Is the Cabibbo-Kobayashi-Maskawa matrix symmetric?

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We examine some of the consequences of having a Cabibbo-Kobayashi-Maskawa (CKM) matrix V with symmetric moduli. We define an asymmetry parameter for a three-generation CKM matrix which can be simply expressed in terms of the eigenstates and the eigenvalues of V. The fact that experimentally the asymmetry is small implies that two of the eigenvalues of V are almost degenerate and/or the eigenstates of V are close to being real. We point out that it is a special feature of three generations that a symmetric $|V|$ implies that V can be made symmetric by appropriate choice of quark field phases. We analyze a recent ansatz by Kielanowski which leads to a symmetric CKM matrix. A simple parametrization for a symmetric CKM matrix is presented and consequences for the top-quark mass and the ratio $|V_{ub}|/|V_{cb}|$ are examined.

I. INTRODUCTION

All presently available data [1] on the Cabibbo-Kobayashi-Maskawa (CKM) matrix V are consistent with having $|V_{ij}|=|V_{ji}|$. In this Brief Report we will analyze some of the consequences of having a symmetric CKM matrix. First, we examine the question whether starting with symmetric moduli one may use the rephasing freedom of the CKM matrix to obtain also symmetric phases. It will be seen that this is indeed the case for three generations, but it does not hold for a larger number of generations. We define an asymmetry parameter for the three-generation CKM matrix which measures how V deviates from being symmetric. This asymmetry can simply be expressed in terms of the eigenstates and the eigenvalues of V, showing that in order for V to be symmetric, two of its eigenvalues should be degenerate and/or its eigenstates should be real. The fact that experimentally the asymmetry is small implies that two of the eigenvalues of V are close to being degenerate and/or the eigenstates of V are close to being real. We also show that $\mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)$, the invariant which controls the strength of CP violation in charged weak currents [2], can be simply expressed in terms of the eigenstates and eigenvalues of the CKM matrix. We present a simple parametrization of a symmetric CKM matrix and analyze the constraints arising from unitarity and present experimental data.

Finally, we examine a recent ansatz by Kielanowski [3] which also leads to a symmetric CKM matrix. We point out that the Kielanowski proposal implicitly assumes a further restriction on the free parameters of the CKM matrix, beyond the symmetry constraint. It is shown that this restriction can be expressed as a simple equation relating the various $|V_{ij}|$. We suggest a variant of the Kielanowski ansatz which also leads to a symmetric CKM matrix.

II. UNITARITY CONSTRAINTS AND SYMMETRIC PHASES

It is well known that the individual phases of V_{ij} have no physical meaning, since under rephasing of the up- and down-quark fields, they transform as

$$V_{ij}\rightarrow V'_{ij}=V_{ij}\exp(\gamma_j-\beta_i)\ . \tag{1}$$

One may wonder whether starting from an arbitrary V it is possible to achieve $\arg V'_{ij}=\arg V'_{ji}$ by an appropriate choice of γ_j,β_i. We will see that in general this is not possible for arbitrary V, but it is possible for a three-generation CKM matrix with symmetric moduli. Indeed, in order to achieve $\arg V'_{ij}=\arg V'_{ji}$, the following relations need to be satisfied:

$$\alpha_{ij}-\alpha_{ji}=\gamma_i-\gamma_j+\beta_i-\beta_j+2n\pi\ , \tag{2}$$

where $\alpha_{ij}=\arg(V_{ij})$. It can be readily verified that in order for Eqs. (2) to have a solution for γ_j,β_i, the V_{ij} have to satisfy the condition

$$\mathrm{Im}(V_{12}V_{23}V_{31}V_{21}^*V_{13}^*V_{32}^*)=0\ . \tag{3}$$

Note that so far we have not specified the number of generations and in particular Eq. (3) is a necessary condition in order to have symmetric phases, for any number of generations ($N\geq 3$). Obviously, for $N>3$ there are other conditions, analogous to Eq. (3), which need also to be satisfied in order to obtain symmetric phases. It is a very special feature of three generations that the condition of Eq. (3) is an automatic consequence of unitarity when $|V_{ij}|=|V_{ji}|$. In order to see this, consider the orthogonality conditions for the first two rows and first two columns of the CKM matrix:

$$V_{11}V_{21}^*+V_{12}V_{22}^*+V_{13}V_{23}^*=0\ , \tag{4}$$

$$V_{11}V_{12}^*+V_{21}V_{22}^*+V_{31}V_{32}^*=0\ . \tag{5}$$

If one multiplies Eq. (4) by V_{21}, Eq. (5) by V_{12}, and assumes $|V_{ij}|=|V_{ji}|$, one obtains, by subtracting the resulting equations,

$$V_{13}V_{23}^*V_{21}-V_{31}V_{32}^*V_{12}=0\ , \tag{6}$$

which in turn implies that Eq. (3) is satisfied. It can be readily verified that for more than three generations Eq. (3) does not follow from unitarity and symmetric moduli do not imply symmetric phases. This point is worth emphasizing. For example, for four generations, even if one had exact knowledge of the moduli of V, with $|V_{ij}|=|V_{ji}|$, this would not imply a symmetric V. (Recent measurements of the Z^0 width at CERN LEP indicate that there are only three light neutrinos in nature. However, four-dimensional CKM matrices may still be relevant in a variety of scenarios, the simplest of which consists of having a fourth generation with a neutral heavy lepton with a mass exceeding 46 GeV.) This feature of the CKM matrix which arises for more than three generations is, of course, closely related to the fact that for more than three generations, an exact knowledge of the moduli of V, in general, does not completely determine V [4].

III. THE EIGENSTATES OF THE CKM MATRIX

The use of the eigenstates of the CKM matrix for the description of weak mixing has been recently advocated [3,5]. Our motivation here is to analyze the restrictions on the eigenstates of V, arising from the assumption that $|V_{ij}|$ is a symmetric matrix. Since the CKM matrix is unitary, it can be diagonalized through a unitary transformation

$$U^{-1}VU=K\ ,\qquad K=\mathrm{diag}(\exp(i\sigma_1),\exp(i\sigma_2),\exp(i\sigma_3))\ , \tag{7}$$

where $\exp(i\sigma_i)$ are the eigenvalues of the CKM matrix, corresponding to eigenstates with components U_{ji} ($j=1,2,3$). The columns of U are three orthonormal eigenvectors of V and are defined up to overall phases for each one of the columns. Indeed if U satisfies Eq. (7), the matrix $U\beta$, with $\beta=\mathrm{diag}(\exp(i\beta_1),\exp(i\beta_2),\exp(i\beta_3))$, will also satisfy it. Obviously, the matrix β has no physical meaning.

We will show that the CKM matrix is symmetric if and only if the matrix U is real, apart from irrelevant overall phases for each one of its columns. From Eq. (7) it is obvious that the reality of U is sufficient in order to have a symmetric V, since for real U, one has $V=UKU^T$. We will show next that reality of U is also a necessary condition in order to have a symmetric V. From Eq. (7) one obtains

$$VU=UK\ , \tag{8}$$

$$U=VUK^{-1}\ . \tag{9}$$

If one takes the complex conjugate of Eq. (9) and takes into account that for a symmetric unitary matrix one has $V^*=V^{-1}$, one gets

$$VU^*=U^*K\ . \tag{10}$$

Equations (8) and (10) indicate that both the columns of U and those of U^* are orthonormal eigenstates of V corresponding to the same eigenvalues. For the moment we will assume that the eigenvalues of V are nondegenerate. It follows then that the columns of U and U^* can only differ by an overall phase and therefore the matrix U is necessarily of the form

$$U=R\gamma\ , \tag{11}$$

where R is a real matrix and $\gamma=\mathrm{diag}(\exp(i\gamma_1),\exp(i\gamma_2),\exp(i\gamma_3))$. We have assumed that the eigenvalues of V are nondegenerate. One can easily verify that if two of the eigenvalues of V are degenerate, then $|V|$ is necessarily symmetric and the eigenstates of V can be chosen to be real.

The relationship between the "effective" reality of the eigenstates of V and the symmetric character of V can be expressed in a more explicit way. First note that for three generations the assumption that V has symmetric moduli implies a single constraint on V. This is due to the fact that for three generations, unitarity alone implies that $|V_{12}|^2-|V_{21}|^2=|V_{31}|^2-|V_{13}|^2=|V_{23}|^2-|V_{32}|^2$. It is thus convenient to define an asymmetry parameter A as

$$A\equiv|V_{12}|^2-|V_{21}|^2=|V_{31}|^2-|V_{13}|^2=|V_{23}|^2-|V_{32}|^2\ . \tag{12}$$

From Eq. (7), one can then evaluate A and one obtains

$$A=-4I[\sin(\sigma_1-\sigma_2)+\sin(\sigma_3-\sigma_1)+\sin(\sigma_2+\sigma_3)]\ , \tag{13}$$

where $I=\mathrm{Im}(U_{11}U_{22}U_{12}^*U_{21}^*)$ and $\exp(i\sigma_i)$ are the eigenvalues of V. The result of Eq. (13) has the expected features. The asymmetry vanishes when two of the eigenvalues of V are degenerate and/or when the matrix U is "effectively" real (i.e., $I=0$). The fact that experimentally the asymmetry is small ($A<4\times10^{-4}$) provides an indication that two of the eigenvalues of V are close to being degenerate and/or U is close to being effectively real (i.e., $I\ll1$).

At this point, it should be emphasized that the vanishing of $\mathrm{Im}(U_{11}U_{22}U_{12}^*U_{21}^*)$ by no means implies the vanishing of $J\equiv\mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)$ and, in general, a symmetric CKM matrix does allow for CP violation. In order to see this, we will compute J in terms of the eigenstates and eigenvalues of V. For a general CKM matrix, one obtains, using Eq. (7) and unitarity:

$$J\equiv\mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)=\sum_{i,j,k}|U_{1i}|^2|U_{2j}|^2|U_{3k}|^2\sin(\sigma_1+\sigma_2+\sigma_3-\sigma_i-\sigma_j-\sigma_k) \tag{14}$$

where i,j,k run from 1 to 3 and $\exp(i\sigma_i)$ are the eigenvalues of V. In the case that two of the eigenvalues of V are degenerate (e.g., $\sigma_2=\sigma_3$), one can set them equal to zero without loss of generality, and one obtains, from Eq. (14),

$$J=2|U_{11}|^2|U_{21}|^2|U_{31}|^2(\sin\sigma_1)(1-\cos\sigma_1)\ . \tag{15}$$

One sees from Eq. (15) that although the CKM matrix becomes symmetric when two of its eigenvalues are degenerate, J is in general nonzero. By computing the full CKM matrix, it can be easily verified that in the case of two degenerate eigenvalues, the CKM matrix is completely defined by the three parameters $|U_{11}|,|U_{21}|,\sigma_1$, i.e., by the knowledge of the nondegenerate eigenvalue and the corresponding eigenstate.

IV. PARAMETRIZATION

It is well known that in general, four independent physical parameters are required in order to characterize the CKM matrix for three generations. We have seen that assuming V to be symmetric implies a single constraint and, as a result, one needs three parameters to characterize a three-generation symmetric CKM matrix. Let us consider the parametrization suggested by Branco and Lavoura [6]:

$$|V_{ij}|^2=\begin{pmatrix} 1-\epsilon-ap\epsilon^3 & \epsilon & ap\epsilon^3 \\ \epsilon+a(p-q)\epsilon^3 & 1-\epsilon-a\epsilon^2-a(p-q)\epsilon^3 & a\epsilon^3 \\ aq\epsilon^3 & a\epsilon^2+a(p-q)\epsilon^3 & 1-a\epsilon^2-ap\epsilon^3 \end{pmatrix}, \tag{16}$$

where the four parameters a,p,q,ϵ are defined by

$$|V_{us}|^2=\epsilon, \quad |V_{cb}|^2=a\epsilon^2, \quad |V_{ub}|^2=ap\epsilon^3, \quad |V_{td}|^2=aq\epsilon^3 . \tag{17}$$

This parametrization is obviously rephasing invariant and it is also of the Wolfenstein type [7], in the sense that its matrix elements are written in terms of powers of a small parameter ϵ. It has the further advantage of making it explicit that $|V_{ij}|=|V_{ji}|$ implies a single constraint, namely $p=q$. One can express the strength of CP violation, namely $J\equiv\mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*)$ in terms of the three parameters ϵ,a,p. From Eq. (16) and setting $p=q$, one obtains

$$|J|^2=\frac{a^2\epsilon^6}{4}[(-1+4p)-2p\epsilon-p(p+2a)\epsilon^2-2ap^2\epsilon^3-a^2p^2\epsilon^4] . \tag{18}$$

This parametrization is not manifestly unitary. It has been previously shown [8] that in a parametrization of a three-generation CKM matrix through independent moduli, the only nontrivial unitary constraint arises from the requirement $|J|^2\geq 0$. Since $|J|^2$ is a quadratic polynomial in p, with negative p^2 coefficient, in order for $|J|^2$ to be positive, p is constrained to be between its roots. Given the experimental limit on p, only the lower bound is relevant. One obtains, from Eq. (18),

$$p\geq\frac{1}{4}+\frac{1}{8}\epsilon+\left[\frac{1}{16}+\frac{a}{8}\right]\epsilon^2 , \tag{19}$$

where we have neglected terms in ϵ^3 and higher powers of ϵ. From the experimental value [1] of $|V_{us}|$,

$$|V_{us}|=0.2205\pm0.0018 , \tag{20}$$

one can derive the following unitary bounds on p and $V_{ub}|V_{cb}$, both applicable only to a symmetric CKM matrix:

$$p>0.256, \quad \frac{|V_{ub}|}{|V_{cb}|}>0.11 . \tag{21}$$

An experimental bound on p can be deducted from the ratio $|V_{ub}/V_{cb}|$, which may be obtained from the semileptonic decay of B mesons by fitting the lepton energy spectrum as a sum of contributions arising from $b\to u$ and $b\to c$ transitions. The actual deduction of the ratio $|V_{ub}/V_{cb}|$ depends on the theoretical model used to generate the lepton energy spectrum. Combining the experimental and theoretical uncertainties, Gilman, Kleinknecht, and Renk [1] quote

$$|V_{ub}/V_{cb}|=0.09\pm0.04 , \tag{22}$$

which leads to the following experimental bound on p:

$$0.051\leq p\leq 0.356 . \tag{23}$$

It is worth noting that if one combines the unitarity bounds of Eq. (22) with the experimental bounds of Eqs. (19) and (23), both p and $|V_{ub}/|V_{cb}|$ are constrained, in the case of a symmetric CKM matrix, to a rather narrow range of values:

$$0.256\leq p\leq 0.356; \quad 0.11\leq\frac{|V_{ub}|}{|V_{cb}|}\leq 0.13 . \tag{24}$$

V. THE KIELANOWSKI ANSATZ

In a recent paper [3], Kielanowski has proposed a symmetric form for the CKM matrix, with two parameters. As we have emphasized, the most general symmetric CKM matrix requires three parameters. We will point out that in Ref. [3], the author implicitly assumes a restriction on the free parameters of a symmetric CKM matrix. We will show that this restriction can be expressed as a simple equation relating the various $|V_{ij}|$. Let us consider a general CKM matrix parametrized by its eigenstates and eigenvalues as in Eq. (7) and make the choice of quark phases of Ref. [3] which leads to the eigenstates

$$U_{j1}=\begin{pmatrix} c_1 \\ s_1c_2 \\ s_1s_2 \end{pmatrix}, \quad U_{j2}=\begin{pmatrix} -s_1c_3 \\ c_1c_2c_3-e^{i\alpha}s_2s_3 \\ c_1c_3s_2+e^{i\alpha}c_2s_3 \end{pmatrix}, \quad U_{j3}=\begin{pmatrix} s_1s_3 \\ -c_1c_2c_3-e^{i\alpha}s_2c_3 \\ -c_1s_2s_3+e^{i\alpha}c_2c_3 \end{pmatrix}, \tag{25}$$

with eigenvalues $\exp(-i2\pi/3)$, $\exp(i2\pi/3)$, and 1, respectively. At this stage, one has a general CKM matrix

$V=UKU^{\dagger}$. We have seen that reality of U is a necessary and sufficient condition for having a symmetric CKM matrix. Within the parametrization of Eq. (25), this would correspond to putting $\alpha=0$. Kielanowski has arbitrarily set $s_3=0$ and for this special value of s_3, U becomes real and therefore the CKM is symmetric. However, one no longer obtains a general symmetric CKM matrix. From Eqs. (7) and (25) it can be easily seen that for $s_3=0$, the following equation constrains the moduli:

$$|V_{23}|^2=\frac{3|V_{12}|^2|V_{13}|^2}{(|V_{12}|^2+|V_{13}|^2)^2}-\frac{|V_{12}|^2|V_{13}|^2}{|V_{13}|^2+|V_{12}|)^2} . \quad (26)$$

In terms of the Branco-Lavoura parametrization of Eq. (16) (with $p=q$), the relation (26) becomes

$$a\epsilon^2=\frac{3ap\epsilon^5}{(ap\epsilon^3+\epsilon)^2}-\frac{ap\epsilon^4}{ap\epsilon^3+\epsilon} . \quad (27)$$

Solving for p one obtains

$$p=\frac{1}{3}+\frac{1}{9}\epsilon+\left[\frac{1}{27}+\frac{2a}{9}\right]\epsilon^2 , \quad (28)$$

where we have neglected higher powers in ϵ. Using the central experimental value for $\epsilon\equiv|V_{us}|^2$, one obtains

$$p=0.339 . \quad (29)$$

Since we have started from a manifestly unitary parametrization, it is not surprising that this value of p is consistent with the unitarity constraint of Eq. (21). What is remarkable is that it is also consistent with the experimental upper bound on p, given by Eq. (23).

At this stage it is interesting to note that one arrives at an analogous situation if in Eq. (25) one sets $s_2=0$ instead of $s_3=0$. It can be readily verified that again the matrix U becomes real and therefore the corresponding V is symmetric. Again, the resulting V is not the most general symmetric CKM matrix and one arrives at the following constraint among the moduli:

$$|V_{12}|^2=\frac{|V_{13}|^2|V_{23}|^2}{[|V_{13}|^2+|V_{23}|^2]^2}\times\{2-(|V_{13}|^2+|V_{23}|^2)+[1-4(|V_{13}|^2+|V_{23}|^2)]^{1/2}\} . \quad (30)$$

As before, Eq. (28) leads to a constraint on the value of p which is given by

$$p\approx 0.344 . \quad (31)$$

It is amusing that the value of p does not differ much from the value obtained for $s_3=0$ and it is also consistent with the bounds of Eq. (24).

A large top-quark mass is in general required by symmetric CKM matrices. In a recent paper [9], Rosner has examined the consequences for m_t, from Kielanowski's ansatz. Using constraints arising from both the observed B-$\bar{B}$ mixing and the strength of CP violation in the K^0-$\bar{K}^0$ system, Rosner has concluded that $m_t=247\pm37$ GeV. Since the value of p in Kielanowski's ansatz is already very close to the upper experimental limit on p, the above interval for m_t is essentially the lowest range one can obtain from m_t, in a general symmetric CKM matrix. It is interesting to note that a recent analysis [10] of B-$\bar{B}$ mixing, ϵ and ϵ'/ϵ, within the context of the standard model, indicates two favorable regions for m_t, namely $m_t\simeq100\pm30$ GeV and $m_t\approx210\pm40$ GeV. A symmetric CKM matrix is obviously only consistent with this second region of values for m_t.

VI. SUMMARY AND CONCLUSIONS

Next we summarize our main conclusions.

(1) All presently available experimental data are consistent with $|V_{ij}|=|V_{ji}|$. For three generations, symmetric moduli lead, through unitarity, to the vanishing of $\mathrm{Im}(V_{12}V_{23}V_{31}V_{21}^*V_{13}^*V_{32}^*)$. This in turn implies that if $|V|$ is symmetric, then it is always possible to choose the phases of the quark fields so that V is also symmetric.

(2) A symmetric CKM matrix implies that either two of its eigenvalues are degenerate and/or its eigenstates are real. In general, a symmetric CKM matrix can be specified by three parameters. We have defined an asymmetric parameter A for the CKM matrix which can be simply expressed in terms of its eigenstates and eigenvalues and whose experimental value is bounded to be close to zero ($A<4\times10^{-4}$).

(3) A symmetric CKM matrix requires a rather large value for the top-quark mass and a restriction of the ratio V_{ub}/V_{cb} to the narrow range $0.11\le(|V_{ub}|/|V_{cb}|)\le0.13$.

We have left open the important question of how to obtain, in a natural way, Yukawa couplings leading to a symmetric CKM matrix.

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