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Parity anomaly in three dimensions via fermion-number fractionalization in two dimensions

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A new approach, peculiar in its simplicity, to the parity anomaly of relativistic fermions coupled to a static external gauge field, in 2+1 dimensions, clarifies the relationship to the fermion number fractionalization in 1+1 dimensions. From this correspondence we get the parity-anomaly behavior at finite temperature.

I. INTRODUCTION

The parity anomaly in (2+1)-dimensional quantum electrodynamics [1] means that from the theory defined by

$$\mathcal{L}=\bar{\psi}(i\not{D}-m)\psi \tag{1}$$

it follows that the fermion current has a vacuum expectation value

$$\langle J^\mu\rangle=\frac{m}{|m|}\frac{e^2}{8\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}+O(m^{-1}) \tag{2}$$

which shows the breaking of parity symmetry.

The interest in this result increased when some relationship with the condensed-matter physics of two-dimensional systems was suggested [2], the quantum Hall effect being a particular example. The usual derivation of (2) is based on an accurate analysis of the fermion determinant.

If model (1) is studied at finite temperature, (2) can be obtained when $\beta\to\infty$, that is, at the low-temperature limit. Finite-temperature effects have been studied [3] by general methods. However, the results in the literature were obtained only at the low-temperature limit [4].

There is, nevertheless (as we will show), a strong relationship between this anomaly and the fermion number fractionalization in 1+1 dimensions, which gives us a straightforward way to compute the parity anomaly at finite temperature. Our approach consists in the transformation of (1), at finite temperature, into a model made up of an infinity of uncoupled fermions in 1+1 dimensions. We, then, recognize $\langle J^\mu\rangle$ as a sum of currents defined in 1+1 dimensions. These currents, on the other hand, can be obtained using the method of Goldstone and Wilczek [5] (GW).

In the following we will review briefly the GW method and, then, show our approach to the parity anomaly.

II. THE GW METHOD

Some models in field theory may exhibit fractional charge [4–6]. The GW method is a general way to obtain, using perturbation theory, the fermionic current. Let us consider, following GW, the Lagrangian

$$\mathcal{L}=i\bar{\psi}\not{\partial}\psi+\bar{\psi}(\varphi_1+i\varphi_2\gamma^5)\psi \tag{3}$$

defined in 1+1 dimensions. Let x_0 be a point such that $\varphi_1(x_0)\neq 0$ and $\varphi_2(x_0)=0$. The first step of the GW method is the computation of $\langle J^\mu(x_0)\rangle=\langle 0|\bar{\psi}(x_0)\gamma^\mu\psi(x_0)|0\rangle$ by means of an expansion in gradients, where one considers that the fields φ_1 and φ_2 were introduced adiabatically in the fermion system. Then, looking at the value of $\langle J^\mu(x_0)\rangle$, we put it in a form which is chirally symmetric, that is, invariant when $\varphi_1\to-\varphi_2$ and $\varphi_2\to\varphi_1$. The final result is

$$\langle J^\mu(x)\rangle=(2\pi)^{-1}\epsilon^{\mu\nu}\epsilon^{ab}\frac{\varphi_a\partial_\nu\varphi_b}{\varphi^2}. \tag{4}$$

III. PARITY ANOMALY AT FINITE TEMPERATURE

Let us study (1) at finite temperature. In order to perform a more complete analysis, we add to (1) a chemical-potential term, $u\bar{\psi}\gamma^0\psi$. The partition function (in Euclidean space) [7] is given by

$$Z=\int D\bar{\psi}D\psi\exp(-S[\bar{\psi},\psi]), \tag{5}$$

where

$$S[\bar{\psi},\psi]=\int_0^\beta dt\int d^2x\,\bar{\psi}(\not{D}-m-u\gamma^0)\psi \tag{6}$$

and

$$\not{D}=\gamma^0\frac{\partial}{\partial t}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}-ie\gamma^\mu A_\mu, \tag{7}$$

$$\bar{\psi}=\psi^\dagger\gamma^1, \tag{8}$$

$$\gamma^0=\sigma_3,\ \gamma^1=\sigma_1,\ \gamma^2=\sigma_2\ \text{(Pauli matrices)}. \tag{9}$$

Because of the antiperiodic condition for fermions, we can express the fields ψ and $\bar{\psi}$ as

$$\psi(x,t)=\sum_n e^{i\omega_n t}\psi_n(x), \tag{10}$$

$$\bar{\psi}(x,t)=\sum_n e^{-i\omega_n t}\bar{\psi}_n(x), \tag{11}$$

where $\omega_n=(2n+1)\pi/\beta$, $n=0,\pm1,\pm2,\ldots$.

Substituting (10) and (11) in (5) and integrating over t, we get

$$Z=\int\prod_n D\psi_n D\bar{\psi}_n\exp(-S[\{\psi_n\},\{\bar{\psi}_n\}]), \tag{12}$$

where

$$S[\{\psi_n\},\{\bar{\psi}_n\}]=\beta\sum_n\int d^2x[\bar{\psi}_n(\nabla\!\!\!/-m)\psi_n-ieA_\mu\bar{\psi}_n\gamma^\mu\psi_n+(i\omega_n-u)\bar{\psi}_n\gamma^0\psi_n] \quad (13)$$

$$=\beta\sum_n\int d^2x[\bar{\psi}_n(\nabla\!\!\!/-m)\psi_n-ieA_i\bar{\psi}_n\gamma^i\psi_n-ieA_0\bar{\psi}_n\gamma^0\psi_n+(i\omega_n-u)\bar{\psi}_n\gamma^0\psi_n]\,. \quad (14)$$

In the above expression, $\nabla\!\!\!/=\gamma^1\partial/\partial x^1+\gamma^2\partial/\partial x^2$.

Since $(\gamma^1)^2=(\gamma^2)^2=1$, we may consider (14) as the Euclidean action of a theory which contains an infinity of fermions in 1+1 dimensions. To make this more explicit, let us, then, change our notation:

$$\gamma^1\to\alpha^0\,, \quad (15a)$$

$$\gamma^2\to\alpha^1\,, \quad (15b)$$

$$\gamma^0=\sigma_3=-i\sigma_1\sigma_2=-i\gamma^1\gamma^2\to -i\alpha^0\alpha^1\equiv-i\alpha^5\,, \quad (15c)$$

and also

$$A_1\to a_0,\ x_1\to y_0\,, \quad (16a)$$

$$A_2\to a_1,\ x_2\to y_1\,. \quad (16b)$$

The identification of (14) with a system of fermions now becomes clear. Using the above conventions, we have

$$S=\beta\sum_n\int d^2y[\bar{\psi}_n(D\!\!\!\!/_E-m)\psi_n-eA_0\bar{\psi}_n\alpha^5\psi_n+(iu+\omega_n)\bar{\psi}_n\alpha^5\psi_n]\,, \quad (17)$$

where

$$D\!\!\!\!/_E=\alpha^0\frac{\partial}{\partial y^0}+\alpha^1\frac{\partial}{\partial y^1}-ie(\alpha^0a_0+\alpha^1a_1)\,.$$

The corresponding theory, in Minkowski space, is given by

$$S=\beta\sum_n\int d^2y\{\bar{\psi}_n iD\!\!\!\!/\psi_n-\bar{\psi}_n[m+i(eA_0-iu-\omega_n)\alpha^5]\psi_n\}\,. \quad (18)$$

At this point, we stress the similarity between action (18) and the one obtained from the Lagrangian (3). The difference is that in (18) there is a covariant derivative. However, this would contribute with terms proportional to a_μ to the current (4), which must be absent due to the gauge invariance. For a while we neglect the problems associated with the chemical potential, that is, we use (4) to get

$$\langle j^\mu\rangle\equiv\beta\sum_n\langle\bar{\psi}_n\alpha^\mu\psi_n\rangle \quad (19)$$

$$=\frac{1}{2\pi}\sum_n\frac{\epsilon^{\mu\nu}m\partial_\nu(eA_0)}{m^2+(eA_0-iu-\omega_n)^2}+\epsilon^{\mu\nu}\partial_\nu\Lambda\,. \quad (20)$$

The $\epsilon^{\mu\nu}\partial_\nu\Lambda$ added in (20) should also, in principle, be added to (4), as it is compatible with current conservation. Λ depends on the fields. Yet, this is a term of order m^{-1} that, as we will see, depends on β as β^α, $0\le\alpha\le1$, when $\beta\to\infty$. The field Λ is assumed well behaved, that is, $\Lambda(x)\to0$ when $x^\mu\to\pm\infty$, and does not contribute to the fermion number. The sum in (20) may be performed exactly [8]:

$$\sum_n\frac{1}{m^2+(eA_0-iu-\omega_n)^2}=\frac{\beta}{4i|m|}\{\tan[\tfrac{1}{2}eA_0\beta+\tfrac{1}{2}i\beta(|m|-u)]-\tan[\tfrac{1}{2}eA_0\beta-\tfrac{1}{2}i\beta(|m|+u)]\}\,. \quad (21)$$

The above quantity is, in general, not real. This follows from the use of result (4), whenever φ_2, in (3), has a nonzero imaginary part. This will occur for $u\neq0$ and $A_0\neq0$. The result is real for $u=0$, for any A_0, and for $A_0=0$, for any u. In the remaining cases, we may simply take the real part of (21) (since $\langle J^\mu\rangle$ has to be real) and rewrite (4) as

$$\langle J^\mu\rangle=(2\pi)^{-1}\epsilon^{\mu\nu}\epsilon^{ab}\frac{1}{2}\left(\frac{\varphi_a\partial_\nu\varphi_b}{\varphi^2}+\text{c.c.}\right)\,.$$

Let us now consider the true current in the (2+1)-dimensional gauge theory. We have

$$\langle J^\mu\rangle=e\langle\bar{\psi}\gamma^\mu\psi\rangle$$

$$=\sum_{n,m}e\langle\bar{\psi}_n\gamma^\mu\psi_m\rangle\exp[it(\omega_m-\omega_n)]\,. \quad (22)$$

Since $\langle J^\mu\rangle$ does not depend on t, its value is given by

$$\langle J^\mu\rangle=e\sum_n\langle\bar{\psi}_n\gamma^\mu\psi_n\rangle\,. \quad (23)$$

Now, using (20) and (21), we obtain

$$\langle J^1\rangle=e\sum_n\langle\bar{\psi}_n\gamma^1\psi_n\rangle=e\sum_n\langle\bar{\psi}_n\alpha^0\psi_n\rangle=\frac{e}{\beta}\langle j^0\rangle$$

$$=\frac{m}{|m|}\frac{e^2}{8\pi}(\partial_2A_0)\text{Im}\{\tan[\tfrac{1}{2}eA_0\beta+\tfrac{1}{2}i\beta(|m|-u)]-\tan[\tfrac{1}{2}eA_0\beta-\tfrac{1}{2}i\beta(|m|+u)]\}+\frac{e}{\beta}\partial_2\Lambda \quad (24a)$$

where Im denotes the imaginary part, and also

$$\langle J^2\rangle=e\sum_n\langle\bar{\psi}_n\gamma^2\psi_n\rangle=e\sum_n\langle\bar{\psi}_n\alpha^1\psi_n\rangle=\frac{e}{\beta}\langle j^1\rangle$$

$$=-\frac{m}{|m|}\frac{e^2}{8\pi}(\partial_1A_0)\text{Im}\{\tan[\tfrac{1}{2}eA_0\beta+\tfrac{1}{2}i\beta(|m|-u)]-\tan[\tfrac{1}{2}eA_0\beta-\tfrac{1}{2}i\beta(|m|+u)]\}-\frac{e}{\beta}\partial_1\Lambda\,. \quad (24b)$$

To compute $\langle J^0 \rangle$ we use the fact that, according to (18),

$$\langle J^0 \rangle = -\frac{i}{\beta}\frac{\delta \ln Z}{\delta A_0} = \frac{e}{\beta}\frac{\delta}{\delta A_0}\left[\int a^\mu(x')\langle j_\mu(x')\rangle d^2x'\right]. \tag{25}$$

Considering now (25), (24a), and (24b), we see that

$$\langle J^0 \rangle = \frac{e}{\beta}\frac{\delta}{\delta A_0(x)}\left[\int a^\mu(x')\left[\frac{e\beta}{8\pi}\frac{m}{|m|}\epsilon_{\mu\nu}[\partial^\nu A_0(x')]F(A_0(x')) + \epsilon_{\mu\nu}\partial^\nu \Lambda(x')\right]d^2x'\right], \tag{26}$$

where

$$F(A_0(x')) = \mathrm{Im}\{\tan[\tfrac{1}{2}eA_0\beta + \tfrac{1}{2}i\beta(|m|-u)] - \tan[\tfrac{1}{2}eA_0\beta - \tfrac{1}{2}i\beta(|m|+u)]\}$$

and the indices μ, ν, above, need to be interpreted with care, to avoid confusion.

From (26) we get, using the original notation for the coordinates,

$$\langle J^0 \rangle = \frac{m}{|m|}\frac{e^2}{8\pi}(\partial_1 A_2 - \partial_2 A_1) \times \mathrm{Im}\{\tan[\tfrac{1}{2}eA_0\beta + \tfrac{1}{2}i\beta(|m|-u)] - \tan[\tfrac{1}{2}eA_0\beta - \tfrac{1}{2}i\beta(|m|+u)]\} + O(m^{-1}). \tag{27}$$

Let us examine more carefully the results (24a), (24b), and (27). Studying the case $u=0$, we find that when $\beta \to \infty$ the currents will be expressed as

$$\langle J^0 \rangle = \frac{m}{|m|}\frac{e^2}{4\pi}(\partial_1 A_2 - \partial_2 A_1) + O(m^{-1}), \tag{28a}$$

$$\langle J^1 \rangle = \frac{m}{|m|}\frac{e^2}{4\pi}\partial_2 A_0 + O(m^{-1}), \tag{28b}$$

$$\langle J^2 \rangle = -\frac{m}{|m|}\frac{e^2}{4\pi}\partial_1 A_0 + O(m^{-1}). \tag{28c}$$

However, when $\beta \to \infty$ Lorentz invariance is restored, which allows us, from (28), to say that, in the more general case of fields A_μ depending on time, we will have

$$\langle J^\mu \rangle = \frac{m}{|m|}\frac{e^2}{8\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} + O(m^{-1}), \tag{29}$$

which is the well-known result.

We, now, come back to the discussion of the dependence of Λ with β. The terms $O(m^{-1})$ in (28) are, as we can easily see, dependent on Λ/β. Assuming, therefore, that $\langle J^\mu \rangle$ is well defined for all values of β, we see that Λ behaves like β^α, $0 \le \alpha \le 1$, when $\beta \to \infty$.

IV. CONCLUSIONS

The (2+1)-dimensional gauge-field theory was studied as a theory of an infinity of fermions in 1+1 dimensions. We derived the parity anomaly at finite temperature in the case of static gauge fields. We note that the method employed here, to pass from 2+1 to 1+1 dimensions, may be useful to study other (2+1)-dimensional systems, once there is a great amount of information on (1+1)-dimensional models.

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