sigma_{r-\delta}$ is an $r=$const hypersurface which lies "above" p'.) Let q_δ denote the intersection point of the null geodesic emanating from p' with $\Sigma_{r-\delta}$. Then, for δ sufficiently small, the maximum of the θ coordinate of the intersection of the outgoing null congruence with $\Sigma_{r-\delta}$ will be achieved at q_δ. Hence, Eq. (9) holds for the convergence of the outgoing null congruence at q_δ. However, the Raychaudhuri equation (see, e.g., [3]) implies that ρ is nondecreasing along each null geodesic. Hence, we obtain

$$\rho(p') \leq \frac{m/(r-\delta) - 1}{[2m(r-\delta) - (r-\delta)^2]^{1/2}} - \frac{\cot\theta(q_\delta)}{r} \,. \tag{12}$$

In the limit $\delta \to 0$, we have

$$\theta(q_\delta) \to \theta(p') < \theta_0(r) \,. \tag{13}$$

Thus, we obtain

$$\rho(p') < 0 \,, \tag{14}$$

giving the desired conclusion that T' is not outer trapped and, hence, not trapped.

Now let M' denote the union of $I^-(\gamma)$ with the region of extended Schwarzschild spacetime which lies outside the black hole. Then clearly M' contains no trapped surfaces. Furthermore, if we view M' as a spacetime in its own right, it is globally hyperbolic. [Proof: We have $M' = I^-(\gamma \cup B)$, where B denotes the boundary of the black-hole region of Schwarzschild spacetime. However, any region of a globally hyperbolic spacetime which can be expressed as the chronological past of any subset of the spacetime is itself globally hyperbolic; see, lemma 2.1 of [5] for a proof.] Hence, given any $x \in M'$, there exists a

Cauchy surface $\mathcal{C}$ for M' passing through x. However, any such $\mathcal{C}$ also must be a Cauchy surface for the entire Schwarzschild spacetime M. (Proof: If not, there would exist a timelike curve which is past and future inextendible in M and lies entirely within $M-M'$. However, $M-M'$ is a subset of the black-hole region, and there does not exist any past inextendible timelike curve contained in the black-hole region of extended Schwarzschild spacetime.) Finally, the past of $\mathcal{C}$ in M is contained in M' and, hence, contains no trapped surfaces. Thus, given $\varepsilon > 0$, we choose $x \in M'$ to lie within the black hole at coordinate radius $r \leq \varepsilon$. Then, any Cauchy surface $\mathcal{C}$ for $M-M'$ which passes through x satisfies the two properties claimed in the theorem.□

Although the above proof shows that no outer trapped or outer marginally trapped surfaces exist in $I^-(\gamma)$ within the black hole ($r \leq 2m$), it might appear from Eqs. (9) and (10) that the causal past of γ, $J^-(\gamma)$, could contain a marginally outer trapped surface on any Σ_r hypersurface with $r < 2m$. However, this is not the case because if we choose a surface with $\theta_{\max} = \theta_0$, the terms in Eq. (8) involving the second derivatives of f will make a strictly negative contribution to ρ, so again we have $\rho < 0$ at the point p where $\theta = \theta_{\max}$. Nevertheless, $J^-(\gamma)$ does contain one marginally trapped surface: the bifurcation two-sphere of the event horizon, at $r = 2m$.

Now, let S be any spacelike slice of extended Schwarzschild spacetime whose past includes γ; i.e., S intersects the black-hole singularity at the TIP defined by γ. Since $I^-(S)$ must include an open neighborhood of the bifurcation two-sphere, it follows that $I^-(S)$ will contain trapped surfaces. However, by choosing S to lie very close to the boundary $\dot{I}^-(\gamma)$ of the past of γ, there should be no difficulty in ensuring that all of the outer trapped surfaces in $I^-(S)$ within the black hole must enter a small neighborhood of the bifurcation two-sphere.

Consider, now, a spacetime corresponding to the spherical gravitational collapse of a body to a Schwarzschild black hole. Then the bifurcation two-sphere of extended Schwarzschild spacetime will be "covered up" by the collapsing matter. In such a spacetime, let γ be a timelike curve which enters the black-hole singularity in the vacuum region outside the collapsing matter. Then, by choosing a spacelike slice S, which lies close to $\dot{I}^-(\gamma)$ while inside the black hole (but becomes asymptotically flat at infinity), there should be no difficulty in ensuring that $I^-(S)$ contains no outer trapped surfaces. Thus, we believe that in *any* spacetime describing spherical collapse to a Schwarzschild black hole, one can find a portion of the spacetime, namely, $I^-(S)$, which has all the qualitative features of the portions of the spacetimes obtained numerically by Shapiro and Teukolsky.

Of course, the spacetimes constructed by Shapiro and Teukolsky are highly nonspherical, and undoubtedly many details regarding the causal structure of their spacetimes near the singularity will differ from the spherical case. Furthermore, there are additional reasons for believing that a black hole might not form in their examples, specifically, the similarity of their spacetimes to that of a collapsing infinite cylinder. However, our analysis indicates that the absence of outer trapped surfaces in the portions of the spacetimes which they construct is entirely consistent with the formation of a black hole.

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[1] S. L. Shapiro and S. A. Teukolsky, Phys. Rev. Lett. **66**, 994 (1991).

[2] S. L. Shapiro and S. A. Teukolsky, Am. Sci. **79**, 330 (1991).

[3] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[4] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge Univ. Press, Cambridge, 1973).

[5] B. S. Kay and R. M. Wald, Phys. Rep. **207**, 49 (1991).