Spectrum-generating algebra for stringlike mesons: Mass formula for $q\bar{q}$ mesons

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We construct a mass formula for the stringlike properties of $q\bar{q}$ mesons, based on the spectrum-generating algebra $U(4)\otimes SU_s(2)\otimes SU_f(n)\otimes SU_c(3)$, with $U(4)\supset SO(4)$ dynamic symmetry. We determine the parameters appearing in this mass formula from fits to 57 well-established mesons in the Particle Data Group summary table. The average deviation for the mass squared of these mesons is 5.7% for our fit of all meson families combined. The mass formula allows us to distinguish meson states which are not $q\bar{q}$ type, such as $a_0(980)$.

I. INTRODUCTION

Hadronic structure is characterized by an interplay of phenomena arising from the complicated nature of the strong interaction. It encompasses dynamical situations ranging from highly relativistic to nonrelativistic. In the last decade, quantum chromodynamics (QCD) has emerged as the theory of strong interaction. In this theory, all hadronic properties (masses, decay widths, form factors,. . .) should be calculable in terms of a few parameters such as the QCD coupling constant α_s, quark masses M_i, and charges e_i (in the standard model, the flavor index i runs from one to six). Unfortunately, only partial analytic solutions of QCD are currently available [1] in $1+1$ dimensions, for the nonperturbative domain. Although considerable progress has been made by lattice calculations [2], this program of study is far from complete. As an alternative to solving directly the dynamical equations of QCD, several modes of hadronic structure have been constructed. Some of these emphasize the "collective" hadronic aspect: examples of this class are the bag model [3] and the string model [4]; others stress the "single-particle" nature of hadronic structure, as in the case of the nonrelativistic [5] or semirelativistic [6] quark model. In the latter, a Schrödinger-type equation is solved, with potentials that are suggested by QCD.

In 1965, Dothan, Gell-Mann and Ne'eman [7] and, independently, Barut and Böhm [8] suggested a different approach to hadron spectroscopy. In this approach, an algebra $\mathcal{G}$ [called spectrum-generating algebra SGA] is chosen, and all operators relevant to hadron structure are expanded onto elements of $\mathcal{G}$. In the special case in which the operators appearing in the expansion are invariant (Casimir) operators of the algebra $\mathcal{G}$ and its subalgebras $\mathcal{G}',\mathcal{G}'',\ldots$, one has a dynamic symmetry (DS). One can then solve the problem analytically in closed form. In the particular case of the energy operator, mass formulas, characteristic of the DS, arise. By acting with the transition operator $\hat{T}$ on the states (representations of $\mathcal{G}\supset\mathcal{G}'\supset\cdots$), one can also obtain, in closed form, transition matrix elements and thus decay widths. The method is quite general, and can be applied to both nonrelativistic and relativistic situations, though the expansion of the operators in terms of elements of the algebras is different in the two cases.

Recently, it has been suggested [9, 10] that SGA's for any combination of quarks, antiquarks and gluons can be constructed by taking products of appropriate space and internal algebras. In the simple case [10] of a quark and antiquark bound in a meson, the suggested space algebra is U(3,1), originating from the fact that one wants to include within the same representation of $\mathcal{G}$ all states, corresponding to rotations and vibrations of a string with quarks at its ends. In view of the difficulty in dealing with noncompact algebras, we prefer to use, in this article, the compact form U(4). There is a correspondence between the infinite-dimensional discrete representations of U(3,1) and those of U(4) when the dimension of the representations of U(4) goes to infinity. Thus, U(3,1) [or its compact form U(4)] describes the quantized geometric excitations of the string. With this article, we begin a systematic investigation of hadronic properties in terms of the SGA

$$\mathcal{G}=U(4)\otimes SU_s(2)\otimes SU_f(6)\otimes SU_c(3)\ , \tag{1}$$

for $q\bar{q}$ mesons, and its generalizations to multiquark and multigluon configurations, subscripts s,f,c in (1) standing for spin, flavor, and color, respectively.

Although the application of the SGA to the $q\bar{q}$ mesons appears, on the face of it, somewhat trivial, we do it for at least two reasons: (1) to set the stage for more complex calculations, such as the case of qqq baryons, and of strong, electromagnetic and weak decay widths [11] of hadrons, for which the use of an SGA is of crucial importance; (2) to emphasize the fact that the method simply becomes an expansion in terms of quantum numbers defining the representations of SGA and its subalgebras, thus producing simple formulas that can be easily compared with experiments. In other words, the presence of a symmetry gives relations between properties of hadronic states, which are, to a great extent, independent of the particular values of the model parameters. This formulation of the hadronic structure problem in terms of algebras allows one to test in a straightforward way features of QCD, without numerical solutions of equations of Schrödinger or Bethe-Salpeter type. Despite the complications of the basic strong interaction in the nonperturbative QCD domain, the spectra of $q\bar{q}$ mesons display simple features, which we can discern in the present approach, at a level of accuracy better than what one would expect at the outset. Thus, the entire phenomenology of $q\bar{q}$ mesons can be described in the present approach in terms of a few parameters. In this paper, we discuss meson masses. Decay widths will be presented in a separate paper [11].

An outline of the remainder of this article is the following. We start with a brief review of the properties of $\mathcal{G}$ (Sec. II), and apply the method of SGA to the construction of a mass formula for the $q\bar{q}$ mesons (Sec. III). This formula is then used to analyze the experimental mass spectrum of mesons, both light and heavy (Sec. IV). A summary of our conclusions is presented in Sec. V.

II. MODELS FOR STRINGLIKE MESONS

We begin with a brief outline of the model on which we base our study. The Fock-space representation of a meson M can be written as

$$|M\rangle = |q\bar{q}\rangle + |q\bar{q}q\bar{q}\rangle + \cdots + |q\bar{q}g\rangle + \cdots + |gg\rangle + \cdots \ , \tag{2}$$

where q ($\bar{q}$) denotes a quark (antiquark) and g a gluon. The stringlike configuration of the first two components of the state $|M\rangle$ is shown in Fig. 1. The quarks and gluons in (2) must be combined in such a way as to give the appropriate form quantum numbers of the meson $|M\rangle$. In the schematic form (2), we have suppressed the amplitudes for each component, as we are going to analyze only the $q\bar{q}$ configuration in this article. We shall return to more complex configurations in subsequent articles.

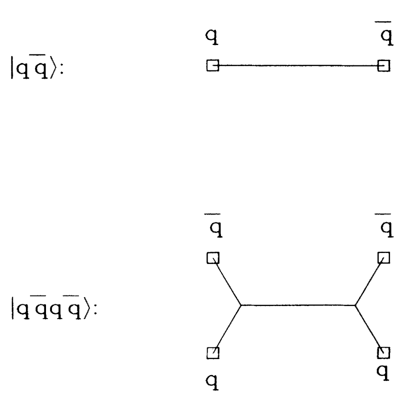

FIG. 1. Illustration of the stringlike configurations of the $q\bar{q}$ and $q\bar{q}\,q\bar{q}$ mesons.

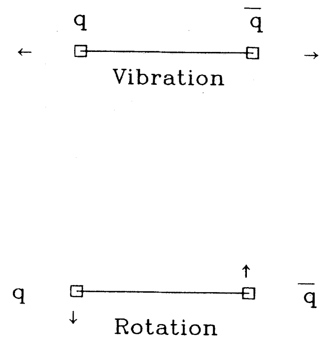

FIG. 2. Rotational and vibrational degrees of freedom of the $q\bar{q}$ string configuration.

The $q\bar{q}$ configuration can perform rotations and vibrations (Fig. 2) characterized by the quantum numbers L and v. We do not consider here bending and twisting vibrations, since these are expected to lie at higher energies. The relatively simple geometric problem of Fig. 2 is somewhat complicated by the fact that quarks have "internal" degrees of freedom, color, spin ($s=1/2$) and flavor ($i=1,\ldots,6$). Here we apply the method of SGA to study both geometric and internal excitations of the $q\bar{q}$ configurations.

A. Spectrum-generating algebra of geometric excitations

Since the purpose of this article is to present a detailed analysis of meson masses, we describe here only briefly the method of SGA. More details can be found in Ref. [9]. The SGA $\mathcal{G}$ is the one in terms of which all operators are expanded. Denoting generically the elements of $\mathcal{G}$ by G_α, one writes all operators O as

$$O = f(G_\alpha), \quad G_\alpha \in \mathcal{G} \ . \tag{3}$$

In this article, we consider only the mass-squared operators M^2. The transition operators are treated in Ref. [11]. If the M^2 operator contains only particular combinations of the operators G_α and their powers, those of the invariant Casimir operators of $\mathcal{G}$ and its subalgebras $\mathcal{G} \supset \mathcal{G}' \supset \mathcal{G}'' \supset \cdots$, denoted here generally by C_i, i.e., if

$$M^2 = f(C_i) \ , \tag{4}$$

one has a dynamic symmetry (DS) and the eigenvalues of M^2 can be written in closed form, yielding mass formulas. The properties of $\mathcal{G}$ are (a) it must contain, in a single representation, all states of the systems and (b) all hadronic operators must be expressible in terms of its elements. For hadrons, the algebra $\mathcal{G}$ must describe both geometric and internal degrees of freedom. We write $\mathcal{G}$ as

$$\mathcal{G} = \mathcal{R} \otimes \mathcal{G}_{sf} \otimes \mathcal{G}_c \ , \tag{5}$$

where $\mathcal{R}$ denotes the geometric part of the algebra describing the string excitations, $\mathcal{G}_{sf}$ represents the internal spin-flavor part, and $\mathcal{G}_c$ the color degree of freedom. Since color does not play any nontrivial role in the classification scheme for the $q\bar{q}$ mesons, we shall henceforth delete it. It has been suggested [9] that U(4) be taken as the SGA of geometric excitations and all states of the $q\bar{q}$ mesons belong to a single irreducible representation of U(4), characterized by the Young tableau

$$[N]=[N,0,0,0]=\square\square\square\cdots\ , \tag{6}$$

where there are N boxes on the right. The meaning of the quantum number N will become apparent in Sec. III. The representation (6) is totally symmetric, corresponding to the fact that the string excitations (rotations and vibrations) are *bosonic* in nature.

We now summarize properties of U(4) and its representations relevant to the construction of the mass formula for mesons. The algebra of U(4) can be split in two ways that contain the angular momentum algebra SO(3):

$$\mathrm{U}(4)\begin{matrix}\nearrow \mathrm{U}(3)\supset\mathrm{SO}(3)\supset\mathrm{SO}(2)\quad(\mathrm{I})\ ,\\ \searrow \mathrm{SO}(4)\supset\mathrm{SO}(3)\supset\mathrm{SO}(2)\quad(\mathrm{II})\ .\end{matrix} \tag{7}$$

If we consider the totally symmetric representations (6), states in the chain (I) are characterized by the quantum numbers

$$\left|\begin{matrix}\mathrm{U}(4)\supset & \mathrm{U}(3)\supset & \mathrm{SO}(3)\supset & \mathrm{SO}(2)\\ \downarrow & \downarrow & \downarrow & \downarrow\\ N & n & L & M_L\end{matrix}\right\rangle\ (\mathrm{I})\ . \tag{8}$$

The allowed values of n,L,M_L are given by the reduction of the representation $[N,0,0,0]$ into those of subalgebras of $\mathcal{G}$. One obtains, $n=N,N-1,\ldots,0;L=n-2,\ldots,1$ or 0 (n =odd or even) and $-L\le M_L\le +L$. Here L and M_L denote the orbital angular momentum and its third component, respectively. For the chain (II) we have instead

$$\left|\begin{matrix}\mathrm{U}(4)\supset & \mathrm{SO}(4)\supset & \mathrm{SO}(3) & \mathrm{SO}(2)\\ \downarrow & \downarrow & \downarrow & \downarrow\\ N & \omega & L & M_L\end{matrix}\right\rangle\ (\mathrm{II})\ . \tag{9}$$

The allowed values of ω,L,M_L are given by $\omega=N,N-2,\ldots,1$ or 0 (N=odd or even); $L=\omega,\omega-1,\ldots,0$, and $-L\le M_L\le +L$.

We now briefly mention the connection between the two classification schemes (I) and (II), admitted by the SGA of U(4), and the nonrelativistic quark model [5, 6]. If states of mesons were generated directly by the solution of nonrelativistic Schrödinger-type equations, the chain (I) would be appropriate to problems involving harmonic oscillator potentials, since the degeneracy group of the three-dimensional (3D) harmonic oscillator is U(3). On the other hand, the chain (II) would be appropriate to Coulomb-like problems or those with a linear potential, since the exact degeneracy group of the 3D Coulomb problems is SO(4) and numerical solutions of the Schrödinger equation with a linear potential have approximate degeneracy pattern of SO(4). Indeed, QCD suggests a linear plus Coulomb-like $q\bar{q}$ potential of the form

$$V(r)=-\frac{4}{3}\frac{\alpha_s}{r}+\sigma r\ , \tag{10}$$

where α_s is the strong coupling constant, and σ the string tension. Most importantly, the observation of rotational trajectories for mesons can be easily accommodated within a single representation of SO(4), but not of U(3) (see Sec. III). We therefore, use in this article the SO(4) basis as one for developing the method of SGA for mesons. We should mention here that we have investigated the possibility of using the U(3) basis, and/or combinations of U(3) and SO(4). Indeed, power of the algebraic method lies at the ease in analyzing all possible situations that might occur. Both U(3) and SO(4) basis have been used in the past [12–14]. Our analysis is closer to those of Barut *et al.* [13] and Böhm *et al.* [14], than to that of Bowler *et al.* [12]. By assuming a dynamic SO(4) symmetry for the space part of the $q\bar{q}$ wave functions, we build in, from the outset, some constraints of the QCD nature of the quark interaction. In contrast, the same result is only obtained by delicate interplay of several dynamical effects in solutions of Schrödinger-type equations with a suitable interquark potential.

B. Spectrum-generating algebra of internal excitations

The SGA for the internal hadronic degrees of freedom has been known for years, and is the basis upon which QCD is built. In addition to the color part $\mathcal{G}_c=\mathrm{SU}_c(3)$, one has the spin-flavor part $\mathcal{G}_{sf}$. Following Gell-Mann [15] and Ne'eman [16], and Gürsey and Radicati [17], we take for $n=6$ flavors of quarks in the standard model, $\mathcal{G}_{sf}=\mathrm{SU}_s(2)\otimes\mathrm{SU}_f(6)$. In view of the nonobservation of the top quark up to now, a large mass for this quark in the range 100 to 200 GeV is indicated. This suggests, for our purpose, a decomposition of the $\mathrm{SU}_f(6)$ of the type

$$\mathrm{SU}_f(6)\supset\mathrm{SU}_f(5)\otimes\mathrm{U}_{Y'''}(1)\supset\mathrm{SU}_I(2)\otimes\mathrm{U}_Y(1)\otimes\mathrm{U}_{Y'}(1)\otimes\mathrm{U}_{Y''}(1)\otimes\mathrm{U}_{Y'''}(1)\ , \tag{11}$$

where I is the isospin quantum number representing the symmetry of the (u,d) quark flavors, and Y,Y',Y'',Y''' are hypercharges associated with additional quark flavors (s,c,b,t). In view of the fact the strange quark has a mass comparable to that of the u,d quarks, we also insert, between $\mathrm{SU}_f(5)$ and $\mathrm{SU}_I(2)\otimes\mathrm{U}_Y(1)\otimes\mathrm{U}_{Y'}(1)\otimes\mathrm{U}_{Y''}(1)$, an intermediate step, $\mathrm{SU}_f(3)\otimes\mathrm{U}_{Y'}(1)\otimes\mathrm{U}_{Y''}(1)$, where $\mathrm{SU}_f(3)$ is the Gell-Mann–Ne'eman SU(3)-flavor symmetry [15, 16]. Whether there is a further intermediate step combining the (c,b) quarks into a new $SU_f(2)$ symmetry is an interesting question that we do not address here. Also we do not consider here splitting of the isospin multiplets of $SU_I(2)$. Finally, since the major contribution to the splittings of the flavor symmetry arises from the quark masses, we use the quark basis for flavor degrees of freedom, which we label by attaching a subscript q or $\bar{q}$ to $SU_f(6)$. This basis [18] is conveniently written as $|q_i,\bar{q}_j\rangle$, $i,j=1,\ldots,6$.

C. Total classification scheme

Combining Secs. II B and II C, we write the meson state vectors $|q\bar{q}\rangle$, with SO(4) "space" symmetry, as

$$|q\bar{q}\rangle=|q_i,\bar{q}_j;N,v,L,S,J,M_J\rangle \ . \tag{12}$$

Here the group quantum number ω has been converted to a quantum number v having the physical meaning of a vibrational quantum number (see Sec. III) by

$$v=\frac{N-\omega}{2} \ , \tag{13}$$

and the orbital **L** and **S** have been coupled to total angular momentum **J**. We note that all quantum numbers in (12), apart from N, have a straightforward physical interpretation, and include all possible degrees of freedom of $(q\bar{q})$ mesons. Since N is related to the dimension of the representation (9), it has the meaning of the maximum number of states that can be accommodated within one representation.

It is possible to construct states (12) by acting with creation and annihilation operators on a vacuum state. This is done by introducing a bosonic realization of U(4) in terms of four boson operators $b_\alpha^\dagger, b_\alpha$ ($\alpha=1,\ldots,4$), divided into a scalar $\sigma^\dagger,\sigma$ (with $J^P=0^+$) and a vector $\pi_\mu^\dagger,\pi_\mu$, $\mu=0,\pm1$ (with $J^P=1^-$). The generators of U(4) can be written explicitly in terms of σ and π operators [11]. The boson operators b_α represent the string quanta. The internal part can also be realized in terms of creation and annihilation of operators of fermionic type. Denoting by $a_{\kappa,i}^\dagger$ $(a_{\kappa,i})$ and $\bar{a}_{\kappa,i}^\dagger$ $(\bar{a}_{\kappa,i})$ the creation operators for quarks and antiquarks with spin component κ and flavor i, one can write the state (12) as

$$|q\bar{q}\rangle=\sum_{\substack{\kappa,\kappa'\\ M_S,M_L}}\langle\tfrac{1}{2},\kappa,\tfrac{1}{2},\kappa'|S,M_S\rangle\langle L,M_L,S,M_S|J,M_J\rangle \times\frac{1}{N!}(b_\alpha^\dagger b_{\alpha'}^\dagger\cdots)_{M_L}^{(L)}a_{\kappa',i}^\dagger\bar{a}_{\kappa',j}^\dagger|0\rangle \ , \tag{14}$$

i.e., a state with one quark and one antiquark and a certain number of quanta of string excitations. The bosonic operators are coupled in such a way as to produce representations of SO(4), and explicit formulas are known for their construction.

III. MASS FORMULA

We now construct a mass formula consistent with the spectrum-generating algebra and dynamic symmetry discussed in the previous section.

A. Mass formula for geometric excitations

If we assume a dynamic U(4)⊃SO(4) symmetry, the mass formula for geometric excitations must be constructed in terms of the Casimir operators of (9). The Casimir operators of U(4) are not relevant, since they contribute a constant term. The algebra of SO(4) has two invariants. Denoting the generators of SO(4) by **L** and **D**, the two invariants are [11, 19]

$$\mathcal{C}(\mathrm{SO}(4))=\mathbf{L}^2+\mathbf{D}^2, \quad \mathcal{C}'(\mathrm{SO}(4))=\mathbf{L}\cdot\mathbf{D} \ . \tag{15}$$

The eigenvalues of (15) in the representation (9) are

$$\langle\mathcal{C}\rangle=\omega(\omega+2), \quad \langle\mathcal{C}'\rangle=0 \ . \tag{16}$$

We introduce the vibrational quantum number v, Eq. (13), and subtract, for convenience, a constant term $N(N+2)$ from $\mathcal{C}$. We then have

$$\langle\mathcal{C}-N(N+2)\rangle=\omega(\omega+2)-N(N+2) =-4(N+1)\left[v-\frac{1}{N+1}v^2\right] \ . \tag{17}$$

The algebra of SO(3) has only one invariant:

$$\mathcal{C}(\mathrm{SO}(3))=\mathbf{L}^2 \ , \tag{18}$$

with eigenvalues $L(L+1)$. We do not break the L degeneracy by introducing invariants of SO(2). Thus the mass formula must be a functional only of (15) and (18).

In contrasting the mass formula, we consider the mass squared operator M^2, which is more appropriate for relativistic situations. Johnson and Thorn [20] and Bars and Hanson [20] have suggested that one expects a linear dependence of M^2 on L. This is *very different* from the usual nonrelativistic rigid string, for which the rotational energy grows [21] as $L(L+1)$, and is a crucial property of soft QCD strings. It implies that the string elongates, as it rotates. The elongation of the string is proportional to $\sqrt{L}$. Also, 't Hooft has shown, by explicit calculation [1] in $1+1$ dimensions, that one expects a linear dependence of M^2 on v. In order to reproduce these QCD results within the SGA approach, we write the M^2 operator as

$$M^2=M_0^2+A'[\mathcal{C}(\mathrm{SO}(4))-N(N+2)] +B\{[\mathcal{C}(\mathrm{SO}(3))+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2}\} \ . \tag{19}$$

We emphasize here that dynamic-symmetric arguments do not fix uniquely the functional form of M^2 in terms of the relevant Casimir invariants. The functional form used in (19) is motivated by QCD arguments, and its usefulness is tested by the quality of the fit to the experimentally determined meson masses. The eigenvalues of (19) are

$$M^2(v,L)=M_0^2-4(N+1)A'\left[v-\frac{1}{N+1}v^2\right] +B\{[L(L+1)+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2}\} . \quad (20)$$

Here the value of $(N/2)$ represents the total number of vibrational states in the representation $[N]$. In view of confinement, as expressed, for example, by the potential (10), the total number of bound states is infinite. We thus must take in our description $N\to\infty$. In practice it is sufficient to take N large enough to include all known and unknown states up to a maximum value of L and v. The maximum v is from (9), $v_{\max}=N/2$ or $(N-1)/2$, while the maximum L in each SO(4) representation is $N-2v$. The observed maximum number of L is $\simeq 5$, and the observed maximum value of v is $\simeq 4$. We take $N=100$. In the limit of large N, Eq. (20) reduces to

$$M^2(v,L)=M_0^2+Av+BL , \quad (21)$$

with $A=-4A'(N+1)$.

B. Mass formula for excitations involving internal degrees of freedom

For internal excitations, we have two parts, the spin part and the flavor part. The Casimir operator of $SU_s(2)$ is

$$\mathcal{C}(SU_s(2))=\mathbf{S}^2 , \quad (22)$$

with eigenvalues $S(S+1)$. The spin and orbital momentum must then be coupled to $\mathbf{J}=\mathbf{L}+\mathbf{S}$, as in (12). Denoting by $SU_J(2)$ the combined algebra

$$SU_s(2)\otimes SO_L(3)\supset SU_J(2) , \quad (23)$$

we have

$$\mathcal{C}(SU_J(2))=\mathbf{J}^2 , \quad (24)$$

with eigenvalues $J(J+1)$. There is no simple QCD argument to tell whether the dependence on S and J is linear on quadratic. Perturbation arguments involving one-gluon exchange suggest that the spin part appears in the potential approach, for a quark and antiquark of equal mass, with three terms: a spin-spin term of the type

$$V_S=V_S(r)\mathbf{S}_q\cdot\mathbf{S}_{\bar{q}} , \quad (25)$$

a spin-orbit term of the type

$$V_{SO}=V_{SO}(r)(\mathbf{S}_q+\mathbf{S}_{\bar{q}})\cdot\mathbf{L} , \quad (26)$$

and a tensor interaction

$$V_T=V_T(r)S_{12}(q,\bar{q}) . \quad (27)$$

With exception of the tensor interaction, which is nondiagonal in the L,S,J basis, the other two terms can be written in terms of the Casimir operators of total $SU_S(2)$ and $SU_J(2)$ as

$$V_S=v_S\mathcal{C}(SU_S(2)) ,$$
$$V_{SO}=v_{SO}[\mathcal{C}(SU_J(2))-\mathcal{C}(SU_S(2))-\mathcal{C}(SO(3))] , \quad (28)$$

with eigenvalues

$$\langle V_S\rangle=v_SS(S+1) , \quad (29)$$

and

$$\langle V_{SO}\rangle=v_{SO}[J(J+1)-S(S+1)-L(L+1)] , \quad (30)$$

Thus, these perturbations arguments suggest a quadratic dependence on S and J. Unfortunately, experiments cannot tell whether the dependence on S is linear or quadratic, since there are only two possible values for S for $(q\bar{q})$ mesons, $S=0$ and $S=1$. The test for the J term in the meson mass squared is difficult, particularly for light mesons, since the error on the mass determinations for the relevant mesons is rather large. However, there are indications that the nonrelativistic $J(J+1)$ rule is not a good one for them. In the analysis of the following section, we take, in analogy with (17) a linear dependence on the quantum numbers, i.e.,

$$\begin{aligned}M^2=M_0^2&+A'[\mathcal{C}(SO(4))-N(N+2)]\\&+B\{[\mathcal{C}(SO(3))+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2}\}\\&+C\{[\mathcal{C}(SU_S(2))+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2}\}\\&+D\{[\mathcal{C}(SU_J(2))+\tfrac{1}{4}]^{1/2}-\tfrac{1}{2}\} ,\end{aligned} \quad (31)$$

with eigenvalues

$$M^2(v,L,S,J)=M_0^2+Av+BL+CS+DJ . \quad (32)$$

We shall return to the question of the dependence on S and J in discussing baryons. Finally, we do not consider here effects of the tensor interaction. The QCD-inspired arguments of Gürsey [22] suggest this term to be rather small, at least for light mesons.

The mass spectrum (32) is extremely simple, as shown in Fig. 3. Each SO(4) representation v provides a Regge trajectory with $L=0,1,2,\ldots$. There are an infinite number of such trajectories corresponding to $v=0,1,\ldots$. The slopes of the trajectories are directly related to the coefficient A,B,C,D in (32).

We now come to the flavor part. In the basis of (12), the mass formula is in general, a matrix

$$(M^2)_{ij,i'j'}\equiv\langle q_i\bar{q}_j|M^2|q_{i'}\bar{q}_{j'}\rangle \quad (33)$$

that we take, for simplicity, to be real (36×36 real symmetry matrix for six flavors) [we have suppressed from (33) other quantum numbers, v,L,S,J,M_J]. If a dynamic symmetry corresponding to (11) exists, the matrix must be diagonal and of the type

$$M^2(i,j;v,L,S,J)=(M_0^2)_{ij}+A_{ij}v+B_{ij}L +C_{ij}S+D_{ij}J . \quad (34)$$

Following common usage, we shall call a combination ij a family (Table I). Analysis of the experimental situation can be done either for each individual family or simultaneously for all families.

If we attempt a simultaneous fitting of masses for all families, we have to face two problems: quark masses M_i

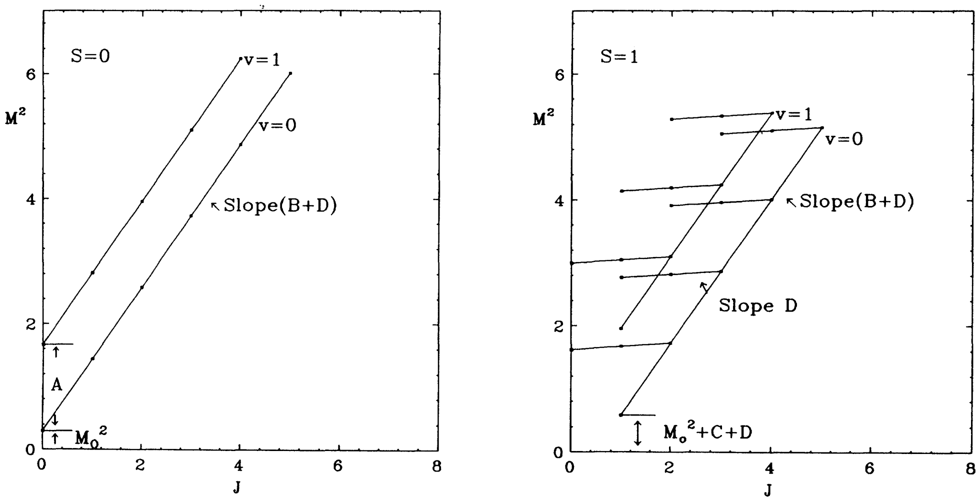

FIG. 3. Mass spectrum, represented by Eq. (32), for $S=0$ and $S=1$ mesons. Trajectories for $v=0$ and $v=1$ are separately shown.

($i=1,\ldots,6$) are widely different; light mesons behave relativistically, while heavy mesons are nonrelativistic. In order to take into account these effects, we must assume a phenomenological dependence of the coefficient in Eq. (33) on the quark masses. In the spirit of a simple expansion of M^2 in terms of quantum numbers, we introduce the quantity $M_{ij}=M_i+M_j$, where M_i and M_j are the constituent masses of quark i and antiquark j and expand all coefficients in M_{ij}, keeping only the first-order term:

$$A_{ij}=a+a'M_{ij},\quad B_{ij}=b+b'M_{ij}\ ,$$
$$C_{ij}=c+c'M_{ij},\quad D_{ij}=d+d'M_{ij}\ , \tag{35}$$
$$(M_0^2)_{ij}=eM_{ij}+(M_{ij})^2\ .$$

This simple parametrization should allow us to go from relativistic to nonrelativistic situations in a simple manner. The rationale is that we should use the mass operator itself, rather than M^2, for nonrelativistic situations. If we then add to M_0 an interaction term U, we obtain

$$M=M_0+U,\quad M^2=M_0^2+2M_0U+U^2\ . \tag{36}$$

If M_0 is small, we are in a relativistic situation; if M_0 is large, we are in a nonrelativistic domain. Both can be approximately covered by the parametrization of (35).

C. Improvements of the mass formula

The mass formula (34), with the flavor dependence parametrized as in (35), describes the experimental situation quite accurately, with the exception of the pseudoscalar nonet, π, K, η, and η'. In order to provide a precise description of this nonet, we need additional correction terms. We introduce the first of these two terms in a purely phenomenological way. This one relates to the fact that the π, K and octet combination of η and η' have unusually low masses (see Figs. 4–7). We include this observation by adding to the M^2 matrix a term of the type

TABLE I. Meson families.

Name	Notation	Quark content
Light unflavored ($I=1$)	π family	$u\bar{d},(u\bar{u}-d\bar{d})/\sqrt{2},d\bar{u}$
Light unflavored ($I=0$)	η family	$c_1(u\bar{u}+d\bar{d})+c_2(s\bar{s})$
Strange	K family	$u\bar{s},\bar{s}u,s\bar{d},\bar{d}s$
Charmed	D family	$c\bar{d},c\bar{u},\bar{c}u,\bar{c}d$
Charmed strange	D_s family	$c\bar{s},\bar{c}s$
Bottom	B family	$u\bar{b},d\bar{b},\bar{d}b,\bar{u}b$
$c\bar{c}$	$c\bar{c}$ or ψ family	$c\bar{c}$
$b\bar{b}$	$b\bar{b}$ or Υ family	$b\bar{b}$

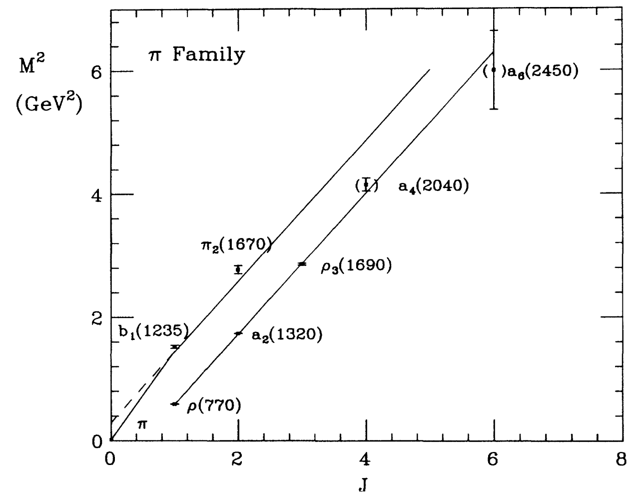

FIG. 4. Mass spectrum, Eq. (32), for the π family for $v=0$. The upper curve is for $S=0$ mesons and the lower curve is for the $S=1$ mesons. The dashed line indicates the straight trajectory from which π meson deviates. Experimental values of masses are indicated, and uncertain states are shown in parentheses.

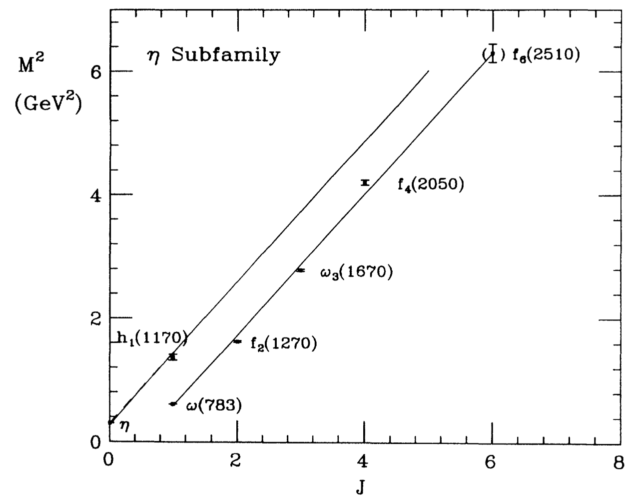

FIG. 6. Same as in Fig. 4, for the η subfamily. The deviation of the η meson from the straight trajectory is not significant here.

$$\langle q_i\bar{q}_j;v,L,S,J|M'^2|q_{i'}\bar{q}_{j'};v',L',S',J'\rangle=-f\delta_{v0}\delta_{L0}\delta_{S0}\delta_{J0}\delta_{v'0}\delta_{L'0}\delta_{S'0}\delta_{J'0}\begin{cases}\langle q_i\bar{q}_j|\delta_8|q_{i'}\bar{q}_{j'}\rangle & \text{if } i,j,i',j'=u,d,s,\\ 0 & \text{otherwise ,}\end{cases} \tag{37}$$

where δ_8 restricts this term to the ground-state octet of $SU_f(3)$.

The second correction term arises from the fact that the quark-antiquark pair can annihilate into gluons and reappear as another $q\bar{q}$ pair [23]. This correction term arises directly from QCD and it can be calculated in perturbation theory. In order to account for this effect, we introduce a term

$$\langle q_i\bar{q}_j;v,L,S,J|M''^2|q_{i'}\bar{q}_{j'};v',L',S',J'\rangle=\delta_{vv'}\delta_{SS'}\delta_{JJ'}\delta_{LL'}\delta_{ij}\delta_{i'j'}H_{ii'}(v,L,S,J)\ , \tag{38}$$

where $H_{ii'}$ is, in general, a 6×6 matrix, for six flavors, depending on v, L, S, and J. We take, in this article, a simple form for the above, corresponding to the fact that annihilation mostly occurs in $v=0, L=0, S=0$ states, and it is par-

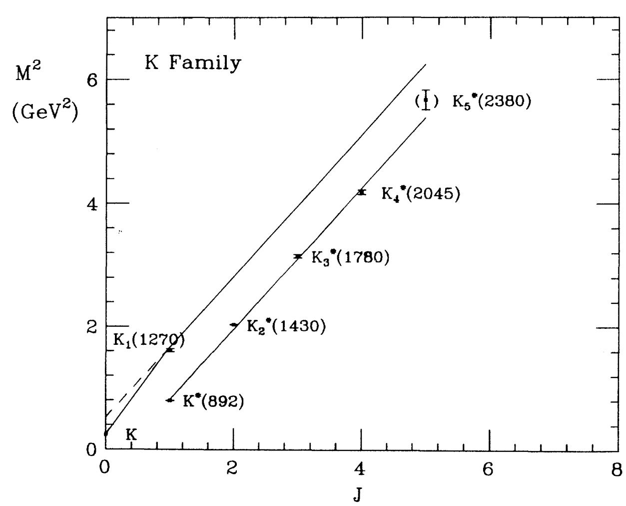

FIG. 5. Same as in Fig. 4, for the K family. The dashed line shows the continuation of the straight trajectory from which the K meson deviates.

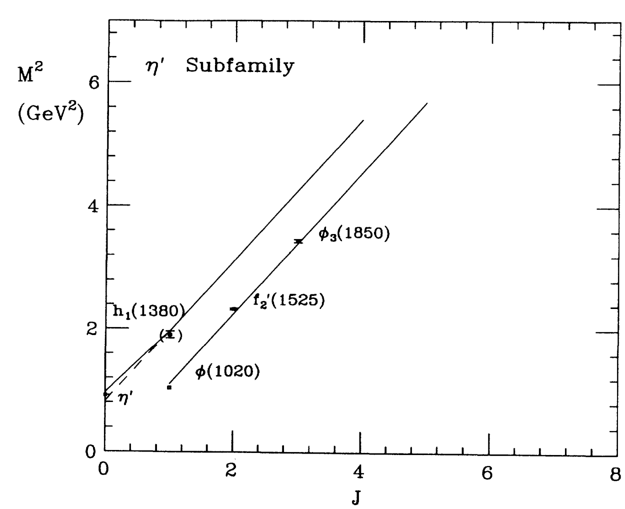

FIG. 7. Same as in Fig. 6. The η' subfamily.

ticularly important for light mesons:

$$\langle q_i\bar{q}_j;v,L,S,J|M''^2|q_{i'}\bar{q}_{j'};v',L',S',J'\rangle = h\,\delta_{v0}\delta_{L0}\delta_{S0}\delta_{J0}\delta_{v'0}\delta_{L'0}\delta_{S'0}\delta_{J'0}\begin{cases}\delta_{ij}\delta_{i'j'} & \text{if } i,j,i',j'=u,d,s\ ,\\ 0 & \text{otherwise}\ .\end{cases} \tag{39}$$

The terms (38) and (39) greatly improve the quality of our fit to the low-lying meson masses.

IV. ANALYSIS OF EXPERIMENTAL DATA

The main purpose of the present article is to analyze the experimental data on meson masses and assess the extent to which the QCD-based mass formula of the previous section describes the data. Large deviations from it would be considered an indication of new physics coming in (for example, intrusion of more complex quark and/or gluon configurations beyond the $q\bar{q}$ one that we have considered thus far). It may also indicate a failure of the parametrization discussed above. We base our analysis on the 1990 compilation of masses of the Particle Data Group (PDG), as given in their summary table [24]. We divide the data set into families, as done by the PDG (Table I).

Summarizing our earlier discussion, our mass formula has the form

$$\langle q_i\bar{q}_j|M^2|q_{i'}\bar{q}_{j'}\rangle = \delta_{ii'}\delta_{jj'}\left[(M_{ij})^2+eM_{ij}+(a+a'M_{ij})v\left[1-\frac{1}{N+1}v\right]+(b+b'M_{ij})L+(c+c'M_{ij})S + (d+d'M_{ij})J\right]+\langle M'^2\rangle_{ij,i'j'}+\langle M''^2\rangle_{ij,i'j'}\ , \tag{40}$$

where the last two terms are given by Eqs. (37) and (39), respectively. Thus, we have *fifteen* parameters to be determined from a fit to the masses of mesons: (1) the constituent quark masses $M_u=M_d$, M_s, M_c, and M_b (we do not consider isospin splitting); (2) the slopes of the mass trajectories, a, a', b, b', c, c', d, d', and e; (3) correction terms f and h. We select 57 well-established states from the Particle Data Group (PDG) meson summary table, assign quantum numbers (v,L,S) to these states and use their masses as experimental inputs to determine the parameters defining Eq. (40). We precede the discussion of the fit by some remarks concerning our general procedure.

A. Choice of meson data for fitting

In the PDG summary table [24], there are 86 mesons listed. Of these, members of a given isospin multiplet are considered only once in our fit, since we are ignoring isospin splittings. Altogether, we select 57 well-established states as inputs for fitting of the parameters of Eq. (40). These states are indicated in tables for all families. Below we discuss reasons for our specific exclusions of states, appearing in the PDG meson summary table (MST), from our fits.

(a) We exclude $\pi(1300)$ which has relatively large error on its mass determination. It has an uncertainty of 15% in the mass squared, compared to accuracy of better than 8% for others we use.

(b) We exclude states which have questionable $q\bar{q}$ configuration. Examples of this category are the 0^{++} states, $a_0(980)$, $f_0(975)$, and $f_0(1240)$. Likewise $f_1(1420)$, and $f_2(1720)$ do not fit in well as $q\bar{q}$ states. By forcing a fit including this class of states worsens the fit considerably. This is our best criterion of exclusion.

(c) We eliminate from our fits those states for which quantum number assignments are uncertain. Thus, in the D and D_S families, we remove $D_1(2420)^0$ and $D_{S1}(2556)^{\pm}$, because we cannot make unique spin assignments for these states.

(d) We reject states above $D\bar{D}$ and $B\bar{B}$ thresholds in the Ψ and Υ families, since there are substantial coupled-channel effects which make $q\bar{q}$ description of these states inaccurate in a simple fashion such as ours.

B. Assignment of quantum numbers and fitting procedure

Quantum numbers J^{PC}, where P,C are parity and charge-conjugation quantum numbers, are assigned in the PDG tables from experiments. We consider masses, decay modes and experimentally given J^{PC} to fix the quantum numbers S, L, and v. States with uncertain quantum numbers are excluded from the fit.

In the PDG meson summary table, there are some mesons with very precise masses, while others have masses known to have much less precision. In order *not* to overbias our fit by the former, we define a function

$$\chi^2=\sum_k\left|\frac{(M_k^{\text{th}})^2-(M_k^{\text{expt}})^2}{(M_k^{\text{expt}})^2}\right|\ , \tag{41}$$

and use the CERN routine MINUIT [25] to minimize it. This routine provides us a variety of procedures to search for minima in a multidimensional parameter space. We take care that the parameters have fitted values not overly sensitive to their initial values at the beginning of the search.

Fit A. We divide mesons into light (π,η,K families) and heavy mesons (D,D_S,B,Ψ, and Υ families) and perform separate fits to their masses. The correction terms (36) and (37) apply mostly to the light mesons and are omitted in the heavy-meson fit. For the latter, we fix M_u and M_s from the fit to light mesons. The values of the parameters obtained are shown in Table II, and the resulting masses in Tables III–X, column marked $M^2(AA)$.

TABLE II. Parameters of the mass formula (40), as determined from our fits to meson masses. In the fit B, ranges of parameters M_u and M_s are restricted to be 0.22–0.30 and 0.40–0.48 GeV, respectively. See the text for other details of these fits.

Parameter	Fit A for light Mesons	Fit A for heavy mesons	Fit B
M_u	0.271 GeV	0.271 GeV	0.250 GeV
M_s	0.454 GeV	0.454 GeV	0.400 GeV
M_c		1.434 GeV	1.468 GeV
M_b		4.719 GeV	4.706 GeV
a	1.374 GeV^2	2.612 GeV^2	0.933 GeV^2
a'		0.513 GeV	0.730 GeV
b	1.092 GeV^2	0.934 GeV^2	0.827 GeV^2
b'		0.560 GeV	0.445 GeV
c	0.118 GeV^2	0.113 GeV^2	0.076 GeV^2
c'		0.000 GeV	0.017 GeV
d	0.052 GeV^2	0.311 GeV^2	−0.034 GeV^2
d'		0.010 GeV	0.131 GeV
e		0.261 GeV	0.274 GeV
f	0.274 GeV^2		0.366 GeV^2
g	0.132 GeV^2		0.115 GeV^2

TABLE III. Quality of fit of our mass formula (40), as judged by masses of mesons in the π family. In Tables III through X, category (a) represents mesons with $q\bar{q}$ structure from the Particle Data Group (PDG) meson summary table, *included* in our fit; category (b) represents mesons with $q\bar{q}$ structure, *predicted* by our mass formula, and corresponding candidates from experimental studies, where available. Category (c), if applicable, are meson states which are *excluded* from our fits, as being plausible non-$q\bar{q}$ candidates. Quantum numbers v,L,S,J are our model assignments. States marked with daggers appear in the meson full listings of the PDG tables, but not in the summary table. The column marked $M^2(Q)$ represents the quark-model predictions of Godfrey and Isgur [6]. The columns marked $M^2(AA)$ and $M^2(AB)$ represent our results from fits A and B, respectively.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^{PC}
				(a)				
π	0.019±0.000	0.023	0.020	0.021	0	0	0	0^{-+}
$\rho(770)$	0.590±0.001	0.593	0.581	0.589	0	0	1	1^{--}
$b_1(1235)$	1.520±0.025	1.488	1.437	1.468	0	1	0	1^{+-}
$a_1(1260)$	1.588±0.076	1.538	1.673	1.638	0	1	1	1^{++}
$a_2(1320)$	1.738±0.002	1.716	1.725	1.670	0	1	1	2^{++}
$\rho(1450)$	2.103±0.023	2.103	1.941	1.873	1	0	1	1^{--}
$\pi_2(1670)$	2.772±0.067	2.822	2.581	2.550	0	2	0	2^{-+}
$\rho_3(1690)$	2.859±0.017	2.822	2.869	2.751	0	2	1	3^{--}
$\rho(1700)$	2.890±0.068	2.756	2.765	2.688	0	2	1	1^{--}
				(b)				
$\pi(1300)$	1.690±0.260	1.690	1.654	1.672	1	0	0	0^{-+}
$^{\dagger}\pi(1770)$	3.133±0.106	3.534	2.986	2.931	2	0	0	0^{-+}
$^{\dagger}a_4(2040)$	4.149±0.106	4.040	4.012	3.833	0	3	1	4^{++}
$^{\dagger}a_3(2050)$	4.326±0.166	4.203	3.961	3.801	0	3	1	3^{++}
$^{\dagger}\pi_2(2100)$	4.410±0.630	4.537	3.941	3.835	1	2	0	2^{-+}
			5.273	5.094	2	2	0	2^{-+}
$^{\dagger}a_6(2450)$	6.003±0.637		6.300	5.996	0	5	1	6^{++}
$^{\dagger}a_0(1320)$	~1.742	1.188	1.621	1.606	0	1	1	0^{++}
$^{\dagger}\rho(2150)$	~4.623	4.000	3.274	3.133	2	0	1	1^{--}
		4.623	4.125	3.973	1	2	1	1^{--}
			4.580	4.366	3	0	1	1^{--}
$^{\dagger}\rho_3(2250)$	~5.063	4.537	4.229	4.036	1	2	1	3^{--}
		5.617	5.052	4.850	0	4	1	3^{--}
			5.562	5.296	2	2	1	3^{--}
$^{\dagger}\rho_5(2350)$	~5.523	5.290	5.156	4.914	0	4	1	5^{--}
			6.516	6.199	1	4	1	5^{--}
				(c)				
$a_0(980)$	0.967±0.005	Decay modes: $\eta\pi, K\bar{K}$ seen						0^{++}

TABLE IV. Mesons of the η family. In the last column, our assumed flavor mixing types are explicitly indicated.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^{PC}	Mixing type
			(a)						
η	0.301±0.001	0.240	0.269	0.257	0	0	0	0^{-+}	$\theta_{\text{fit }A}=-23°$
$\eta'(958)$	0.917±0.000	0.865	0.970	0.967	0	0	0	0^{-+}	$\theta_{\text{fit }B}=-19°$
$\omega(783)$	0.611±0.000	0.608	0.581	0.589	0	0	1	1^{--}	$u\bar{u}$
$\phi(1020)$	1.039±0.000	1.040	1.111	1.110	0	0	1	1^{--}	$s\bar{s}$
$h_1(1170)$	1.369±0.049	1.488	1.437	1.468	0	1	0	1^{+-}	$u\bar{u}$
$f_2(1270)$	1.623±0.013	1.638	1.725	1.670	0	1	1	2^{++}	$u\bar{u}$
$f_1(1285)$	1.644±0.013	1.538	1.673	1.638	0	1	1	1^{++}	$u\bar{u}$
$\eta(1295)$	1.677±0.010	1.613	1.654	1.672	1	0	0	0^{-+}	$u\bar{u}$
$\omega(1390)$	1.935±0.050	2.132	1.941	1.873	1	0	1	1^{--}	$u\bar{u}$
$f_1(1510)$	2.286±0.012	2.190	2.203	2.293	0	1	1	1^{++}	$s\bar{s}$
$f_2'(1525)$	2.326±0.015	2.341	2.255	2.364	0	1	1	2^{++}	$s\bar{s}$
$\omega(1600)$	2.541±0.038	2.756	2.765	2.688	0	2	1	1^{--}	$u\bar{u}$
$\omega_3(1670)$	2.782±0.017	2.822	2.869	2.751	0	2	1	3^{--}	$u\bar{u}$
$\phi(1680)$	2.822±0.168	2.856	2.471	2.612	1	0	1	1^{--}	$s\bar{s}$
$\phi_3(1850)$	3.437±0.026	3.610	3.399	3.619	0	2	1	3^{--}	$s\bar{s}$
$f_4(2050)$	4.198±0.041	4.040	4.012	3.833	0	3	1	4^{++}	$u\bar{u}$
			(b)						
†$f_4(2220)$	4.951±0.027	4.840	4.542	4.873	0	3	1	4^{++}	$s\bar{s}$
†$f_6(2510)$	6.300±0.151		6.300	5.996	0	5	1	6^{++}	$u\bar{u}$
†$h_1(1380)$	1.904±0.055	2.161	1.967	2.113	0	1	0	1^{+-}	$s\bar{s}$
$2\,^1S_0$		2.403	2.183	2.361	1	0	0	0^{-+}	$s\bar{s}$
$3\,^1S_0$			2.986	2.931	2	0	0	0^{-+}	$u\bar{u}$
$3\,^1S_0$			3.516	3.832	2	0	0	0^{-+}	$s\bar{s}$
$4\,^1S_0$			4.292	4.165	3	0	0	0^{-+}	$u\bar{u}$
$4\,^1S_0$			4.822	5.274	3	0	0	0^{-+}	$s\bar{s}$
$^3D_1$		3.534	3.295	3.476	0	2	1	1^{--}	$s\bar{s}$
$2\,^1P_1$		3.168	2.797	2.753	1	1	0	1^{+-}	$u\bar{u}$
$2\,^1P_1$		4.040	3.327	3.615	1	1	0	1^{+-}	$s\bar{s}$
$^3P_0$		1.188	1.621	1.606	0	1	1	0^{++}	$u\bar{u}$
$^3P_0$		1.850	2.151	2.222	0	1	1	0^{++}	$s\bar{s}$
$2\,^3P_0$		3.168	2.981	2.891	1	1	1	0^{++}	$u\bar{u}$
$2\,^3P_0$		3.960	3.511	3.724	1	1	1	0^{++}	$s\bar{s}$
$3\,^3P_0$			4.314	4.150	2	1	1	0^{++}	$u\bar{u}$
$3\,^3P_0$			4.844	5.195	2	1	1	0^{++}	$s\bar{s}$
$2\,^3P_1$		3.312	3.033	2.923	1	1	1	1^{++}	$u\bar{u}$
$2\,^3P_1$		4.121	3.563	3.795	1	1	1	1^{++}	$s\bar{s}$
$3\,^3P_1$			4.366	4.182	2	1	1	1^{++}	$u\bar{u}$
$3\,^3P_1$			4.896	5.267	2	1	1	1^{++}	$s\bar{s}$
$2\,^3P_2$		3.312	3.085	2.955	1	1	1	2^{++}	$u\bar{u}$
$2\,^3P_2$		4.162	3.615	3.866	1	1	1	2^{++}	$s\bar{s}$
$^3F_2$		4.203	3.909	3.769	0	3	1	2^{++}	$u\bar{u}$
$^3F_2$		5.018	4.439	4.730	0	3	1	2^{++}	$s\bar{s}$
$3\,^3P_2$			4.418	4.214	2	1	1	2^{++}	$u\bar{u}$
$3\,^3P_2$			4.948	5.338	2	1	1	2^{++}	$s\bar{s}$
$^1D_2$		2.822	2.581	2.550	0	2	0	2^{-+}	$u\bar{u}$
$^1D_2$		3.572	3.111	3.367	0	2	0	2^{-+}	$s\bar{s}$
$^3D_2$		2.890	2.817	2.719	0	2	1	2^{--}	$u\bar{u}$
$^3D_2$		3.648	3.347	3.547	0	2	1	2^{--}	$s\bar{s}$
$f_0(975)$	0.952±0.006		Decay modes: $\pi\pi$, then $K\bar{K}$ dominant					0^{++}	
$f_1(1420)$	2.031±0.004		Decay modes: $K\bar{K}\pi$ dominant					1^{++}	
†$f_0(1240)$	1.538±0.074		Decay modes: $K\bar{K}$ seen					0^{++}	
†$f_2(1430)$	~2.045		Decay modes: $K\bar{K}$ and $\pi\pi$ seen					2^{++}	

In these tables, states included in the fit are shown, together with states predicted, and a comparison with semirelativistic quark-model (QM) calculation of Godfrey and Isgur [6], thus far the best accounting of meson masses.

Fit B. In the spirit of our earlier report [10], we have here a global fit of all mesons—light and heavy. We restrict M_u and M_s in the range 0.22–0.30 and 0.40–0.48 GeV, respectively. The results are shown in Tables III–X, in the column $M^2(AB)$.

C. Analysis of the results

We find that the quality of both fits A and B are such that they can be used to assess the nature of the unknown states in meson spectra. The average deviation is 5.8% in the light-meson sector and 2.4% in the heavy-meson sector for fit A; the average deviation is 5.7% for fit B. For light mesons, our results are comparable to, or better than, the semirelativistic quark-model results [6], while they are worse for heavy mesons. This is largely due to the fact that we are parameterizing the mass dependence of M^2 in a simple linear form, Eq. (40). Better descriptions of heavy mesons can be obtained by more elaborate expansions.

We use as a criterion for assessing the nature of unknown meson states, their deviation from the fit. If a state deviates by more than four average deviations, we question its assignment as a $q\bar{q}$ state. From this conservative measure, three states in the π and η family, $a_0(980)$, $f_0(975)$, and $f_0(1240)$, cannot be assigned $q\bar{q}$ status: the first two have been suggested to be $K\bar{K}$ molecules [26]. In the π family, the fact that $a_0(1320)$, $a_1(1260)$, and $a_2(1320)$ are quite close, in experiment and in our fit, indicates that the spin-orbit interaction, between q and $\bar{q}$ is very weak, a point that should be explored further. For the newly reported state $a_0(1320)$ (Ref. [27]), our prediction is much closer to experiment than that of QM. For the states $\rho(2150)$, $\rho_3(2250)$, and $\rho_5(2350)$, our predictions are quite different from those of QM. In the η family, we cannot accommodate $f_1(1420)$ and $f_2(1720)$ as 1^{++} and 2^{++} states, in agreement with similar conclusions from the QM. We cannot take $f_0(1240)$ to be $s\bar{s}\,{}^3P_0$, $f_2(1430)$ cannot be $q\bar{q}$, and $f_2(1565)$ is too low to be the $(1/\sqrt{2})(u\bar{u}+\bar{d}d)2\,{}^3P_2$ state. For the heavy mesons, our inclusion of a quark-mass decoupling term improves the fit.

TABLE V. Mesons of the K family.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^P
			(a)					
K	0.246±0.000	0.221	0.251	0.235	0	0	0	0^-
$K^*(892)$	0.795±0.000	0.810	0.813	0.827	0	0	1	1^-
$K_1(1270)$	1.613±0.025	1.796	1.669	1.768	0	1	0	1^+
$K^*(1370)$	1.869±0.148	2.496	2.173	2.220	1	0	1	1^-
$K_1(1400)$	1.966±0.020	1.904	1.905	1.943	0	1	1	1^+
$K_0^*(1430)$	2.042±0.017	1.538	1.853	1.892	0	1	1	0^+
$K_2^*(1430)$	2.032±0.004	2.045	1.957	1.995	0	1	1	2^+
$K^*(1680)$	2.816±0.215	3.168	2.997	3.059	0	2	1	1^-
$K_2(1770)$	3.126±0.050	3.276	3.048	3.111	0	2	1	2^-
$K_3^*(1780)$	3.147±0.028	3.204	3.100	3.163	0	2	1	3^-
$K_4^*(2045)$	4.182±0.037	4.452	4.244	4.330	0	3	1	4^+
			(b)					
†$K_0^*(1950)$	3.783±0.117	3.572	3.213	3.285	1	1	1	0^+
†$K_2^*(1980)$	3.912±0.158	3.764	3.317	3.388	1	1	1	2^+
		4.623	4.140	4.227	0	3	1	2^+
†$K_5^*(2380)$	5.674±0.157	5.712	5.388	5.498	0	4	1	5^-
†$K(1460)$	~2.132	2.103	1.885	1.994	1	0	0	0^-
†$K_2(1580)$	~2.496	3.168	2.812	2.936	0	2	0	2^-
†$K_1(1650)$	2.723±0.165	3.610	3.029	3.162	1	1	0	1^+
		3.725	3.265	3.336	1	1	1	1^+
†$K(1830)$	~3.349	4.080	3.218	3.359	2	0	0	0^-
†$K_2(2250)$	5.049±0.076	4.973	4.172	4.330	1	2	0	2^-
		5.108	4.408	4.504	1	2	1	2^-
†$K_3(2320)$	5.401±0.112	4.494	3.956	4.104	0	3	0	3^+
		4.623	4.192	4.279	0	3	1	3^+
			5.316	5.497	1	3	0	3^+
			5.552	5.672	1	3	1	3^+
†$K_4(2500)$	6.200±0.100	5.808	5.100	5.272	0	4	0	4^-
		5.954	5.336	5.447	0	4	1	4^-
			6.460	6.665	1	4	0	4^-
			6.696	6.840	1	4	1	4^-

TABLE VI. Mesons of the D family.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	s	J^P
				(a)				
D	3.485±0.002	3.534	3.354	3.422	0	0	0	0^-
D^*	4.034±0.006	4.162	3.907	3.825	0	0	1	1^-
$D_2^*(2460)^0$	6.049±0.011	6.250	6.123	5.608	0	1	1	2^+
				(b)				
$D_1(2420)^0$	5.876±0.029	5.954	5.570	5.205	0	1	0	1^+
		6.200	5.795	5.417	0	1	1	1^+

In order to emphasize the considerable accuracy of our mass formula for light mesons, we plot in Figs. 4–7, the experimental M^2 values for states in the π, K, and η families, as a function of J, along with our theoretical curves. The linearity of the Regge trajectories is clearly seen, especially for the ϕ, K^*, and ω trajectories. We also see the role of the first-correction term in the π and K masses. If this term would not have been included we would have obtained masses given by the dashed lines in Figs. 4 and 5.

If we write

$$M^2=M_0^2+\left[\frac{1}{\alpha'}\right]J \ , \tag{42}$$

the slopes $(1/\alpha')$ of the "rotational" Regge trajectories we obtain, from fit B, are

$$\pi,\rho;\frac{1}{\alpha'}=1.081\ \text{GeV}^2,\quad K,K^*:\frac{1}{\alpha'}=1.168\ \text{GeV}^2\ . \tag{43}$$

These slopes are slightly different from each other, because of the dependence of the coefficients of Eq. (35) on quark masses, and of the fact that ρ and K^* have spin $S=1$, while π and K have $S=0$.

Johnson and Thorn [20] give the slope of the Regge trajectories in the bag model as

$$\alpha'=\frac{1}{16\pi^{3/2}}\left[\frac{3}{2}\right]^{1/2}\frac{1}{\sqrt{\alpha_s}}\frac{1}{\sqrt{B}}\ , \tag{44}$$

where B is the bag pressure, and α_s the strong coupling constant. With $B^{1/4}=0.146$ GeV, $\alpha_s=0.55$, Eq. (44) gives

$$\alpha'=0.87\ \text{GeV}^{-2},\quad (1/\alpha')=1.15\ \text{GeV}^2\ , \tag{45}$$

in agreement with (43).

We also plot, in Fig. 8, the quantity

$$\Delta_L(M^2)=M^2(L=1)-M^2(L=0)\ , \tag{46}$$

as a function of M_{ij}. From this figure one can see that the linear dependence of the coefficients A,B,C,D of Eq. (35) is approximately verified over a large range of values. We find here no difference between the physics of light quarks and that of heavy quarks. The slopes of rotational Regge trajectories at the latter scale are, from fit B

$$\psi:\ \frac{1}{\alpha'}=2.48\ \text{GeV}^2,\quad \Upsilon:\ \frac{1}{\alpha'}=6.21\ \text{GeV}^2\ . \tag{47}$$

Similarly we plot, in Figs. 9–12, the M^2 values for states in the π, K, and η families as a function of v. The linearity of the "vibrational" Regge trajectories appears to be there, although not as clearly as for "rotational" Regge trajectories. We also see once again the role of the first-correction term in the π and K masses. If we write

$$M^2\cong M_0^2+\left[\frac{1}{\beta'}\right]v\ , \tag{48}$$

we obtain the slope of the "vibrational" Regge trajectories, from fit B,

$$\pi,\rho:\ \frac{1}{\beta'}=1.298\ \text{GeV}^2;\quad K,K^*:\ \frac{1}{\beta'}=1.408\ \text{GeV}^2\ . \tag{49}$$

From 't Hooft's calculations [1], one can reasonably infer

TABLE VII. Mesons of the D_s family.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^P
				(a)				
$D^\pm$	3.876±0.003	3.920	4.059	4.001	0	0	0	0^-
D_s^*	4.453±0.008	4.537	4.614	4.429	0	0	1	1^-
				(b)				
$D_{s1}(2536)^\pm$	6.434±0.004	6.401	6.379	5.870	0	1	0	1^+
		6.605	6.605	6.087	0	1	1	1^+

TABLE VIII. Mesons of the B family.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^P
				(a)				
B	27.86±0.02	28.20	26.20	25.92	0	0	0	0^-
				(b)				
$^\dagger B^*$	28.42±0.05	28.84	26.79	26.86	0	0	1	1^-

that $1/\beta'$ be proportional to the strong coupling constant α_s and some quark-mass squared scale m^2,

$$\frac{1}{\beta'} \simeq 4\pi^2 \alpha_s m^2 \ . \tag{50}$$

Using $\alpha_s = 0.55$ and $m^2 = (M_u)^2 \simeq (0.25 \text{ GeV})^2$, one estimates $1/\beta' \simeq 1.36 \text{ GeV}^2$, which is the same order of magnitude as that in (48).

We next plot, in Figs. 13 and 14, the "vibrational" Regge trajectories for the ψ and Υ families. We note that experimental masses no longer fall exactly linear on trajectories, but their trajectories are slightly bent, a feature also anticipated in the 't Hooft calculation [1]. The departure of the trajectories from linearity may indicate a breaking of the SO(4) symmetry. Another possible explanation is that the effective value of N, in our formula, appropriate for heavy mesons, is not $N \rightarrow \infty$ ($N = 100$ used in Figs. 13 and 14), but much smaller. This may be due to coupling with break-up channels that effectively terminate the rotational and vibrational bands. The U(4)⊃O(4) gives the following dependence on N [Eq. (20)]:

$$M^2(v) = M_0^2 - 4(N+1)A' \left[v - \frac{1}{N+1} v^2 \right] \ . \tag{51}$$

with $N \simeq 20$, we find the "vibrational" trajectories to be significantly bent [29], so as to actually describe the vibrational spectra of ψ and Υ observed so far. More experimental work is needed to extend these studies of the excited vibrational states, to clarify the picture.

Finally we comment on the dependence of the spin splittings on quark flavors. We do this by plotting, in Fig. 15, the observed spin splittings

$$\Delta_S(M^2) = M^2(S=1) - M^2(S=0) \ , \tag{52}$$

as a function of M_{ij}, together with the results of fits A and B. The dependence of Δ_S on M_{ij} is rather complex, since it emerges from the interplay of two effects: the correction in term f which affects the π and K splittings, and the change from a relativistic to a nonrelativistic regime, which we parametrize in a linear fashion. One can see, from Fig. 15, that the only spin splittings that we do

TABLE IX. Mesons of the $c\bar{c}$ family.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^{PC}
				(a)				
$\eta_c(1S)$	8.878±0.010	8.821	8.979	9.424	0	0	0	0^{-+}
$J/\psi(1S)$	9.591±0.001	9.610	9.544	10.03	0	0	1	1^{--}
$\chi_{c0}(1P)$	11.66±0.01	11.83	11.74	11.81	0	1	1	0^{++}
$\chi_{c1}(1P)$	12.32±0.00	12.32	12.08	12.16	0	1	1	1^{++}
$\chi_{c2}(1P)$	12.65±0.00	12.60	12.42	12.51	0	1	1	2^{++}
$\psi(2S)$	13.59±0.00	13.54	13.59	13.07	1	0	1	1^{--}
				(b)				
$\psi(3770)$	14.21±0.02	14.59	14.62	14.30	0	2	1	1^{--}
$\psi(4040)$	16.32±0.08	16.81	17.55	16.06	2	0	1	1^{--}
$\psi(4160)$	17.30±0.17	17.56	18.67	17.34	1	2	1	1^{--}
$\psi(4415)$	19.49±0.05	19.80	21.43	18.98	3	0	1	1^{--}
$^\dagger\eta_c(2S)$	12.92±0.04	13.10	13.02	12.47	1	0	0	0^{-+}

TABLE X. Mesons of the $b\bar{b}$ family.

Meson	M^2 (expt)	$M^2(Q)$	$M^2(AA)$	$M^2(AB)$	v	L	S	J^{PC}
				(a)				
$\Upsilon(1S)$	89.50±0.00	89.49	92.17	92.84	0	0	1	1^{--}
$\chi_{b0}(1P)$	97.22±0.03	97.02	97.98	96.65	0	1	1	0^{++}
$\chi_{b1}(1P)$	97.85±0.01	97.61	98.38	97.85	0	1	1	1^{++}
$\chi_{b2}(1P)$	98.27±0.01	98.01	98.79	99.05	0	1	1	2^{++}
$\Upsilon(2S)$	100.47±0.01	100.00	99.54	100.56	1	0	1	1^{--}
$\chi_{b0}(2P)$	104.76±0.02	104.65	105.35	104.38	1	1	1	0^{++}
$\chi_{b1}(2P)$	105.17±0.01	105.06	105.76	105.58	1	1	1	1^{++}
$\chi_b 2(2P)$	105.45±0.01	105.27	106.16	106.78	1	1	1	2^{++}
$\Upsilon(3S)$	107.23±0.01	107.12	106.77	108.13	2	0	1	1^{--}
				(b)				
$\Upsilon(4S)$	111.94±0.07	113.00	113.85	115.55	3	0	1	1^{--}
$\Upsilon(10860)$	118.05±0.17	118.37	120.79	122.81	4	0	1	1^{--}
$\Upsilon(11020)$	121.42±0.18	123.21	127.57	129.92	5	0	1	1^{--}

not account for well enough are those of the B and ψ mesons. It could well be that the coefficient C is not described by a linear dependence on M_{ij}, while the other coefficients A, B, and D are.

D. Predictions

The fact that the mass formula (34) and (35), or (40) accounts for the observed meson states well, allows one to make reliable predictions for unobserved states. For example, we have

$$M_{a_2}^2 - M_\rho^2 = M_{\rho_3}^2 - M_{a_2}^2 \; . \tag{53}$$

Inserting the values from Table III, we find that the left-hand side of (53) is (1.15 ± 0.00) GeV^2, while the right-hand side is (1.12 ± 0.02) GeV^2, i.e., the dynamic SO(4) symmetry is, for all purposes, unbroken in the light mesons. Using the mass formula, we can predict

$$M_{\rho_5}^2 = (5.16\pm0.08)\ \text{GeV}^2 \; , \tag{54}$$

where ρ_5 is the unknown state with $v=0$, $S=1$, $L=4$, $J^{PC}=5^{--}$. A complete list of all predicted states can be obtained from us upon request. We estimate our error to be rather small for light mesons, but to increase somewhat for heavy mesons.

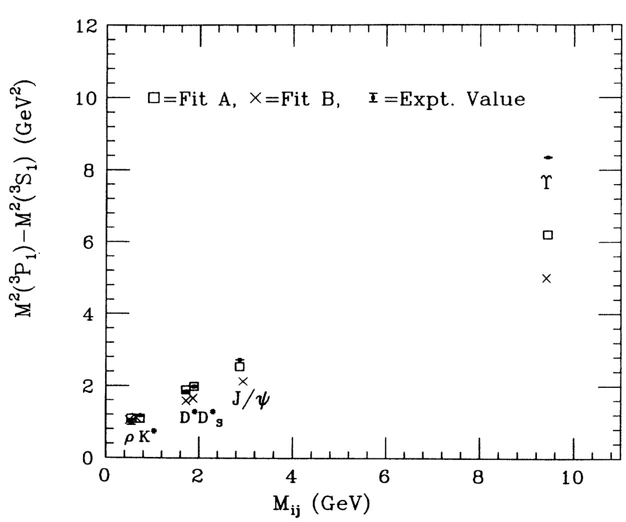

FIG. 8. The plot of $\Delta_L(M^2)=M^2(L=1)-M^2(L=0)$ as a function of $M_{ij}=M_i+M_j$. Fits A and B are described in the text.

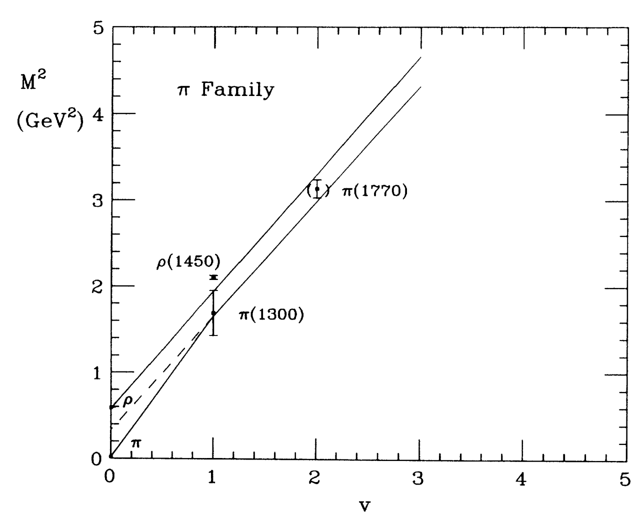

FIG. 9. M^2 for the π family as functions of v, for $S=0$ and $S=1$ states. The dashed line is the straight trajectory.

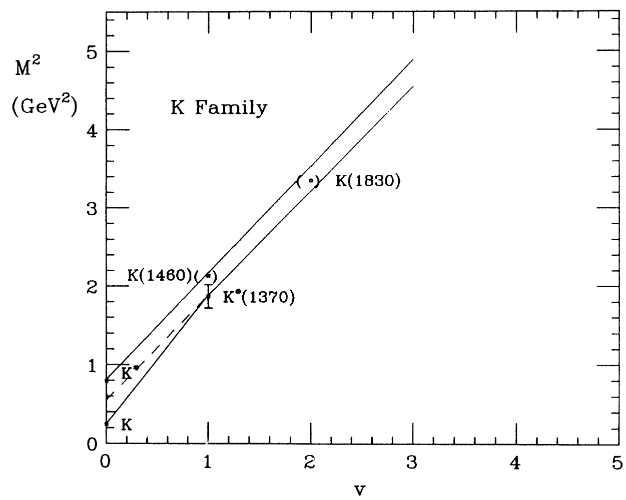

FIG. 10. Same as in Fig. 9. The K family.

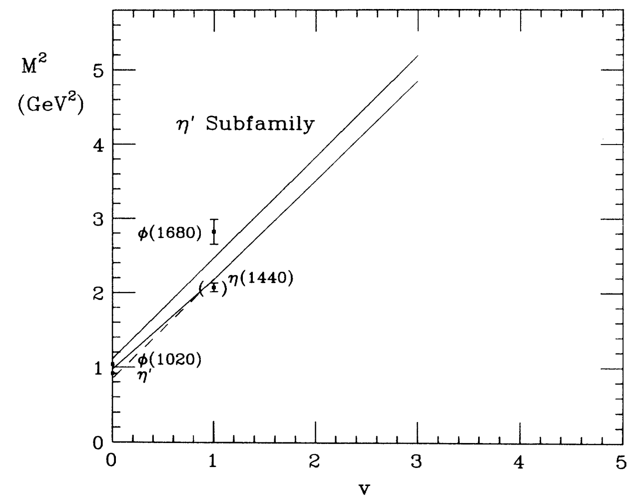

FIG. 12. Same as in Fig. 9. The η' subfamily.

As an example, we show, in Fig. 16, our predicted spectrum of D and D_s families (based on fit A), with states plotted from the lowest state of each family. We estimate our error here to be of the order ± 100 MeV in the masses.

V. CONCLUSIONS

We have presented here an analysis of meson masses based on stringlike $(q\bar{q})$ configurations, described by the spectrum-generating algebra $U(4) \otimes SU_s(2) SU_f(6) \otimes SU_c(3)$, with $U(4) \supset SO(4)$ dynamic symmetry. For the problem described here, viz., masses of $q\bar{q}$ mesons, the main role of the SGA is to construct states and to provide the quantum numbers, onto which the mass formula is expanded; it also produces a simple, and yet reasonably accurate, classification scheme within which well-established $q\bar{q}$ mesons can be accommodated. Particularly important is the evidence for a dynamic SO(4) symmetry. The observed mesons state do, indeed, fall into Regge trajectories, representations of SO(4), linearity of

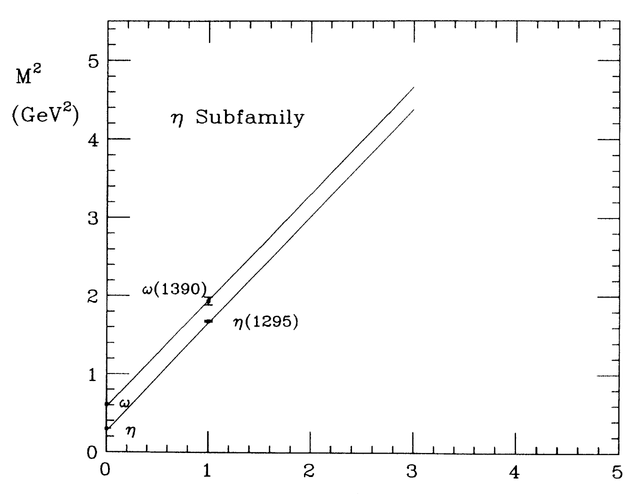

FIG. 11. Same as in Fig. 9. The η subfamily.

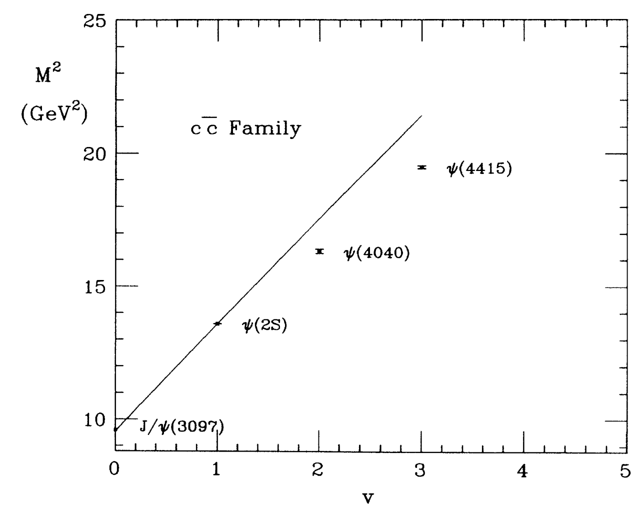

FIG. 13. M^2 for the $c\bar{c}$ family as function of v. Notice the deviations from the straight theoretical trajectory.

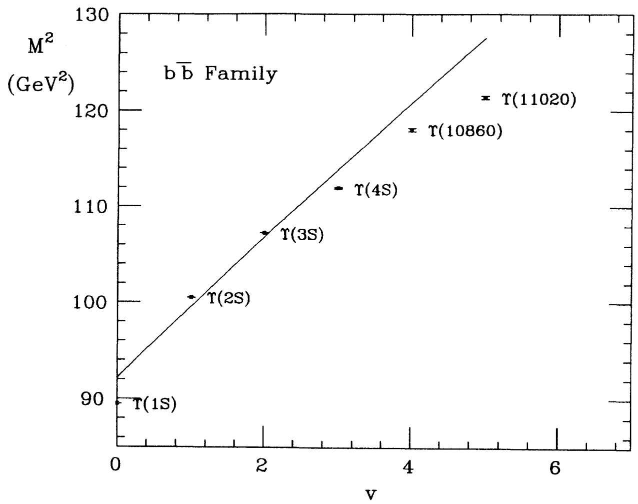

FIG. 14. Same as in Fig. 13. The $b\bar{b}$ family.

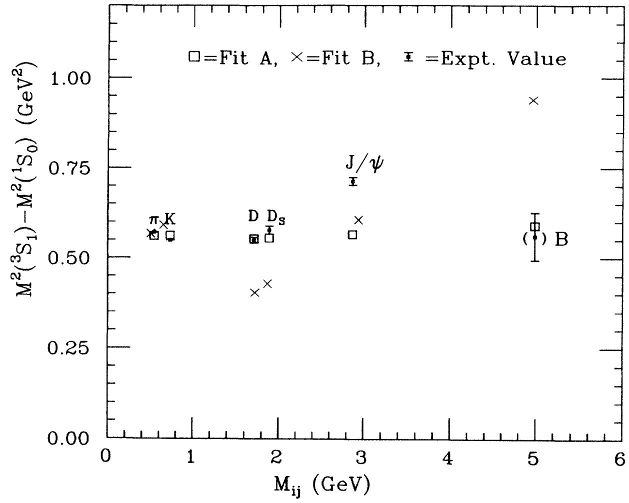

FIG. 15. Hyperfine splittings as functions of M_{ij}.

which is satisfied to a high accuracy for light mesons. The goodness of the SO(4) symmetry for this space part of the hadron wave functions seems comparable to that of the internal part of the wave functions (Gell-Mann–Ne'eman and Gürsey-Radicati symmetries). The accuracy of the mass formula of Sec. III implies that one can safely use it as a testing ground for future experimental predictions, and as a starting point for more elaborate analysis of the $(q\bar{q})$ configuration, and, indeed, of all the configurations of Fig. 1. From this point of view, the mass formula (34) and (35) can be viewed in the same manner as the Weizsäcker mass formula of nuclear physics, or, as a Landau expansion [28] in terms of the quantum numbers.

A major difference between the situation described here and that encountered in other systems in physics is the fact that the M^2 operator, in the present case, is linear in the quantum numbers, rather than quadratic. This is a peculiar feature, which has its physical origin in the fact that $q\bar{q}$ meson of QCD stretches, as it rotates. As described in Sec. III, this feature can be incorporated easily in the SGA approach, while it must arise from some complicated numerical interplay of various effects in models based on the Schrödinger-type equations. We thus think that the method discussed here provides a much simpler framework for attacking hadron spectroscopy, than methods based on Schrödinger-type equations with residual "potentials."

Finally, the classification scheme of this paper can be used to calculate other properties of mesons, in particu-

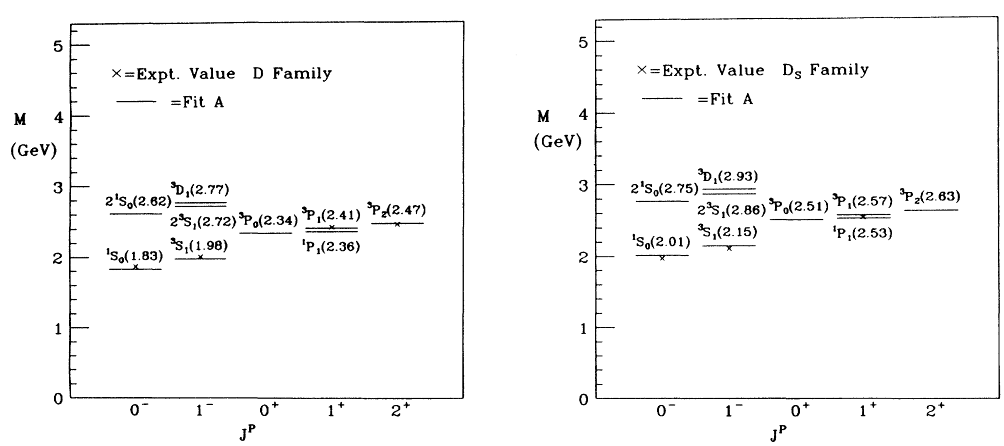

FIG. 16. Our predicted spectrum of D and D_s meson families, based on the fit A.

lar, strong, electromagnetic and weak decay widths, which will be reported in a parallel paper [11]. The recognition of the potentially important role of spectrum-generating algebra and dynamic symmetries for the space part of the hadronic wave functions opens the way for a simple, yet accurate, description of hadronic problems.

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