Generalized Tremaine-Gunn limits for bosons and fermions

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The Tremaine-Gunn phase-space constraint giving lower limits on the mass of dark-matter particles that were once in kinetic equilibrium is generalized to arbitrary decoupling temperatures and chemical potentials for fermions as well as bosons. It is stressed that even in the cases where no *exact* limit exists (e.g., for bosons with a chemical potential approaching the mass) a *statistical* limit can be obtained. In most cases particle masses of several eV or more are necessary to explain the dark matter in galaxy halos.

I. INTRODUCTION

One of the methods often used for constraining the mass of elementary-particle candidates for the dark matter in galaxies is the phase-space constraint due to Tremaine and Gunn [1]. Originally the method was constructed with the aim of placing lower limits on the mass of neutrino candidates for dark matter, but the method was generally applicable to relativistically decoupling fermions with a zero chemical potential. Madsen [2] generalized the limits to bosons, realizing the statistical nature of the method for bosons with mass equal to chemical potential. A few comments on nonrelativistic decoupling were included in the latter investigation as well.

The present paper is devoted to a general derivation of lower mass limits from conservation of fine-grained phase-space density for bosons as well as fermions with arbitrary masses, chemical potentials, and decoupling temperatures. The classical Tremaine-Gunn limit is the special case of fermions with a zero chemical potential and relativistic decoupling. Other analytical limits are derived as well, and the general cases are illustrated in the figures.

The paper is organized as follows. Section II contains a derivation of the Tremaine-Gunn limit for a general distribution function and comments on the possible loop holes in the method related to the assumption of isothermality of the final state. Section III summarizes the general mass constraints from phase-space conservation, and gives some analytical limits. Section IV is devoted to the relation between the mass limits derived and the cosmic density parameter. Section V shows how statistical limits can be obtained in cases where no exact limits exist, and how statistical limits in general are stronger than the exact limits. Section VI contains the conclusion.

II. THE TREMAINE-GUNN LIMIT AND ITS LIMITATIONS

Consider an isotropic gas with a distribution function given so that $4\pi p^2 n(p)dp$ is the number density of particles with momenta between p and $p+dp$. The *occupation number* is then given as $f(p)=h^3 n(p)/g$, where h is Planck's constant and g is the number of helicity states. For a particle species in kinetic equilibrium the occupation number is given by a Fermi-Dirac or Bose-Einstein distribution,

$$f(p)=\{\exp[(E-\mu)/T]\pm 1\}^{-1}\ , \qquad (1)$$

with μ being the chemical potential, and the energy related to momentum and mass via $E^2=p^2+m^2$ (unless otherwise noted we work in units where $c=k_B=1$).

Denote by $N(f)$ the fraction of particles with an occupation number exceeding f. Using a method introduced by Tremaine, Hénon, and Lynden-Bell [3], Madsen [2] utilized conservation of fine-grained phase-space density during dissipationless evolution to calculate the distribution of the maximal fraction of particles, $N(\varphi)$, that can be arranged to have a coarse-grained phase-space density exceeding φ. Whereas $N(\varphi)$ is general exceeds $N(f)$, a crucial observation is that φ cannot exceed an eventual maximum value of f.

This observation was the basis of the Tremaine-Gunn argument [1], which in its simplest version says that the coarse-grained phase-space density of neutrinos in galaxy halos cannot exceed the maximum value of the initial fine-grained density, which in the case of one type of relativistically decoupling neutrino with zero chemical potential equals $g/2h^3$ (in the subsequent derivation we shall use $f_{\max}g/h^3$ for the maximal fine-grained phase-space density). Assuming the dark-matter distribution to be an isothermal sphere with core radius $r_c=(9\sigma^2/4\pi G\rho_c)^{1/2}$, where ρ_c is the central density and σ is the Maxwellian one-dimensional velocity dispersion, the corresponding maximum phase-space density is $\rho_c m^{-4}(2\pi\sigma^2)^{-3/2}$, where m is the particle mass. Requiring this maximum to be less than $f_{\max}g/h^3$ leads to

$$m>m_{\min}\equiv\left[\frac{9h^3}{2(2\pi)^{5/2}f_{\max}gG\sigma r_c^2}\right]^{1/4}$$
$$=38\ \text{eV}\ \sigma_{100}^{-1/4}r_{10}^{-1/2}g^{-1/4}f_{\max}^{-1/4}\ . \qquad (2)$$

Here $\sigma_{100}=\sigma/100\ \text{km}\,\text{s}^{-1}$ and $r_{10}=r_c/10$ kiloparsecs (kpc).

Equation (2) has been applied to a number of galaxies by a number of authors for placing lower bounds on the mass of dark-matter neutrinos (effectively $f_{\max}=1$ for neutrino plus antineutrino with $\mu=0$). "Typical" neutrino mass limits derived for spiral galaxies are in the range 5–40 eV, whereas the presence of dark matter in dwarf spheroidals (still a somewhat controversial issue on the observational side) would correspond to limits in the range 100–300 eV. For comparison the upper cosmological bound for a single stable flavor is [4] $m_\nu \leq 91.5$ eV $\Omega h_0^2 g^{-1}$ (Ω is the present mean density in units of the critical density, and h_0 is the Hubble parameter in units of 100 km s^{-1} Mpc^{-1}, with $0.4<h_0<1$), and the experimental limit on the mass of the electron antineutrino is close to 10 eV [5]. Different authors have drawn different conclusions on the viability of neutrino dark matter from these numbers [1,6–13].

Before continuing with a generalization of the Tremaine-Gunn limit it should be emphasized that the application of Eq. (2) and the more general results below is subject to a number of problems, which means that one should regard the limits as order-of-magnitude estimates, rather than precise values.

(1) It is very difficult to subtract the luminous contribution from most rotation curves, thereby isolating the dark-matter contribution. Only a quite small number of galaxies are modeled in such detail, that it makes sense to extract well-defined parameters for the dark halo. Better observations and detailed modeling have shown that many of the galaxies for which neutrino mass limits have been published (including some used by the present author) are unsuited for that purpose [14,10].

(2) The dark matter is (by definition) not directly observable. The actual coarse-grained phase-space distribution cannot be seen, and there is no guarantee that the assumption of an isothermal sphere, which is a basis of Eq. (2), is correct. In fact, it is not even known whether the dark matter has an isotropic velocity distribution. As discussed in Refs. [7,9,12,13], deviations from isotropy may significantly change mass limits, thereby weakening the Tremaine-Gunn constraint.

(3) If an isothermal sphere is assumed for the dark matter, there are still problems related to the determination of the core radius and velocity dispersion. Using measured values for the luminous component is not necessarily correct. In fact, there are good reasons for believing that the dark matter is more extended than the luminous matter, so use of the luminous core radius makes the neutrino mass limits too strong. Similar problems relate to the velocity dispersion. For instance, the neutrino mass determined from dwarf spheroidals may be reduced to acceptable values if the dark halos are much more extended than the stellar components [6,8,15].

(4) Finally one should incorporate knowledge about the complete fine-grained distribution function, and the corresponding bounds on the coarse-grained distribution, instead of only comparing maximum phase-space densities. An attempt in this direction was made by Madsen and Epstein [6], who introduced the concept of a maximally compact sphere of neutrinos consistent with the neutrino distribution function at decoupling and the spherical Jeans equation (the construction is not a steady-state solution, but nevertheless gives useful instantaneous limits). In Section V we shall return to the importance of the fact that in some cases a large fraction of particles have occupation numbers much below $f_{\max}$, so that at least statistically mass limits increase relative to Eq. (2).

In the rest of this paper we shall compare the fine-grained phase-space distribution of dark-matter particles with the isothermal sphere, and therefore neglect the possible consequences of anisotropy and other deviations from isothermality [item (2) above]. We shall not comment further on the observational problems discussed as items (1) and (3), whereas some consequences of considering the complete distribution of f rather than just $f_{\max}$ [item (4)] will be described in Sec. V. But the reader should be aware of the problems described and not use the generalized Tremaine-Gunn mass limits uncritically.

III. MAXIMUM OCCUPATION AND MINIMUM MASS

The distribution of fine-grained occupation numbers for a species in kinetic equilibrium is given by Eq. (1) at the temperature of decoupling, T_D. In particular, since the particle energy is at least equal to the mass, the *maximum* occupation number is given by

$$f_{\max}=\{\exp[(m-\mu)/T_D]\pm 1\}^{-1} . \tag{3}$$

For fermions, there are no *a priori* restrictions on the chemical potentials. For bosons, $\mu \leq m$, and for bosons and antibosons annihilating into photons ($\bar{\mu}=-\mu$), $-m \leq \mu \leq m$ to avoid negative occupation numbers for particles as well as antiparticles. (Sometimes one works in terms of the *kinetic* chemical potential, $\mu_k \equiv \mu - m$. In the present paper we shall not use μ_k, but note that the chemical potential used in Ref. [2] is in fact μ_k.)

Some useful limits can be extracted from Eq. (3). For example, for *bosons,*

$$f_{\max}\to\begin{cases}[(m-\mu)/T_D]^{-1} & \text{if } (m-\mu)/T_D\to 0 ,\\ \exp[-(m-\mu)/T_D] & \text{if } (m-\mu)/T_D\to\infty ,\end{cases} \tag{4}$$

so that the minimum mass goes like

$$m_{\min}\to 38\ \text{eV}\ \sigma_{100}^{-1/4} r_{10}^{-1/2} g^{-1/4} \times\begin{cases}[(m-\mu)/T_D]^{1/4} & \text{if } (m-\mu)/T_D\to 0,\\ \exp[(m-\mu)/4T_D] & \text{if } (m-\mu)/T_D\to\infty .\end{cases} \tag{5}$$

For *fermions,*

$$f_{\max}\to\begin{cases}1 & \text{if } (m-\mu)/T_D\to -\infty,\\ \frac{1}{2} & \text{if } (m-\mu)/T_D\to 0 ,\\ \exp[-(m-\mu)/T_D] & \text{if } (m-\mu)/T_D\to\infty ,\end{cases} \tag{6}$$

with

$$m_{\min}\to 38\ \text{eV}\ \sigma_{100}^{-1/4} r_{10}^{-1/2} g^{-1/4} \times\begin{cases}1 & \text{if } (m-\mu)/T_D\to -\infty ,\\ 2^{1/4} & \text{if } (m-\mu)/T_D\to 0 ,\\ \exp[(m-\mu)/4T_D] & \text{if } (m-\mu)/T_D\to\infty .\end{cases} \tag{7}$$

Note that fermions and bosons, as expected, behave similarly in the nonrelativistic limit $[(m-\mu)/T_D\to\infty]$, that the boson mass limits diverge for $(m-\mu)/T_D\to 0$, and that the fermion mass limits only depend very weakly on $(m-\mu)/T_D$ in the degenerate limit. The original Tremaine-Gunn limit [1] (for one flavor without antiparticles) corresponds to Eq. (7) for $(m-\mu)/T_D \to 0$ ($\mu=0$ and $m/T_D << 1$). The general results for $m_{\min}$ as given by Eqs. (2) and (3) are illustrated in Fig. 1.

IV. CONTRIBUTION TO THE COSMIC DENSITY

The present contribution to the mean mass density of the Universe from a particle species decoupling at temperature T_D is given by $\rho_x = m(R_D/R_0)^2 \int 4\pi p^2 n(p)dp$, where the integral over the distribution is to be calculated at decoupling, and R_D and R_0 are the scale factors at decoupling and the present, respectively. Using entropy conservation on the form $g_{*s}T^3R^3=$ constant, where $g_{*s}=\sum_{\text{bosons}} g_i(T_i/T)^3+\frac{7}{8}\sum_{\text{fermions}} g_i(T_i/T)^3$, and introducing the density parameter $\Omega_x=\rho_x/\rho_{\text{crit}}$ ($\rho_{\text{crit}}=8.099h_0^2\times 10^{-47}$ GeV^4), one finds

$$\Omega_x\rho_{\text{crit}} = m\frac{gg_{*s0}}{2\pi^2 g_{*sD}}T_0^3\int_{(m-\mu)/T_D}^{\infty}\frac{[(x+\mu/T_D)^2-(m/T_D)^2]^{1/2}(x+\mu/T_D)}{\exp(x)\pm 1}dx \ . \tag{8}$$

The integral can be evaluated analytically in certain limits, and takes the following values:

$$\begin{aligned}
&2\zeta(3), \quad \text{boson}, \quad T_D \gg m,\mu \ ,\\
&3\zeta(3)/2, \quad \text{fermion}, \quad T_D \gg m,\mu \ ,\\
&(\mu^2-m^2)^{3/2}/3T_D^3, \quad \text{fermion}, \quad T_D \ll \mu - m \ ,\\
&2\exp(\mu/T_D), \quad \text{fermion and boson}, \quad \mu<0; |\mu| \gg T \gg m \ ,\\
&2\pi^2(m/2\pi T_D)^{3/2}\exp[-(m-\mu)/T_D], \quad \text{fermion and boson}, \quad m, m-\mu \gg T_D \ ,
\end{aligned} \tag{9}$$

where Riemann's function is $\zeta(3)\approx 1.202$.

For a present temperature $T_0=2.75$ K $T_{2.75}$ one gets the limits

$$\frac{\Omega_x h_0^2 g_{*sD}}{gg_{*s0}T_{2.75}^3} = \begin{cases} 2.00\times 10^{-2} m_{\text{eV}}, \\ 1.50\times 10^{-2} m_{\text{eV}}, \\ 2.77\times 10^{-3} m_{\text{eV}}(\mu^2-m^2)^{3/2}/T_D^3, \\ 1.66\times 10^{-2} m_{\text{eV}}\exp(\mu/T_D), \\ 1.04\times 10^{-2} m_{\text{eV}}(m/T_D)^{3/2}\exp[-(m-\mu)/T_D] \ , \end{cases} \tag{10}$$

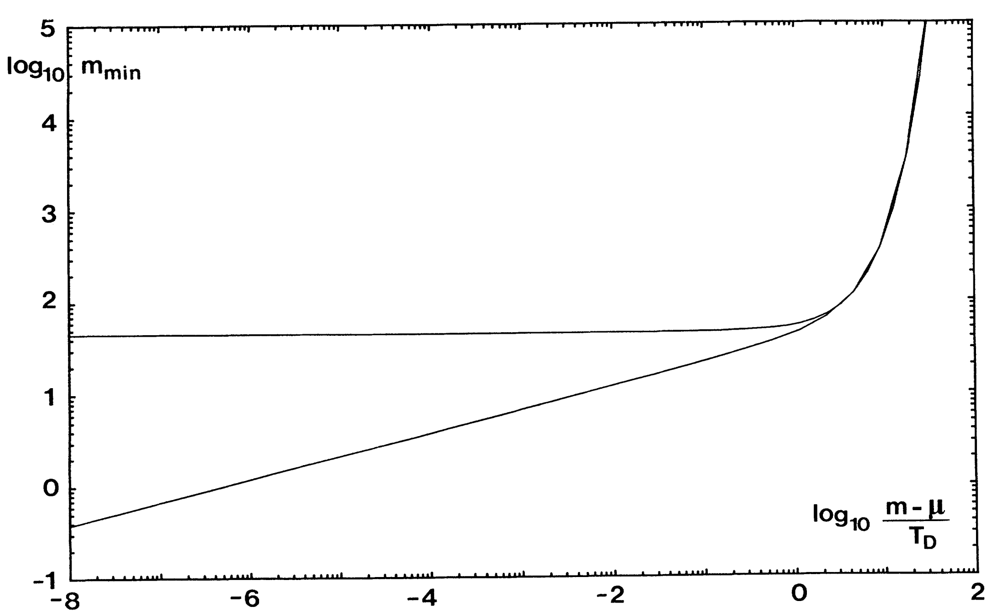

FIG. 1. Generalized Tremaine-Gunn mass limits as given by Eqs. (2) and (3). $m_{\min}$ in units of eV $\sigma_{100}^{-1/4} r_{10}^{-1/4} g^{-1/4}$ is plotted as a function of $(m-\mu)/T_D$ for fermions (upper curve) and bosons (lower curve).

for the same cases given in Eq. (9). (Here $m_{\rm eV}$ is the particle mass in eV.)

Combining Eqs. (2), (3), and (8) it is possible to calculate the minimal value of the density parameter required if the dark-matter particles obey the generalized Tremaine-Gunn constraints derived above. Results are shown in Figs. 2 and 3, where the value of $\Omega_x h_0^2 T_{2.75}^{-3} g^{-3/4} g_{*sD} g_{*s0}^{-1} \sigma_{100}^{1/4} r_{10}^{1/2}$ is plotted as a function of m/T_D for different values of μ/T_D. For the same sets of parameters Figs. 4 and 5 show the corresponding value of the minimum mass required (in units of eV $\sigma_{100}^{-1/4} r_{10}^{-1/2} g^{-1/4}$).

A few trends are apparent. First of all, particles decoupling in the extremely nonrelativistic regime ($m/T_D \gg 1$) generally have no problems with obeying the phase-space constraints. As seen from Figs. 2 and 3 the contribution to the density parameter decreases strongly (like $(m/T_D)^{3/2}\exp[-3(m-\mu)/4T_D]$) with increasing m/T_D, which means that particles must be much more massive than $m_{\rm min}$ to play a role as dark matter, and if they are, then the phase-space constraint is amply satisfied. For fermions problems increase with increasing chemical potential, and in the relativistic decoupling regime the density contribution grows with $(\mu/T_D)^3$ for $\mu \gg T_D$, meaning that only weakly or nondegenerate fermions in the eV mass range can be the dark matter in this regime. The exact limits depend on the size of the galaxy under investigation: The dwarf spheroidal problem comes in via the factor $\sigma_{100}^{-1/4} r_{10}^{-1/2}$; the smaller the galaxy, the higher the minimal mass and contribution to the density parameter. Also, the limits depend on g_{*sD}/g_{*s0}. For decoupling in the temperature range 1–100 MeV this factor is of order 3, but for *very* early decoupling it may reach a value near 30.

For bosons the situation is more complex in that the minimal contribution to the density parameter is not a monotonic function of m/T_D for a fixed μ/T_D. The contribution is negligible in the nonrelativistic regime as well as for $\mu \to m$. Problems are biggest for mildly nonrelativistic decoupling, but as demonstrated in Sec. V mass limits for relativistic decoupling increase significantly if one recognizes that only a limited fraction of bosons have occupation numbers near $f_{\rm max}$.

The figures all show results for a single particle species with the given chemical potential. If antiparticles are distinct from the particles, and if particles and antiparticles can annihilate into photons, then the antiparticles will be distributed according to Eq. (1) with chemical potential $\bar{\mu} = -\mu$. For $\mu = \bar{\mu} = 0$ antiparticles are as abundant as particles; the total contribution to Ω_x will double,

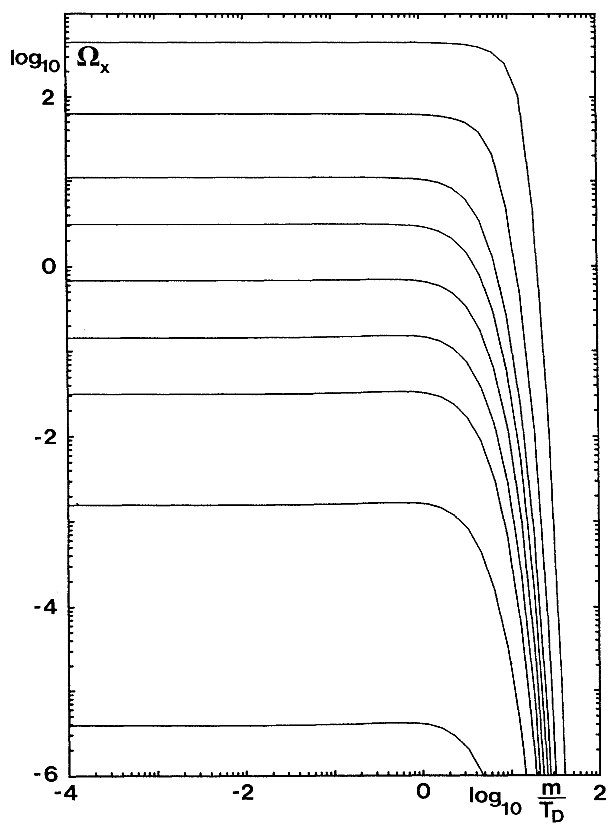

FIG. 2. Minimal value of the density parameter Ω_x for fermions obeying Eq. (2), as a function of m/T_D for $\mu/T_D = -16$, -8, -4, -2,0,2,4,8,16 (lower to upper curve). Ω_x is given in units of $h_0^{-2} T_{2.75}^{3} g^{3/4} g_{*sD}^{-1} g_{*s0} \sigma_{100}^{-1/4} r_{10}^{-1/2}$.

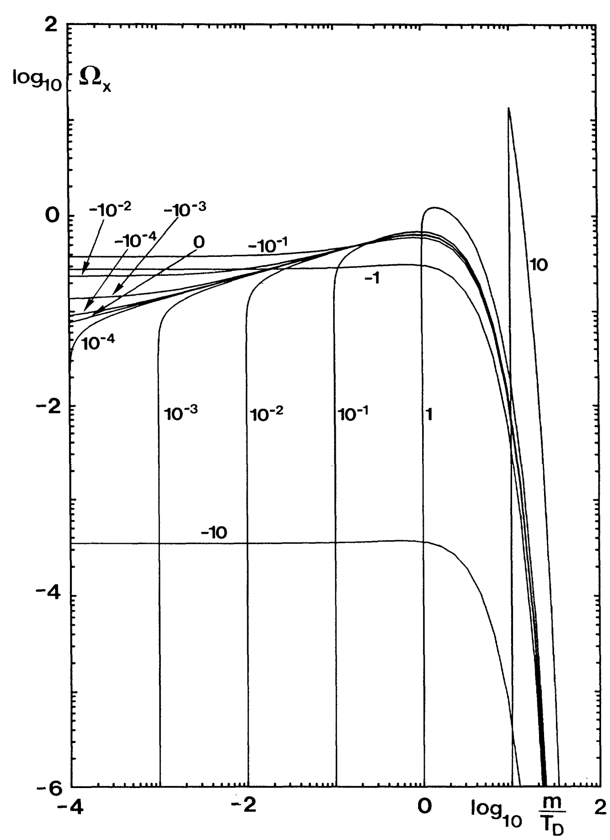

FIG. 3. As Fig. 2, but for bosons with μ/T_D as indicated.

and the mass limits derived from phase-space constraints will be reduced by a factor of $2^{-1/4}$. For $\mu > 0$ the antiparticles will be less abundant and changes will be less than the factors of 2 and $2^{-1/4}$ mentioned previously [16].

V. STATISTICAL TREMAINE-GUNN LIMITS

As illustrated in Figs. 1 and 2 in Ref. [2] only a minor fraction of particles have occupation numbers close to f_{max}. Observations indicate that the dark matter in galaxies contribute at least 10% (and probably more) of the total dark-matter content in the Universe. It is therefore necessary to utilize particles from a significant fraction of the distribution, meaning that the *typical* occupation number of dark-matter particles, in particular for bosons, may be much smaller than f_{max}. This again means that the minimum mass and the contribution to Ω_x increase, strengthening problems with obeying the generalized Tremaine-Gunn constraints.

The limits described in previous sections are *exact* in the sense that they can be applied to individual galaxies. The stronger limits discussed in the present section are *statistical* and should as such be applied to samples of galaxies. The distinction lies in the possibility of hand-picking high-occupation-number particles for the halos of a few galaxies, thereby making it possible to explain the dark matter with particle masses near the exact m_{min}. This, however, cannot be done (especially for bosons) for a large sample of galaxies due to the limited number of high-f particles. (And, of course, in practice nature may have difficulties in making such selections at all, even though the particles with the highest occupation numbers also have the lowest momenta, and as such will play an important part in structure formation. In fact, as discussed in Ref. [17], the low-momentum bosons may play a decisive role in a hot dark-matter boson scenario.)

Figures 6 and 7 show the value of $\Omega_x h_0^2 T_{2.75}^{-3} g^{-3/4} g_{*sD} g_{*s0}^{-1} \sigma_{100}^{1/4} r_{10}^{1/2}$ for the minimum particle mass obeying the Tremaine-Gunn constraint if f_{max} in Eq. (2) is substituted by the (lower) f value corresponding to the fraction of particles $N(f)$ having occupation numbers between f and f_{max} [18]. Figures 8 and 9 give the values of m_{min}. The *exact* limits presented in previous sections correspond to the fraction of particles going to 0. Figures 6–9 assume $\mu/T_D = 0$, and curves are given for a series of m/T_D values.

Notice that the statistical mass limits for fermions are only slightly different from the exact limits. For bosons the situation is quite different. For instance, relativisti-

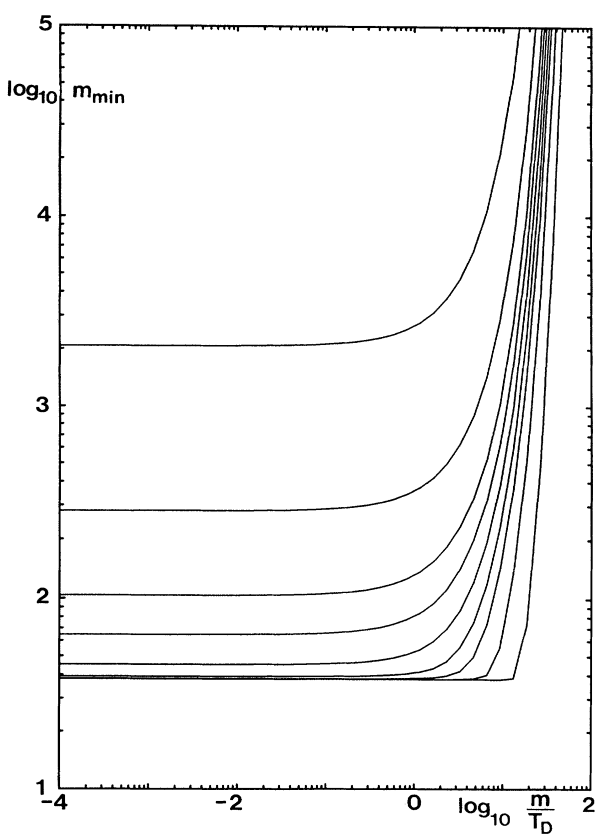

FIG. 4. Minimum fermion mass (in units of eV $\sigma_{100}^{-1/4} r_{10}^{-1/2} g^{-1/4}$) obeying Eq. (2) for the parameter sets used in Fig. 2 (upper to lower curve).

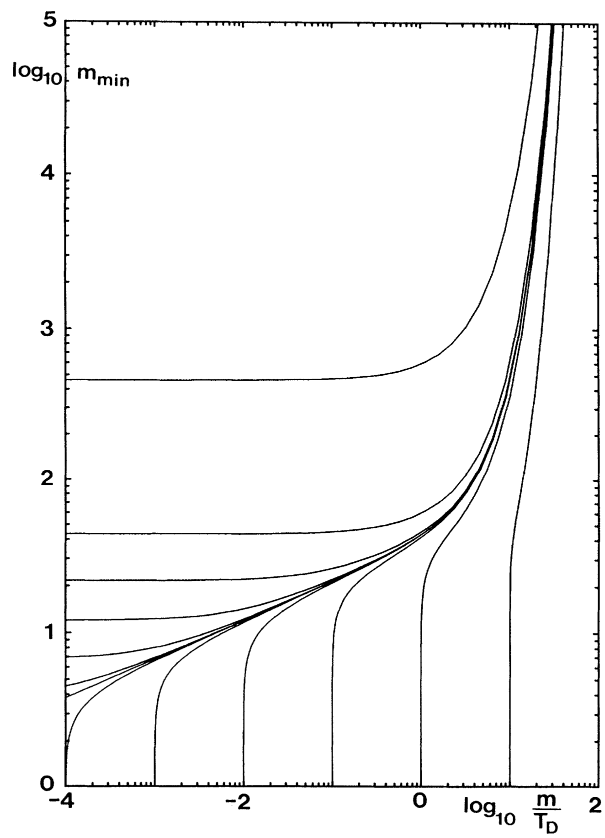

FIG. 5. Minimum boson mass (in units of eV $\sigma_{100}^{-1/4} r_{10}^{-1/2} g^{-1/4}$) obeying Eq. (2) for the parameters used in Fig. 3. Values for μ/T_D are -10, -1, -10^{-1}, -10^{-2}, -10^{-3}, -10^{-4}, 0, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10 from upper to lower curve.

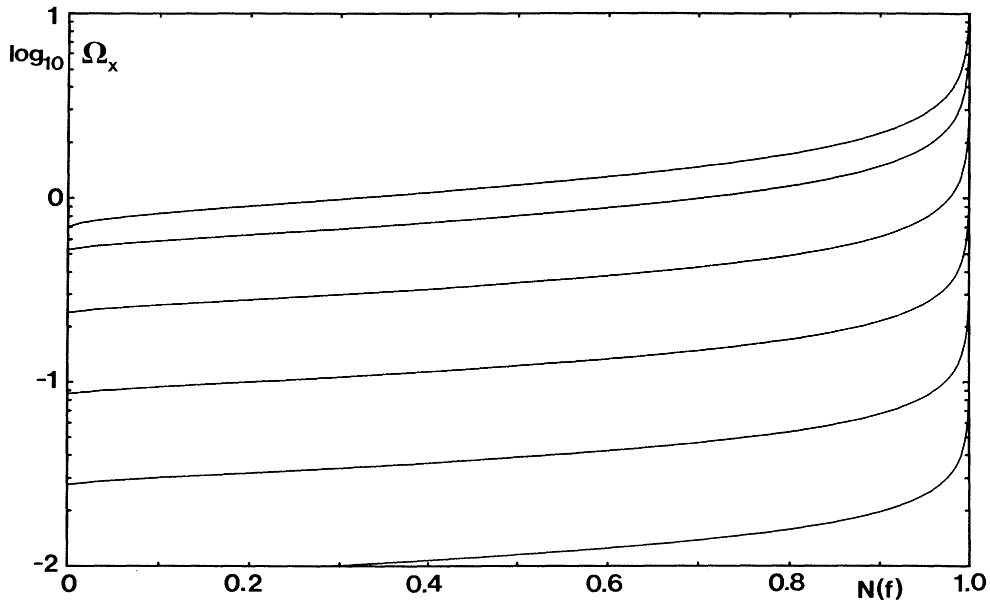

FIG. 6. Minimum value of Ω_x (in units of $h_0^{-2}T_{2.75}^3 g^{3/4} g_{*sD}^{-1} g_{*s0} \sigma_{100}^{-1/4} r_{10}^{-1/2}$) for fermions with $\mu/T_D=0$ as a function of the particle fraction $N(f)$ with occupation numbers between f and f_{max}, assuming the minimum mass to be given by the lowest occupation number in the interval. m/T_D takes values from 0 to 10 in steps of 2 (upper to lower curve).

cally decoupling bosons have very low exact m_{min} limits, and for $m/T_D \to 0$ no exact limit exists. The statistical limits are dramatically higher. If one assumes that 10 or 20% of all particles should be used in galaxy halos, statistical mass limits for bosons are not much smaller than corresponding values for fermions.

VI. CONCLUSION

The present paper has been devoted to the derivation of minimum mass limits for dissipationless particles constituting the dark matter in galaxies and other bound structures. The coarse-grained distribution of dark-matter particles was supposed to be an isothermal sphere, and limits were derived by comparing the maximum phase-space density of such a coarse-grained distribution with the maximal fine-grained phase-space density of the particle distribution (so-called *exact* limits), or with the characteristic fine-grained density of a given fraction of particles *(statistical* limits).

From the mass limit, chemical potential, and decoupling temperature, the minimal contribution to the cosm-

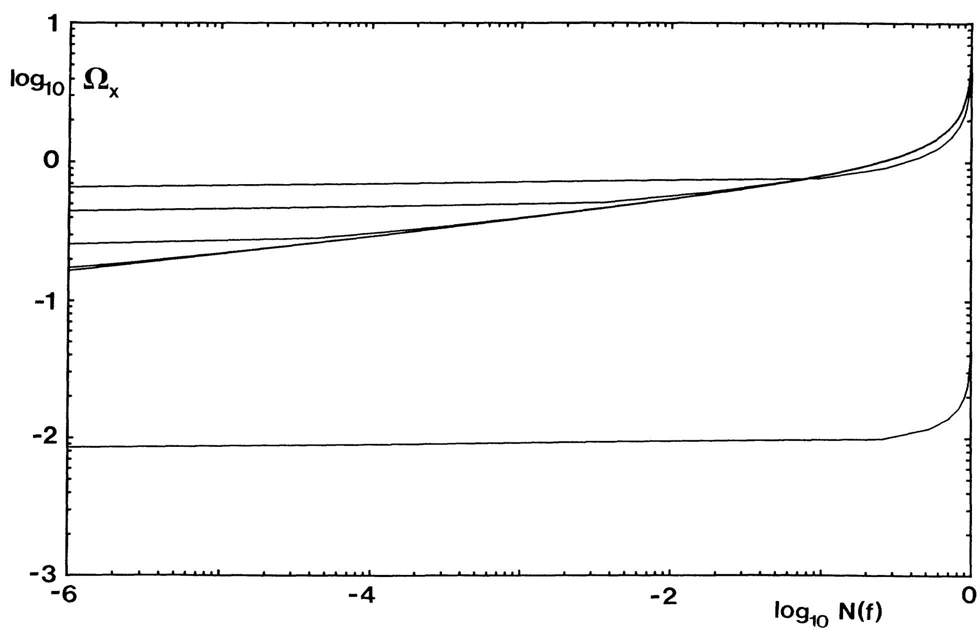

FIG. 7. As Fig. 6, but for bosons with $m/T_D = 1, 10^{-1}, 10^{-2}, 10^{-3}, 0, 10$ (upper to lower curve).

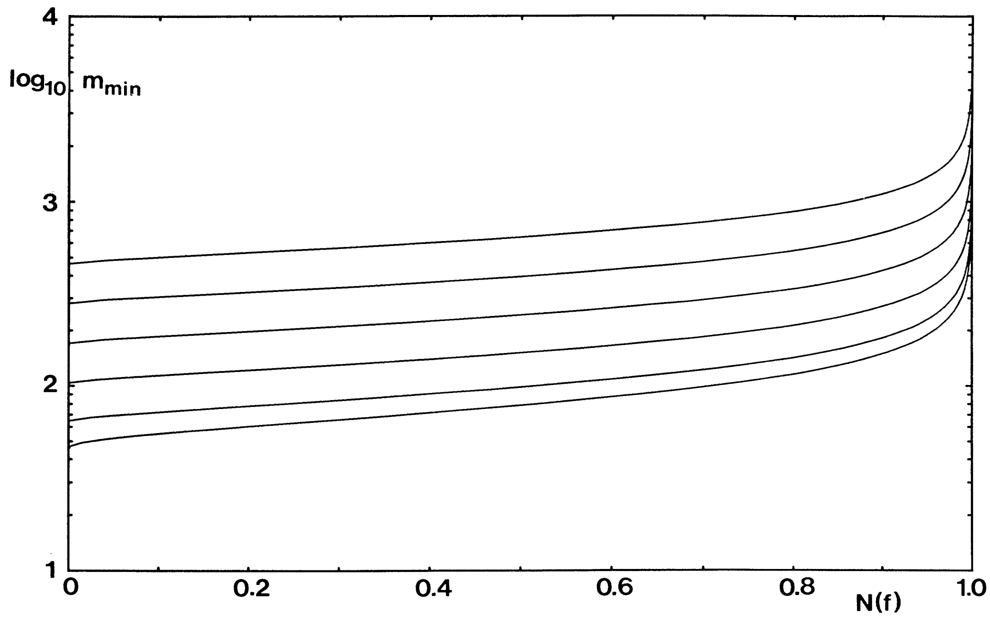

FIG. 8. $m_{\rm min}$ in units of eV $\sigma_{100}^{-1/4} r_{10}^{-1/2} g^{-1/4}$ as a function of the fermion fraction $N(f)$. Parameters as for Fig. 6 (lower to upper curve).

density parameter was derived. To be a realistic dark-matter candidate, a given particle type should not contribute more than $\Omega_x h_0^2 \approx 1$ for particle mass equal to $m_{\rm min}$. If the minimal contribution derived is much smaller than this value, it means that the particle is allowed to be much more massive than $m_{\rm min}$, thereby contributing the necessary density without violating the phase-space constraints. Good dark-matter candidates in this respect are therefore particles decoupling in the strongly nonrelativistic regime, particles with negative chemical potential and relativistic decoupling (provided that they do not have antiparticles with equal but positive chemical potential), and bosons with $(m-\mu)/T_D \rightarrow 0$. However, in particular limits for bosons are strengthened if one considers the statistical limits described in Sec. V, rather than the exact limits. This is because the few low-momentum bosons have much higher (in some cases divergent) occupation numbers than is typical for the distribution in gen-

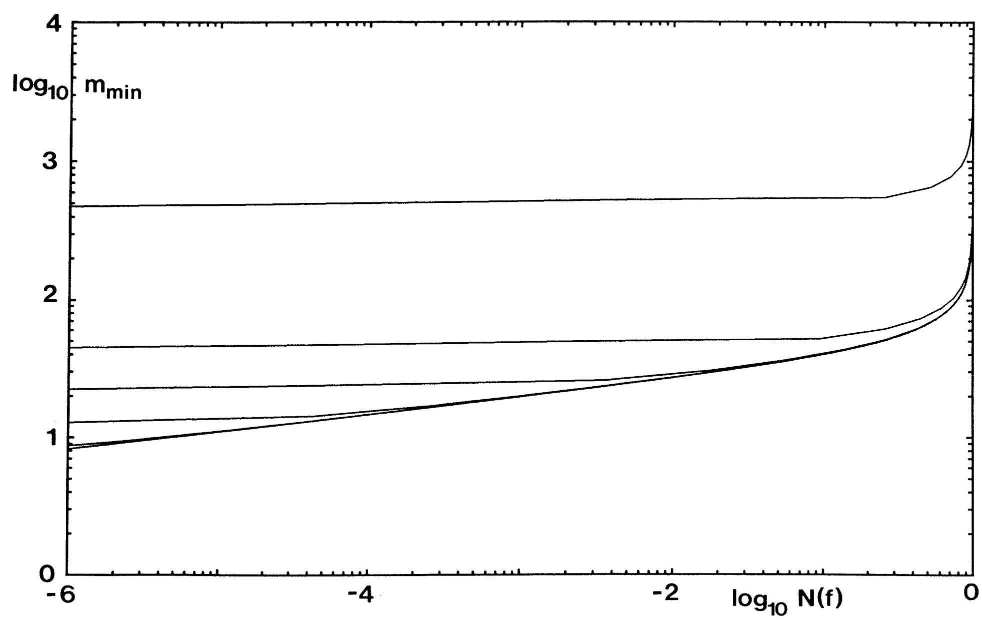

FIG. 9. As Fig. 8, but for bosons with $m/T_D = 0, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10$ (lower to upper curve).

eral.

To use the limits in practice, one should be aware of the potential loopholes described in Sec. II. First of all, the galaxies to be used must be very well modeled, and the luminous contribution to the density properly subtracted. Second, the core radius and velocity dispersion of the dark matter must be inferred. And finally, one must remember that the coarse-grained distribution of dark matter is not necessarily isothermal, or even isotropic. Therefore the phase-space limits derived should merely be used as an order-of-magnitude estimate.

On the particle physics side it is crucial to remember that limits apply only to particles that were once in kinetic equilibrium (e.g., not to nonthermal axions). One also needs independent information on the chemical potential and decoupling temperature of the particle in question (it is implicit in the derivations that decoupling takes place over a fairly restricted time interval).

But apart from the reservations mentioned in the preceding paragraphs, the generalized Tremaine-Gunn constraints derived in this paper should be obeyed by any particle candidates for the dark matter.

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[16] A quick look at Fig. 3 may seem to contradict these statements, but one should remember that bosons and antibosons annihilating into photons have $-m \leq \mu \leq m$.
[17] J. Madsen, Astrophys. J. Lett. **371**, L47 (1991).
[18] Slightly (but not much) less stringent limits are obtained if one uses the mean or median value of the occupation number in the interval, rather than the lowest f value; see Ref. [2].