Spherically symmetric thin walls

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Assuming a continuous ansatz for the metric we solve the Einstein equation for a thin wall directly by using the method of distributions. The same results as the thin-shell formalism of Israel are regained.

Here we report on a simple and direct way of solving the Einstein equation for a spherically symmetric thin shell of mass, with any equation of state. Given any spherically symmetric thin shell with the $(2+1)$-dimensional timelike history Σ, we ask for the metric of the entire spacetime and the dynamics of the shell (see Fig. 1). Assuming the spacetime to be otherwise free, the exterior metric ($g^{+}_{\mu\nu}$) will be Schwarzschild and the interior one ($g^{-}_{\mu\nu}$) flat. This problem has been solved by Israel[1] using the Gauss-Codazzi formalism, relating the surface stress-energy tensor of the shell to the exterior curvature of Σ. Since then this method has been used by authors interested in the dynamics of domain walls.[2,3] The thin-shell formalism of Israel circumvents the application of the theory of distributions.

Now Geroch and Traschen[4] show that, at least for $(2+1)$-dimensional walls, one can even use discontinuous metrics as solutions of the Einstein equation. This leads to nonlinear operations with distribution-valued tensors. On the other hand, the validity of thin-shell idealization has been examined. This is because of the work of Raychaudhuri and Mukherjee[5] indicating that thick walls do have stress-energy tensor components that are orthogonal to the wall. It was required that these components should vanish in the zero-thickness limit. Widrow[6] treats the Einstein-scalar equations for a thick domain wall with planar symmetry. He then takes the zero-thickness limit for his solution and shows that the orthogonal components of the stress-energy tensor vanish in that limit. The same question has been studied for general domain walls by Garfinkle and Gregory.[7] They expand the coupled Einstein-scalar equations that describe the thick gravitating wall in powers of the thickness of the wall. The solutions of the zeroth-order equations reproduce the results of the usual Israel thin-wall approximation for domain walls.

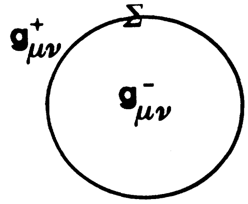

FIG. 1. Spherically symmetric thin shell with $(2+1)$-dimensional timelike history Σ.

The direct method we are going to use has already been applied to plane walls[8] leading to the same results as the Gauss-Codazzi formalism. In the case of a spherically symmetric thin shell we proceed as follows. First we write the exterior Schwarzschild and interior flat metric in the form

$$ds^2_+ = e^{\nu(t)}dt^2 - e^{-\nu(r)}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\varphi^2)$$
$$= \left[1-\frac{2M}{r}\right]dt^2 - \left[1-\frac{2M}{r}\right]^{-1}dr^2 - r^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ , \qquad (1)$$

and

$$ds^2_- = d\bar{t}^2 - d\bar{r}^2 - \bar{r}^2(d\theta^2+\sin^2\theta\varphi^2)\ . \qquad (2)$$

This metric is discontinuous at Σ defined by $r=R(t)$. However, the discontinuity is an artifact of coordinates chosen, and can be removed by an appropriate coordinate transformation. The transformation

$$\bar{r}=a(r,t),\quad \bar{t}=b(r,t)\ , \qquad (3)$$

upon substitution in (2), leads to

$$ds^2_- = (b^2_{,t}-a^2_{,t})dt^2 + 2(b_{,t}b_{,r}-a_{,t}a_{,r})dtdr - (a^2_{,r}-b^2_{,r})dr^2 - a^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ . \qquad (4)$$

Now, we obtain, for the conditions of continuity of the metric at Σ,

$$\beta_t^2-\alpha_t^2=e^{\nu(R)},\quad 2(\beta_t\beta_r-\alpha_t\alpha_r)=0\ ,$$
$$\alpha_r^2-\beta_r^2=e^{-\nu(R)},\quad A=R\ , \qquad (5)$$

where

$$\alpha_r=\left.\frac{\partial a}{\partial r}\right|_R,\quad \alpha_t=\left.\frac{\partial a}{\partial t}\right|_R,\quad \beta_r=\left.\frac{\partial b}{\partial r}\right|_R,\quad \beta_t=\left.\frac{\partial b}{\partial t}\right|_R\ ,$$

and $A=a(R(t),t)$.

These equations can be solved to find

$$\alpha_{\tau}=\gamma^{-2}(\sqrt{\gamma^2+R_{,t}^2}-e^{-\nu(R)}R_{,t}^2)\ ,$$
$$\alpha_t=\gamma^{-2}R_{,t}(e^{\nu(R)}-\sqrt{\gamma^2+R_{,t}^2})\ ,$$
$$\beta_t=\gamma^{-2}(e^{\nu(R)}\sqrt{\gamma^2+R_t^2}-R_{,t}^2)\ ,$$
$$\beta_r=\gamma^{-2}R_{,t}(1-e^{-\nu(R)}\sqrt{\gamma^2+R_{,t}^2})\ , \tag{6}$$

where

$$\gamma=(e^{\nu(R)}-e^{-\nu(R)}R_{,t}^2)^{1/2}\ . \tag{7}$$

Now, after establishing the continuity of the matrix at Σ we can write it as a distribution:

$$g_{\mu\nu}=g_{\mu\nu}^{+}\theta(r-R)+g_{\mu\nu}^{-}\theta(R-r)\ , \tag{8}$$

where $g_{\mu\nu}^{+}$ and $g_{\mu\nu}^{-}$ can be read off from (1) and (4).

This metric is regular[4] and can be differentiated formally. It turns out that the first derivative of it is proportional to the step function. Its second derivative enters linearly in the expressions for curvature and Ricci tensor. The Ricci tensor vanishes in the exterior and interior of the shell and is just proportional to $\delta(R-r)$. Therefore, it suffices to calculate just terms in the second derivative of the metric which are proportional to δ. This leads to

$$R_{00}=-\tfrac{1}{2}R^{-4}[(X^2UW)_{,t}R_{,t}+R^4e^{\nu}(\nu_{,r}e^{\nu}-U_{,r}+2V_{,t}) -R^{+4}e^{-\nu}U_{,t}R_{,t}]\delta(r-R)\ ,$$
$$R_{01}=+\tfrac{1}{2}R^{-4}[(X^2UW)_{,t} -R^4(e^{\nu}W_{,t}-e^{-\nu}U_{,t})]\delta(r-R)\ ,$$
$$R_{11}=\tfrac{1}{2}R^{-4}[4R^3+(X^2UW)_{,r}-R^4e^{\nu}(-\nu_{,r}e^{-\nu}+W_{,r}) +R^4e^{-\nu}(2V_{,t}+W_{,t}R_{,t})]\delta(r-R)\ , \tag{9}$$
$$R_{22}=\tfrac{1}{2}[e^{\nu}(2R+X_{,t})+e^{-\nu}X_{,t}R_{,t}]\delta(r-R)\ ,$$
$$R_{33}=\sin^2\theta R_{22}\ ,$$

where $U=b_{,t}^2-a_{,t}^2$, $V=b_{,t}b_{,r}-a_{,t}a_{,r}$, $W=b_{,r}^2-a_{,r}^2$, $X=-a^2$.

Now we have to specify the energy-momentum tensor of the shell. We take the shell matter as ideal fluid with σ and τ as surface energy density and tension, respectively. The unit spacelike normal to Σ will be called n^{μ}. The induced three-metric is

$$h_{\mu\nu}=g_{\mu\nu}+n_{\mu}n_{\nu}\ . \tag{10}$$

The surface stress-energy tensor of the shell Σ will then be written as

$$S^{\mu\nu}=\sigma u^{\mu}u^{\nu}+\tau(h^{\mu\nu}-u^{\mu}u^{\nu}),\quad u^{\mu}u_{\nu}=1\ , \tag{11}$$

where the vector field u^{μ} represents the fluid motion on Σ. $S^{\mu\nu}$ is related to the energy-momentum tensor $T^{\mu\nu}$ by

$$S^{\mu\nu}=\int dl\ T^{\mu\nu}\ , \tag{12}$$

where l is the proper distance through Σ in the direction of the normal n^{μ}. For the metric (8) the vectors n^{μ} and u^{μ} are given by

$$(n^{\mu})=(-e^{-\nu}\dot{R},-\gamma^{-1}e^{\nu},0,0)\ , \tag{13}$$

$$(u^{\mu})=(\gamma^{-1},\dot{R},0,0)\ , \tag{14}$$

where the overdot denotes the differentiation with respect to proper time of the observer u^{μ}. Now, from (12) and (13) we can easily obtain the desired relation between $T^{\mu\nu}$ and $S^{\mu\nu}$:

$$T^{\mu\nu}=\gamma S^{\mu\nu}\delta(r-R(t))\ , \tag{15}$$

where γ is given by (7).

Substitution from (8), (10), (11), (13), and (14) into (15) gives

$$(T_{\mu\nu})=\gamma\begin{bmatrix}\gamma^{-2}e^{2\nu}\sigma & -\gamma^{-1}\dot{R}\sigma & 0 & 0\\ -\gamma^{-1}\dot{R}\sigma & e^{-2\nu}\dot{R}^2\sigma & 0 & 0\\ 0 & 0 & -R^2\tau & 0\\ 0 & 0 & 0 & -R^2\sin^2\theta\tau\end{bmatrix} \times\delta(r-R)\ . \tag{16}$$

The Einstein field equation

$$R_{\mu\nu}=\kappa(T_{\mu\nu}-\tfrac{1}{2}Tg_{\mu\nu}) \tag{17}$$

combined with the continuity conditions (6) then leads to

$$\left[\frac{\sigma}{2}-\tau\right]\dot{R}=\frac{d}{ds}(\sqrt{e^{\nu}+\dot{R}^2}-\sqrt{1+\dot{R}^2})\ , \tag{18}$$

$$\frac{\sigma}{2}R=\sqrt{1+\dot{R}^2}-\sqrt{e^{\nu}+\dot{R}^2}\ . \tag{19}$$

The equation of state of shell fluid $\tau=\tau(\sigma)$ added to these field equations determines R, σ, and τ as functions of time. From (18) and (19) we obtain

$$R=R_0\exp\left[-\tfrac{1}{2}\int_{\sigma_0}^{\sigma}\frac{d\sigma}{\sigma-\tau(\sigma)}\right]\ . \tag{20}$$

The equation of state $\tau=\Gamma\cdot\sigma$, where Γ is a constant satisfying $0\leq\Gamma\leq1$, is of special interest. In this case (20) leads to

$$R=R_0\left[\frac{\sigma}{\sigma_0}\right]^{-1/2(1-\Gamma)}\quad\text{or}\quad\frac{\sigma}{\sigma_0}=\left[\frac{R}{R_0}\right]^{-2(1-\Gamma)}\ . \tag{21}$$

For dust shell, $\Gamma=0$, we obtain

$$\sigma\propto\frac{1}{R^2} \tag{22}$$

which is the well-known result of Israel.[1] For domain wall, $\Gamma=1$, we obtain the result of Ipser and Sikivie:[9]

$$\sigma=\text{const}\ . \tag{23}$$

From Eqs. (19) and (1) we can obtain the mass of the shell, M, as a function of R, its derivative and σ:

$$M=\frac{\sigma R^2}{2}\left[\sqrt{1+\dot{R}^2}-\frac{\sigma R}{4}\right]\ , \tag{24}$$

which is the same as that obtained by Israel[1] and Ipser and Sikivie.[9]

The condition of staticity leads obviously to positive pressures. From (18) and (19) we obtain

$$\tau = -\frac{\sigma^2 R}{4(2-\sigma R)} . \quad (25)$$

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