Generalized Riemann ζ-function regularization and Casimir energy for a piecewise uniform string

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The generalized ζ-function techniques will be utilized to investigate the Casimir energy for the transverse oscillations of a piecewise uniform closed string. We find that the ζ-function regularization method can lead straightforwardly to a correct result.

The study of vacuum fluctuations, as embodied in the Casimir effect,[1] has been a subject of extensive research.[2] The Casimir energy may be thought of as the energy due to the distortion of the vacuum. This distortion may be caused either by some background fields, or by the presence of boundaries in the spacetime manifold. Early study of the Casimir effect of a membrane was performed by Kikkawa and Yamasaki,[3] and some work on this subject has been done by us.[4] Recently, Brevik and Nielsen[5] studied the Casimir energy of the transverse oscillations of a piecewise uniform string. They regularized the zero-point energy by means of the exponential cutoff method and concluded that "The regularization procedure in the present problem is more delicate than what one may expect beforehand. Thus, a straightforward application of the Riemann ζ-function regularization method would lead to an incorrect result." In this paper, the generalized Riemann ζ function will be utilized to investigate this topic. We find that, with the application of the generalized Riemann ζ function, the ζ-function regularization method would lead to a correct result.

We start by introducing a generalized Riemann ζ function. The analytical continuation for the Riemann ζ function $\zeta(s)$ is not difficult because of the functional relation or so-called reflection formula. A function which in a sense is a generalization of $\zeta(s)$ is defined by

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \quad (0<a\le 1,\ \mathrm{Re}s>1)\ . \tag{1}$$

This function reduces to $\zeta(s)$ when $a=1$, and to $(2^s-1)\zeta(s)$ when $a=\frac{1}{2}$. For $\mathrm{Re}s>0$, we have

$$\Gamma(s)=(n+a)^s\int_0^{\infty}x^{s-1}e^{-(n+a)x}dx\ . \tag{2}$$

Hence,

$$\Gamma(s)\zeta(s,a)=\int_0^{\infty}\frac{x^{s-1}e^{-(a-1)x}}{e^x-1}dx\ , \tag{3}$$

if the inversion of the order of summation and integration can be justified; and this is guaranteed by the absolute convergence if $\mathrm{Re}s>1$. Now we consider the integral

$$I(s,a)=\int_C\frac{z^{s-1}e^{-az}}{e^z-1}dz\ , \tag{4}$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction excluding the points $\pm 2i\pi, \pm 4i\pi, \ldots$, and returns to positive infinity. Here z^{s-1} is defined as $\exp[(s-1)\ln z]$ when the logarithm is real at the beginning of the contour; thus $I(\ln z,a)$ varies from 0 to 2π round the contour. We can take C to consist of the real axis from ∞ to ρ $(0<\rho<2\pi)$, the circle $|z|=\rho$, and the real axis from ρ to ∞. On the circle, the integral tends to zero with ρ if $\mathrm{Re}s>1$. On making $\rho\to 0$ we therefore obtain

$$\zeta(s,a)=\frac{-e^{i\pi s}\Gamma(1-s)}{2\pi i}\int_C\frac{z^{s-1}e^{-az}}{1-e^{-z}}dz\ . \tag{5}$$

Equation (5) provides the analytic continuation of $\zeta(s,a)$ over the whole plane, and $\zeta(s,a)$ is regular everywhere except for a simple pole at $s=1$ with residue 1. Expanding the loop to infinity, the residues are at $\pm 2m\pi i$; hence, if $\mathrm{Re}s<0$, we have

$$\zeta(s,a)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left[\sin\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}\frac{\cos 2m\pi a}{m^{1-s}}+\cos\tfrac{1}{2}\pi s\sum_{m=1}^{\infty}\frac{\sin 2m\pi a}{m^{1-s}}\right]. \tag{6}$$

If s is an integer, the integrand in $I(s,a)$ is one valued, and $I(s,a)$ can be evaluated by the theorem of residues. We come to the following expressions of $\zeta(s,a)$:

$$\begin{aligned}\zeta(-n,a)&=-\frac{B_{n+1}(a)}{n+1}\ ,\\ \zeta(-1,a)&=-\tfrac{1}{2}(a^2-a+\tfrac{1}{6})\ ,\\ \zeta(2n,1)&=-\tfrac{1}{2}\ ,\\ \zeta(1-2n,1)&=\frac{(-1)^nB_n}{2n}\ ,\end{aligned} \tag{7}$$

where $B_n(a)$ is the Bernoulli polynomial and B_n is the Bernoulli number.

It may be worth emphasizing that the generalized ζ function $\zeta(s,a)$ is the fundamental ζ function associated with the piecewise uniform string, while $\zeta(s)$ is the fundamental ζ function associated with the uniform string.

The composite string model consists of two parts I and II, endowed in general with different lengths ($L_{\rm I}$ and $L_{\rm II}$), tensions ($T_{\rm I}$ and $T_{\rm II}$) and mass densities ($\rho_{\rm I}$ and $\rho_{\rm II}$), although adjusted in such a way that the velocity of sound always equals the velocity of light. Let $\psi=\psi(\sigma,\tau)$ be the transverse displacement of point σ at time τ. Taking into account right- and left-moving waves in regions I and II, we have

$$\psi_{\rm I}=\xi_{\rm I}e^{i\omega(\sigma-\tau)}+\eta_{\rm I}e^{-i\omega(\sigma+\tau)} , \tag{8a}$$

$$\psi_{\rm II}=\xi_{\rm II}e^{i\omega(\sigma-t)}+\eta_{\rm II}e^{-i\omega(\sigma+\tau)} . \tag{8b}$$

By using the conditions of continuity of transverse displacement and elastic force across the two junctions $\sigma=0,L$ ($L=L_{\rm I}+L_{\rm II}$) and $\sigma=L_{\rm I}$, one can get the dispersion equation

$$(1-x)^2\cos(\omega L-2\omega L_{\rm I})-(1+x)^2\cos\omega L+4x=0 , \tag{9}$$

where $x=T_{\rm I}/T_{\rm II}$. With the parameter $y=L_{\rm II}/L_{\rm I}$, it is convenient to rewrite the dispersion equation (9) in the form

$$\sin\omega L_{\rm I}\sin(y\omega L_{\rm I})+F(x)\sin^2\left[\frac{1+y}{2}\omega L_{\rm I}\right]=0 , \tag{10}$$

where $F(x)$ is defined as

$$F(x)=\frac{4x}{(1-x)^2} . \tag{11}$$

For definiteness we take $L_{\rm I}$ to be the smaller one of the two lengths, so that $y\geq 1$. The Casimir energy is a very useful concept; it may be viewed as the "zero-point energy" of the vacuum. In order to calculate the Casimir energy of the transverse oscillations of a piecewise uniform string, we must determine the frequencies ω of the possible transverse oscillations once the quantities x, L, and $L_{\rm I}$ are given. Note that Eq. (11) is invariant under the substitution $x\to 1/x$; hence, we shall only consider x in the interval $0<x\leq 1$ in the following. According to Ref. 5, we define the Casimir energy for the composite string as

$$E=E_{\rm I+II}-E_{\rm uniform} , \tag{12}$$

where $E_{\rm I+II}$ is the zero-point energy for parts I+II, and $E_{\rm uniform}$ is the zero-point energy for a uniform string. Unfortunately, the dispersion relation will not be solved in full generality; we shall solve it in some special cases.

(i) We consider the special case of $x=1$, which corresponds to a uniform string.[6] From Eq. (9) we have $\omega_n=2\pi n/L$ ($n=1,2,3,\dots$), and the Casimir energy is

$$E_{\rm uniform}=2\times\frac{1}{2}\sum_{n=1}^{\infty}\omega_n=\frac{2\pi}{L}\zeta(-1,1)=-\frac{\pi}{6L} , \tag{13}$$

where the factor 2 takes into account that the mode ω_n is degenerate.

(ii) The $x\to 0$ case implies that $T_{\rm I}\to 0$, if $T_{\rm II}$ is assumed finite. From Eq. (9), we get two sequences:

$$\omega_n=\begin{cases}\dfrac{\pi n}{L_{\rm I}} \\ & (n=1,2,3,\dots) . \\ \dfrac{\pi n}{L_{\rm II}}\end{cases} \tag{14}$$

The Casimir energy is

$$\begin{aligned}E&=\frac{\pi}{2L_{\rm I}}\zeta(-1,1)+\frac{\pi}{2L_{\rm II}}\zeta(-1,1)-\frac{2\pi}{L}\zeta(-1,1)\\&=-\frac{\pi}{24L}\left[y+\frac{1}{y}-2\right] .\end{aligned} \tag{15}$$

(iii) In the $y=2J+1$ ($J=1,2,\dots$) cases, the frequency equation can be reduced to

$$\sin^2\omega L_{\rm I}\sum_{j=0}^{J}\left\{\left[\binom{2J+1}{2j+1}+\frac{1}{2}\binom{2J+2}{2j+2}F(x)\right](1-\sin^2\omega L_{\rm I})^{J-j}+\frac{1}{2}\binom{J+1}{j+1}F(x)\right\}(-\sin^2\omega L_{\rm I})^{j}=0 . \tag{16}$$

In the present case, the general structure of the frequency spectrum is found as follows. First, the dispersion equation (16) has one degenerate branch, given by

$$\omega=\frac{\pi n}{L_{\rm I}} . \tag{17}$$

Next, there are J nondegenerate double branches, obtained by solving an algebraic equation of degree J in $\sin^2\omega L_{\rm I}$. Each double branch corresponds to a definite solution for $\sin^2\omega L_{\rm I}$. The frequency spectrum corresponding to such a branch can always be expressed in the form

$$\omega_{nj}L_{\rm I}=\begin{cases}\pi(\beta_j+n) ,\\ \pi(1-\beta_j+n) ,\end{cases} \tag{18}$$

where $n=0,1,2,\dots$ and β_j is a number in the interval $(0,\frac{1}{2}]$, $j=1,2,\dots,J$. The value of β_j is found by explicit calculation from Eq. (16) once x and J are given. The zero-point energy for regions I+II is

$$\begin{aligned}E_{\rm I+II}&=E(\text{degenerate branch})+\sum E(\text{double branches})\\&=2\left[\frac{1}{2}\right]\frac{\pi}{L_{\rm I}}\sum_{n=1}^{\infty}n+\frac{1}{2}\frac{\pi}{L_{\rm I}}\sum_{j=1}^{J}\left[\sum_{n=0}^{\infty}(\beta_j+n)\right.\\&\qquad\left.+\sum_{n=0}^{\infty}(1-\beta_j+n)\right] .\end{aligned} \tag{19a}$$

By using Eqs. (1) and (7), we find

$$E_{\mathrm{I+II}}=\frac{2\pi(J+1)}{L}\zeta(-1,1)+\frac{\pi(J+1)}{L}\sum_{j=1}^{J}[\zeta(-1,\beta_j)+\zeta(-1,1-\beta_j)]=\frac{\pi(2J+1)J}{6L}-\frac{\pi J}{2L}\sum_{j=1}^{J}[\beta_j^2+(1-\beta_j)^2]-\frac{\pi}{6L}\ . \tag{19b}$$

Then the Casimir energy is

$$E=\frac{\pi J(2J+1)}{6L}-\frac{\pi(L+1)}{2L}\sum_{j=1}^{J}[\beta_j^2+(1-\beta_j)^2]\ . \tag{20}$$

In the $y=1$, $x\neq 0$ case, the dispersion relation (10) reduces to $\cos\omega L=1$. The Casimir energy is

$$E=\frac{2\pi}{L}\zeta(-1,1)-\frac{2\pi}{L}\zeta(-1,1)=0\ . \tag{21}$$

(iv) In the $y=2J$ cases, the frequency equation can be reduced to

$$(1-\cos\omega L_{\mathrm{I}})\left\{\sum_{j=0}^{2J-2}(\cos\omega L_{\mathrm{I}})^j+(\cos\omega L_{\mathrm{I}}+1)\sum_{j=0}^{J-2}\begin{bmatrix}2J+1\\2j+1\end{bmatrix}(\cos\omega L_{\mathrm{I}})^{2J-2j-3}(\cos^2\omega L_{\mathrm{I}}-1)^j\right.$$
$$\left.-[1+F(x)]\left[\sum_{j=0}^{2J}(\cos\omega L_{\mathrm{I}})^j+(\cos\omega L_{\mathrm{I}}+1)\sum_{j=0}^{J-1}\begin{bmatrix}2J+1\\2j+1\end{bmatrix}(\cos\omega L_{\mathrm{I}})^{2J-2j-1}(\cos^2\omega L_{\mathrm{I}}-1)^j\right]\right\}=0\ . \tag{22}$$

The structure of the spectrum is similar to (iii). There exists one degenerate branch from the factor $1-\cos\omega L_{\mathrm{I}}$:

$$\omega=\frac{2\pi n}{L_{\mathrm{I}}}\ . \tag{23}$$

Next, there are $2J$ nondegenerate simple branches from the quantity in large curly brackets in Eq. (22). Each of these branches correspond to

$$\omega=\begin{cases}\dfrac{\pi(\beta_j+2n)}{L_{\mathrm{I}}}\ ,\\[2mm] \dfrac{\pi(2-\beta_j+2n)}{L_{\mathrm{I}}}\ ,\end{cases} \tag{24}$$

where $n=0,1,2,\ldots$ and β_j is a number in the interval $(0,1]$, $j=1,2,\ldots,2J$. The value of β_j is found by explicit calculation from Eq. (22) once x and J are given. The zero-point energy for regions I+II is

$$E_{\mathrm{I+II}}=E\ (\text{degenerate branch})+\sum E\ (\text{simple branch})=2\left[\frac{1}{2}\right]\frac{2\pi}{L_{\mathrm{I}}}\sum_{n=1}^{\infty}n+\frac{1}{2}\frac{\pi}{L_{\mathrm{I}}}\sum_{j=1}^{2J}\left[\sum_{n=0}^{\infty}(2n+\beta_j)+\sum_{n=0}^{\infty}(2n+2-\beta_j)\right]\ . \tag{25}$$

By using Eqs. (1) and (7), we have

$$E_{\mathrm{I+II}}=\frac{\pi(2J+1)}{L}\zeta(-1,1)+\frac{\pi(2J+1)}{L}\sum_{j=1}^{2J}[\zeta(-1,\beta_j/2)+\zeta(-1,1-\beta_j/2)]=\frac{\pi J(4J+1)}{3L}-\frac{\pi(2J+1)}{8L}\sum_{j=1}^{2J}[\beta_j^2+(2-\beta_j)^2]-\frac{\pi}{6L}\ . \tag{26}$$

Then the Casimir energy is

$$E=\frac{\pi J(4J+1)}{3L}-\frac{\pi(2J+1)}{8L}\sum_{j=1}^{2J}[\beta_j^2+(2-\beta_j)^2]\ . \tag{27}$$

In conclusion, we have used the generalized Riemann ζ-function regularization method to get the correct expression for the Casimir energy of a piecewise uniform string. This method is rather formal in nature than the exponential cutoff method used in Ref. 5.

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