Light-cone regular vertex in three-dimensional quenched QED

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We study a model Schwinger-Dyson equation in three dimensions based on three-dimensional QED in the confining quenched approximation. We employ an ansatz for the dressed photon-fermion vertex function that does not have a singularity on the light cone. Our ansatz also allows us to study the relationship between the transverse part of the vertex and gauge independence of $\langle\bar{\psi}\psi\rangle$. We find that the transverse part of the vertex is crucial to obtaining $\langle\bar{\psi}\psi\rangle$ independent of the gauge parameter. It modifies both the infrared and ultraviolet behavior of the fermion propagator without upsetting the $1/p^2$ behavior expected of the mass function at large spacelike p^2.

I. INTRODUCTION

The phase structure of quantum electrodynamics (QED) has become the subject of intensive study. It is studied in lattice QED to determine if there is a nontrivial continuum limit (which occurs only if the lattice theory has a phase transition of second or higher order [1]). Also, there are authors [2] who argue that a new phase of QED might provide a means of understanding the e^+e^- peaks observed at GSI, Darmstadt [3,4].

One order parameter of QED that is a measure of a phase transition is the chiral condensate $\langle\bar{\psi}\psi\rangle$, which is trivially related to the fermion propagator $S_F(x)$: $\mathrm{tr}S_F(x=0)=-\langle\bar{\psi}\psi\rangle$. A nonzero value of $\langle\bar{\psi}\psi\rangle$, when the Lagrangian bare mass of the fermion is zero, signals dynamical chiral symmetry breaking (DCSB) and a plot relating $\langle\bar{\psi}\psi\rangle$ to relevant dimensionless parameters in a given model can be used to study the transition to the chirally asymmetric phase. The connection between $\langle\bar{\psi}\psi\rangle$ and $S_F(x)$ suggests that chiral symmetry and its dynamical breakdown may be studied in the continuum using the Schwinger-Dyson equation (SDE) for the fermion self-energy [5,6]. Investigations of the SDE for QED also lead on to the study of lepton mass generation in walking technicolor models [7]. In this connection it has been argued [8] that three-dimensional QED (QED_3) is a natural model for studying the hierarchy problem and thus the SDE for QED_3 has become a subject of study [9,10].

In Minkowski space the renormalized SDE for QED_3 is

$$\Sigma(p)=\not{p}(1-Z_1)+Z_1 ie^2 \times\int \bar{d}^3q\,\Gamma_\mu(p,q)D^{\mu\nu}(p-q;\xi)S_F(q)\gamma_\nu\,, \quad (1.1)$$

where $\bar{d}^3q=d^3q/(2\pi)^3$. In $D=3$ dimensions there are two inequivalent 2×2 representations of the Clifford algebra $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$; hence, to describe spinorial representations of the Lorentz group, two-component spinors are sufficient. However, it is not possible to describe chiral symmetries using two-component spinors [11] and so in any study of DCSB one must necessarily employ four-component spinors and a 4×4 representation of the Clifford algebra, which is a direct sum of the two inequivalent representations referred to above. This ensures that the mass term in the fermion propagator is not parity violating.

In (1.1) ξ is the gauge parameter in the covariant gauge-fixing procedure and $D^{\mu\nu}(p;\xi),\Gamma_\mu(p,q),S_F(p),\Sigma(p)$ are, respectively, the renormalized photon propagator, fermion-photon vertex, fermion propagator and fermion self-energy. The constant Z_1 is the vertex renormalization constant and in deriving (1.1) we have made use of the Ward identity $Z_1=Z_2$, where Z_2 is the wave-function renormalization constant.

A feature of QED_3 that makes it a model relevant to strong interaction physics is that, in the absence of fermion loop contributions to the photon polarization tensor, it is a confining theory. This has been rigorously established [12] for lattice QED_3 and can be seen heuristically by looking at the classical potential:

$$V(\mathbf{x})=\int_{-\infty}^{\infty}dx_3\int \bar{d}^3q\; e^{i(\mathbf{q}\cdot\mathbf{x}-q_3x_3)}e^2D(q) =\int \bar{d}^2q\; e^{i\mathbf{q}\cdot\mathbf{x}}e^2D(\mathbf{q})\,, \quad (1.2)$$

where

$$D^{\mu\nu}(q)=\left[-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right]D(q)-\xi\frac{q^\mu q^\nu}{q^2}\,. \quad (1.3)$$

Using the bare photon propagator in QED_3 we find

$$V(r)=\frac{e^2}{2\pi}\ln(re^2) \quad (1.4)$$

which exhibits logarithmic confinement.

When fermion loop vacuum polarization diagrams are included the feature of confinement is lost. In an approximation [11] valid for $\mathrm{tr}\Sigma(p) \ll p$ one finds that [10] $D(q)$ becomes

$$\frac{1}{q^2[1+(\tilde{\alpha}/|q|)]}, \quad \tilde{\alpha}=\frac{Ne^2}{8} \tag{1.5}$$

with N the number of fermion generations in the model, and the potential is

$$V(r)=-\frac{e^2}{4}[\mathbf{H}_0(\tilde{\alpha})-N_0(\tilde{\alpha}r)] , \tag{1.6}$$

where [13] $\mathbf{H}_0(x)$ is a Struve function and $N_0(x)$ a Neumann function. From (1.6) one can determine that at small r:

$$V(r)\sim \ln(\tilde{\alpha}r) . \tag{1.7}$$

However, at large r one finds now that

$$V(r)\simeq -\frac{e^2}{2\pi}\frac{1}{\tilde{\alpha}r} \tag{1.8}$$

and the theory is no longer confining.

Perturbation expansions in QED_3 are free of ultraviolet divergences (the theory is super-renormalizable); however, diagrams in the massless fermion theory have infrared divergences; for example, the $O(e^2)$ contribution to the photon-fermion vertex diverges as $\ln(m)$, where m is the current mass of the fermion [14]. This problem can be avoided if the fermion vacuum-polarization contribution to the photon polarization tensor is included because it softens the infrared divergence of the photon propagator [see (1.5) above]; however, as pointed out above, the model is then no longer confining. The problem associated with this infrared divergence is that such diagrams contribute to Z_1. If Z_1 is not finite then it becomes difficult to decide whether (1.1) and the other renormalized SDE provide a sensible approach to solving the field theory. This problem is similar to one associated with the ultraviolet divergences in QED_4 [15].

In the massless theory and in D dimensions one finds that

$$Z_1=1+e^2\left[\frac{4}{D}+D-5+\xi\right] \times(\text{regularized divergent integral})+O(e^4) . \tag{1.9}$$

It is clear from (1.9) why Landau gauge, $\xi=0$, is favored in studies of the SDE in $D=4$ dimensions: to $O(e^2)$, at least, $Z_1=1$ and the bare SDE is exactly the same as the renormalized SDE. It is equally clear that in QED_3 this is not the case and, in fact, $Z_1=1$ for $\xi=\frac{2}{3}$; a feature of the SDE for QED_3 that is apparent in Ref. [16]. In this case there is no reason why Landau gauge should be favored over another choice for the gauge parameter.

We now present the following equation as a model SDE:

$$\Sigma(p)=ie^2\int d^3q\; \Gamma_\mu(p,q)D^{\mu\nu}(p-q;\xi)S_F(q)\gamma_\nu . \tag{1.10}$$

With the exception of Ref. [16], where the gauge dependence of the wave-function renormalization is accounted for at $O(1/N^2)$ (N is the number of fermion flavors), this is the equation that has been considered by all authors before us who have studied DCSB in QED_3 using the SDE and questions of (possibly finite) renormalization have been neglected. Working in Landau gauge, as is common, does not obviate the need for Z_1 and the common practice of neglecting it requires justification before this approach can begin to become rigorous. This is why we describe this equation as a model SDE.

In other works the further truncation

$$\Gamma_\mu(p,q)=\gamma_\mu \tag{1.11}$$

is also made although a first attempt to proceed beyond this has recently been made [10]. We do not make this truncation here. Within the context of (1.10) we are interested in developing an understanding of the role and importance of the proper photon-fermion vertex in the SDE.

Pursuant to our discussion of confinement and the inclusion of fermion vacuum polarization diagrams we choose to study (1.10) using the free photon propagator:

$$\begin{gathered} D_{\mu\nu}(k)\rightarrow D^0_{\mu\nu}(k) , \\ D^0_{\mu\nu}(k)=\frac{-g_{\mu\nu}k^2+(1-\xi)k_\mu k_\nu}{k^4} . \end{gathered} \tag{1.12}$$

This is because we are interested in studying a model in which the kernel of the integral equation has a singularity that can lead to confinement.

In this paper we analyze (1.10) with (1.12) and study the fermion condensate obtained from the solution. One of our primary aims is to study the effect of including a model photon-fermion vertex that satisfies the Ward identity and is also regular on the light cone, in contrast with the simple light-cone singular vertex considered elsewhere [10] which is incompatible with perturbative analyses [17]. Further, we study the dependence of $\langle\bar{\psi}\psi\rangle$ on the gauge parameter, ξ, using both a light-cone singular (LCS) and light-cone regular (LCR) vertex which presents an interesting comparison. In addition we study the effect on the condensate of including the transverse part of the photon-fermion vertex; for example, the gauge dependence of $\langle\bar{\psi}\psi\rangle$. In Sec. II we derive the integral equations that are the subject of our study and in Sec. III discuss the analytical and numerical procedures we employ in their solution. We also describe the important features of the solutions. Our conclusions are presented in Sec. IV.

II. LIGHT-CONE REGULAR VERTEX

It has become clear that in studying field theories in the continuum it is necessary to preserve the Ward identity in any approximate or model SDE [10,18–21]. In QED_3 the Ward identity simply relates the fermion propagator to the photon-fermion vertex:

$$(p-q)_\mu \Gamma^\mu(p,q) = S^{-1}(p) - S^{-1}(q) . \tag{2.1}$$

It is implicitly assumed that using a vertex that satisfies this identity is sufficient to preserve gauge covariance of the SDE. This may be true of the exact SDE but is not assured in any model equation.

It is easy to determine that the bare vertex γ_μ satisfies (2.1) when the bare fermion propagator is used. Once this is established, however, it becomes obvious that the bare vertex cannot satisfy this identity when it involves the dressed propagator and is therefore inadequate in any model or approximate SDE. A simple ansatz for the vertex that does satisfy (2.1) is [18,21]

$$\Gamma^\mu(p,q) = \text{transverse part not constrained by (2.1)} + A(p^2)\gamma^\mu + \frac{k^\mu}{k^2}\{[A(p^2)-A(q^2)]\not{q} - [B(p^2)-B(q^2)]\} \tag{2.2}$$

with $k^\mu = (p-q)^\mu$.

This is a particularly simple vertex ansatz to use because in Landau gauge the complicated terms decouple and one is left, effectively, with the vertex [19]

$$\Gamma^\mu(p,q) = A(p^2)\gamma^\mu . \tag{2.3}$$

However, the vertex in (2.2) is singular on the light cone, $k^2=0$. In a detailed study of the analytic properties of the photon-fermion vertex using perturbation theory [17] it has been established that such a kinematic singularity is definitely not present at first order in the photon-fermion vertex and it is argued that this feature should persist to all orders in perturbation theory. The following vertex, with

$$\frac{(p-q)_\mu}{(p-q)^2} \to \frac{(p+q)_\mu}{p^2-q^2}$$

that is

$$\Gamma^\mu(p,q) = [aA(p^2)+(1-a)A(q^2)]\gamma^\mu + \frac{(p+q)^\mu}{p^2-q^2}\{[A(p^2)-A(q^2)][(1-a)\not{p}+a\not{q}] - [B(p^2)-B(q^2)]\} \tag{2.4}$$

not only satisfies the Ward identity but also has the analytic structure prescribed by perturbation theory. In (2.4), a is a parameter that has been included so that we may vary in a simple fashion the transverse part of the vertex which is not constrained by the Ward identity at all.

To proceed with the solution of (1.10) it is usual to perform a formal Wick rotation to Euclidean space. The fact that the singularity structure of the fermion propagator may complicate or even preclude such a rotation of integration contour is usually neglected. It has been established that an approximate Euclidean SDE for QED_4 yields a solution for the fermion propagator that has complex-conjugate branch points [22] and, more recently, that model SDE for QCD also suffer this problem [23]. It is thus apparent that the solution of the Euclidean-space model SDE may not be related to the solution of the naively associated Minkowski-space model SDE via simple analytic continuation.

It becomes important then to consider alternatives to the simple-minded Wick rotation which, after all, was introduced and justified only within the context of a simple scalar bound-state problem. We believe that the answer may lie with the axioms of constructive field theory [24]; in particular, the relations between the Schwinger functions and Wightman functions in configuration space. The idea here is to Fourier transform to configuration space where the axioms demand that for a well-defined quantum field theory one can perform an analytic continuation in the *time* variable ($x_4 \leftrightarrow ix_0$). Of course, we do not advocate a strict constructive-field-theory approach but rather one that, within the constraints of practicality, attempts to make use of some of the fundamental features of constructive quantum field theory.

To proceed we adopt the strategy of defining our model in Euclidean space [25]:

$$\begin{aligned} i\gamma\cdot p[A(p^2)-1]+B(p) &= \int \bar{d}^3q\, e^2 D_{\mu\nu}(p-q)\Gamma_\mu(p,q) \\ &\quad\times \frac{1}{i\gamma\cdot qA[q^2+B(q^2)]}\gamma_\nu , \end{aligned} \tag{2.5}$$

where

$$i\gamma\cdot pA(p^2)+B(p^2) = S^{-1}(p) \tag{2.6}$$

and, in Euclidean space, the free photon propagator is

$$D_{\mu\nu}(k) = \frac{\delta_{\mu\nu}k^2-(1-\xi)k_\mu k_\nu}{k^4} . \tag{2.7}$$

As we discussed in Sec. I we use the bare photon propagator because we wish to consider integral equations in which the kernel manifests singularities associated with confinement. Consequently our studies correspond to quenched QED_3 lattice calculations.

In Euclidean space our photon-fermion vertex is

$$i\Gamma_\mu(p,q)=i[aA(p^2)+(1-a)A(q^2)]\gamma_\mu +L_\mu\{i[A(p^2)-A(q^2)][(1-a)\gamma\cdot p+a\gamma\cdot q] +[B(p^2)-B(q^2)]\}\ . \tag{2.8}$$

To obtain the LCS vertex from (2.8) one uses

$$L_\mu^S=\frac{(p-q)_\mu}{(p-q)^2} \tag{2.9}$$

while the LCR vertex is obtained with

$$L_\mu^R=\frac{(p+q)_\mu}{p^2-q^2}\ . \tag{2.10}$$

Our preferred Euclidean-space model SDE is obtained from (2.5), (2.8), and (2.10). The equation thus obtained represents a confining model with a photon-fermion vertex that has the analytic structure prescribed by perturbation theory.

Here it is important to realize that the longitudinal part of the LCR vertex is exactly the same as the longitudinal part of the LCS vertex: the difference lies completely in the transverse part. This is obvious since (2.9) is purely longitudinal whereas

$$\left[\delta_{\mu\nu}-\frac{(p-q)_\mu(p-q)_\nu}{(p-q)^2}\right]L_\nu^R(p,q)\neq 0\ .$$

So while the transverse part of the LCS vertex is simply

$$\Gamma_\mu^{S-T}(p,q)=[aA(p^2)+(1-a)A(q^2)]\times\left[\gamma_\mu-\frac{(p-q)_\mu}{(p-q)^2}\gamma\cdot(p-q)\right]$$

the transverse part of the LCR vertex is much more complicated. Consequently, even in Landau gauge, the SDE obtained with the LCR vertex is significantly more complex than the one obtained when the LCS vertex is used. Herein, although our preferred vertex is obtained with (2.10), we will also report studies of (2.5) using the vertex obtained with (2.9). In this way the remarks we have made above will become obvious and comparisons may be made.

Using the LCS vertex obtained with (2.9), the SDE of (2.5) yields the following pair of nonlinear, coupled integral equations $[k=(p-q)]$:

$$p^2[A(p^2)-1]=\int d^3q\frac{1}{q^2A(q^2)^2+B(q^2)^2}\times\left[\frac{2}{k^4}p\cdot k q\cdot k[aA(p^2)+(1-a)A(q^2)]A(q^2)+\frac{\xi}{k^4}[q^2p\cdot kA(q^2)-p^2q\cdot kA(p^2)]\times A(q^2)-\frac{\xi}{k^4}p\cdot k[B(p^2)-B(q^2)]B(q^2)\right]\ , \tag{2.11}$$

and

$$B(p^2)=\int d^3q\frac{1}{q^2A(q^2)^2+B(q^2)^2}\left[\frac{2}{k^2}A(q^2)B(q^2)+\frac{\xi}{k^4}[p\cdot kA(p^2)B(q^2)-q\cdot kA(q^2)B(p^2)]+\frac{2a}{k^2}[A(p^2)-A(q^2)]B(q^2)\right]\ . \tag{2.12}$$

The angular integrals in these equations can be evaluated analytically. The first term in large parentheses in (2.11) is zero and we find that

$$p^2[A(p^2)-1]=\frac{\xi}{4\pi^2}\int_0^\infty\frac{q^2dq}{q^2A(q^2)^2+B(q^2)^2}\times\left[\frac{1}{p^2-q^2}\{[q^2A(q^2)-p^2A(p^2)]A(q^2)-[B(p^2)-B(q^2)]B(q^2)\}+\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|\{[q^2A(q^2)+p^2A(p^2)]A(q^2)-[B(p^2)-B(q^2)]B(q^2)\}\right]\ , \tag{2.13}$$

and

$$B(p^2)=\frac{1}{4\pi^2}\int_0^\infty \frac{q^2dq}{q^2A(q^2)^2+B(q^2)^2}\times\left[\frac{\xi}{p^2-q^2}[A(p^2)B(q^2)-A(q^2)B(p^2)]+\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|[4(1-a)A(q^2)B(q^2)+(4a+\xi)A(p^2)B(q^2)+\xi A(q^2)B(p^2)]\right]. \tag{2.14}$$

It is clear that this pair of equations assumes a particularly simple form if one chooses $\xi=0$ for then (2.13) simply becomes the identity

$$A(p^2)=1 \tag{2.15}$$

and (2.14) reduces to the bare vertex equation (3.1) below. As remarked in Sec. I, this is why Landau gauge is often favored in analyses of the SDE. Of course, this neglects the influence of Z_1.

When the LCR vertex obtained with (2.10) is used the equations are much more complicated. One obtains

$$\begin{aligned}p^2[A(p^2)-1]=\frac{1}{4\pi^2}\int_0^\infty \frac{q^2dq}{q^2A(q^2)^2+B(q^2)^2}&\times\left[\xi\left[\frac{1}{p^2-q^2}\{[q^2A(q^2)-p^2A(p^2)]A(q^2)-[B(p^2)-B(q^2)]B(q^2)\}\right.\right.\\&\left.+\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|\{[q^2A(q^2)+p^2A(p^2)]A(q^2)-[B(p^2)-B(q^2)]B(q^2)\}\right]\\&+\frac{2}{p^2-q^2}\left[1-\frac{p^2+q^2}{2pq}\ln\left|\frac{p+q}{p-q}\right|\right]\\&\left.\times\{[(1-a)p^2+aq^2][A(p^2)-A(q^2)]A(q^2)+[B(p^2)-B(q^2)]B(q^2)\}\right],\end{aligned} \tag{2.16}$$

and

$$B(p^2)=\frac{1}{4\pi^2}\int_0^\infty \frac{q^2dq}{q^2A(q^2)^2+B(q^2)^2}\times\left[\frac{1}{p^2-q^2}\left[(\xi-2)+\frac{p^2+q^2}{pq}\ln\left|\frac{p+q}{p-q}\right|\right][A(p^2)B(q^2)-A(q^2)B(p^2)]+\frac{1}{2pq}\ln\left|\frac{p+q}{p-q}\right|[4(1-a)A(q^2)B(q^2)+(4a+\xi)A(p^2)B(q^2)+\xi A(q^2)B(p^2)]\right]. \tag{2.17}$$

In (2.16) and (2.17) the gauge parameter choice $\xi=0$ does simplify the equations but not to the dramatic extent that it did in the LCS case. This may be the reason why the LCR vertex, which is consistent with QED perturbation theory, has been ignored in previous studies of the SDE.

In the absence of a current mass for the fermion these integral equations are scale invariant. The mass scale is set by $\mu=e^2$ and the solution for any value of μ can be obtained from the $\mu=1$ solution by the scale transformation:

$$A(p^2;\mu)=A\left[\frac{p^2}{\mu^2};1\right], \tag{2.18}$$

$$B(p^2;\mu)=\mu B\left[\frac{p^2}{\mu^2};1\right]. \tag{2.19}$$

It suffices therefore to solve the equations for $\mu=1$ and henceforth we set $\mu=1$.

This feature of scale invariance means, of course, that there can be no critical coupling parameter in this model because there are no dimensionless parameters and once a chiral-symmetry-breaking solution exists for one value of e^2 it exists for all values of e^2.

III. NUMERICAL SOLUTIONS OF THE SDE

A. Ultraviolet asymptotic behavior of the fermion propagator

Before proceeding with a numerical solution of (2.13) and (2.14) and (2.16) and (2.17) it is useful to obtain analytically some information about the ultraviolet asymptotic behavior of the solutions. For this analytic study we consider the simplest SDE possible which is obtained from (2.5) with the bare vertex of (1.11). (The effect of vertex dressing will become apparent in our numerical calculations.) In this case the integral equation for $B(p^2)$ is

$$B(p^2)=\frac{2+\xi}{4\pi^2 p}\int_0^\infty dq \ln\left|\frac{p+q}{p-q}\right|\frac{qB(q^2)}{q^2A^2(q^2)+B^2(q^2)} \tag{3.1}$$

and we neglect the equation for $A(p^2)$ because $A(p^2)\equiv 1$ in Landau gauge and for $\xi\equiv 0$ $A(p^2)\approx 1$ for $p^2\sim\infty$.

It is easy to compare the solutions of (3.1) with the result of an operator product expansion (OPE) in QED_3. Following Ref. [26] we find that when propagating in the presence of a condensate $\langle\bar\psi\psi\rangle\neq 0$ the fermion propagator will receive a self-mass contribution of the form

$$-2\pi^3\langle\bar\psi\psi\rangle\delta^3(q) . \tag{3.2}$$

The factor $\delta^3(q)$ is present because, by definition, the condensate does not exchange momentum with the fermion and the numerical factors are simply to ensure appropriate normalization.

Including this term as a perturbative contribution to the fermion propagator we find

$$S(q)=\frac{1}{i\gamma\cdot q}-\frac{2+\xi}{4}\frac{\langle\bar\psi\psi\rangle}{q^4}+\cdots . \tag{3.3}$$

This OPE analysis then predicts that, as $p^2\to\infty$,

$$\frac{4}{2+\xi}p^2\Sigma(p)\to-\langle\bar\psi\psi\rangle . \tag{3.4}$$

The asymptotic behavior of the solution can also be analyzed by using (3.1) to derive an approximate differential equation [19]. A differential equation (DE) valid for large p^2 is obtained by approximating the kernel as follows:

$$\ln\left|\frac{p+q}{p-q}\right|\approx\frac{2q}{p}\theta(p-q)+\frac{2p}{q}\theta(q-p) \tag{3.5}$$

which is a good approximation [19] for $p^2\ll q^2$ or $p^2\gg q^2$. This allows us to derive the following DE:

$$\frac{d}{dp}\left[p^3\frac{d}{dp}B(p)\right]+\frac{2+\xi}{\pi^2}B(p)=0 . \tag{3.6}$$

It should be possible to obtain a DE valid over a greater range of p^2 by using the method developed in Ref. [27]; however, for our purposes (3.6) is sufficient. To obtain more information about the solution we simply solve the integral equation directly.

The solution of this equation that is consistent with the ultraviolet boundary condition $B(p^2)\to 0$ as $p^2\to\infty$ is

$$B(p)=\kappa\frac{1}{p}J_2\left[\left[\frac{4(2+\xi)}{\pi^2 p}\right]^{1/2}\right]$$

$$\approx\kappa\frac{2+\xi}{2\pi^2}\frac{1}{p^2} ,\quad p^2\sim\infty , \tag{3.7}$$

where $J_2(x)$ is a Bessel function of integer order and κ is a constant that cannot be determined by the DE. It can, however, be determined by comparing (3.7) with (3.4) from which we see that

$$\kappa=\frac{\pi^2}{2}\langle\bar\psi\psi\rangle . \tag{3.8}$$

To test these predictions we solved (2.5) numerically [28] with the *bare* vertex specified in (1.11) and our results are presented in Table I. We use the standard definition of the fermion condensate $\langle\bar\psi\psi\rangle=-\mathrm{tr}S_F(x=0)$ which corresponds to

$$\langle\bar\psi\psi\rangle=-\frac{2}{\pi^2}\int_0^\infty dp\frac{p^2B(p^2)}{p^2A^2(p^2)+B^2(p^2)} . \tag{3.9}$$

[In a 4×4 representation of the Dirac matrices $\mathrm{tr}(I)=4$.] It will be seen that the analytic predictions are in excellent agreement with our numerical results.

B. Numerical solutions with light-cone singular vertex

We now proceed with a study of the complete integral equations (2.13) and (2.14) and (2.16) and (2.17). Of interest here is whether the asymptotic behavior of the fermion propagator predicted by the OPE is preserved when the *dressed* vertices are used and also whether the gauge dependence of $\langle\bar\psi\psi\rangle$ can be reduced or eliminated when the transverse parts of the vertex are included. It is clear that any study of the SDE for QED_3 and, in fact, other field theories, that makes pronouncements about chiral-symmetry breaking based upon the observation $\langle\bar\psi\psi\rangle\equiv 0$ is suspect unless the gauge variance of $\langle\bar\psi\psi\rangle$ can be eliminated. The fermion condensate should be a gauge-invariant quantity

TABLE I. Comparison of the asymptotic form of the fermion self-mass with $\langle\bar\psi\psi\rangle$ to test the OPE prediction.

ξ	p (units of e^2)	$\frac{4}{2+\xi}p^2B(p^2)$ (units of 10^{-3} e^4)	$-\langle\bar\psi\psi\rangle$ (units of 10^{-3} e^4)
−0.2	272	2.638	2.638
	511	2.638	
	1000	2.638	
0.0	272	2.316	2.316
	511	2.316	
	1000	2.316	
0.5	272	1.775	1.775
	511	1.775	
	1000	1.775	
1.0	272	1.447	1.447
	511	1.447	
	1000	1.447	
1.2	272	1.352	1.352
	511	1.352	
	1000	1.352	

$$\langle \bar{\psi}\psi \rangle(\xi) = \text{const} \tag{3.10}$$

and a first test of model calculations of $\langle \bar{\psi}\psi \rangle$ should be that they at least preserve this feature. Otherwise the value of $\langle \bar{\psi}\psi \rangle$ obtained is irrelevant since by varying the gauge parameter any value is possible, including $\langle \bar{\psi}\psi \rangle = 0$.

Our first significant numerical study [30] concentrates on the SDE with the LCS vertex, (2.13) and (2.14). We find that the functions $A(p^2)$ and $B(p^2)$ are gauge-parameter dependent as one would naively expect given the explicit ξ dependence of the integral equations. In Fig. 1 we plot $\langle \bar{\psi}\psi \rangle$ as a function of ξ for a range of values of a. From this figure we see that the $\xi=0$ value of $\langle \bar{\psi}\psi \rangle$ obtained with the dressed LCS vertex is the same as the Landau gauge value of $\langle \bar{\psi}\psi \rangle$ in Table I which is obtained with the bare vertex. This can be understood once it is realized that in Landau gauge

$$\Gamma^S_\mu(p,q) D_{\mu\nu}(p-q;\xi=0) = [aA(p^2) + (1-a)A(q^2)] \times \gamma_\mu D_{\mu\nu}(p-q;\xi=0) \; . \tag{3.11}$$

This coupled with (2.15) entails that in Landau gauge the *effective* LCS vertex is exactly the same as the bare vertex. From this it also follows that the Landau gauge equations obtained with the LCS vertex are independent of a.

An important observation that is presented by Fig. 1 is that for $\xi \neq 0$ $\langle \bar{\psi}\psi \rangle$ is quite sensitive to the value ξ. This is generally true, however, for

$$a \approx 3.0 \; ; \tag{3.12}$$

the sensitivity of $\langle \bar{\psi}\psi \rangle$ to the gauge parameter is minimized, at least in the range $0 \le \xi \le 1$. In this case it appears that implicit gauge variance of the solution functions, which appear also in the vertex function, almost completely compensates for the explicit gauge dependence of the photon propagator. Our calculation employs an extremely simple-minded vertex modification; however, this feature is an important one that may survive even if one were to use the exact vertex function obtained as the solution of the vertex equation [15] in the SDE. The calculation demonstrates that the transverse parts of the vertex are extremely important in restoring gauge independence to $\langle \bar{\psi}\psi \rangle$.

C. Numerical solutions with light-cone regular vertex

We have also obtained numerical solutions of (2.16) and (2.17) which contain the LCR vertex [30] and our results are summarized in Figs. 2 and 3. In this case it is not possible to recover the bare vertex at all; a point we illustrate with the plot of $A(p^2)$ in Fig. 2 obtained with $\xi=0$ and a range of values of a. The effect of the transverse part of the vertex is clear for we see in Fig. 2 that, using (2.10),

$$A(p^2) \neq 1 \quad \text{even for } \xi = 0 \; , \tag{3.13}$$

a significant difference from (2.15), the solution obtained with (2.9).

The plot of $\langle \bar{\psi}\psi \rangle$ against ξ in Fig. 3 is again significant. We see in this figure that on the domain $0 \le \xi \le 1$ there is a value of

$$a \approx 0.53 \tag{3.14}$$

for which the explicit gauge dependence of the photon propagator is compensated by the implicit gauge dependence of the solution functions. This feature, which we also encountered in our studies of the LCS vertex, is a nontrivial result and the observations we made in Sec. III B are also appropriate here. We remark that our

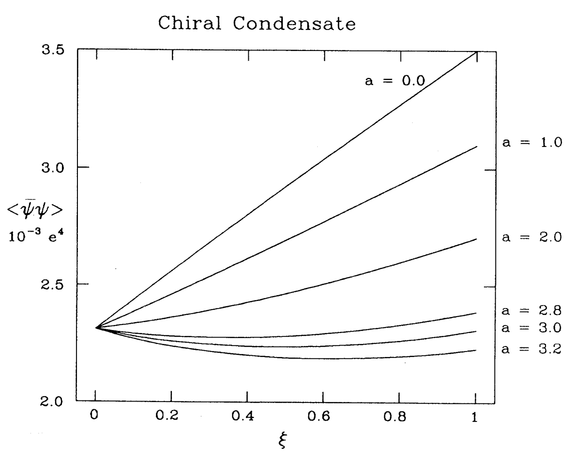

FIG. 1. Plot of $\langle \bar{\psi}\psi \rangle$, obtained from the solutions of the SDE using the light-cone singular vertex, as a function of the gauge parameter ξ for various values of a, the parameter that varies the transverse part of the vertex. A value of $a=3.0$ minimizes the sensitivity of $\langle \bar{\psi}\psi \rangle$ to ξ.

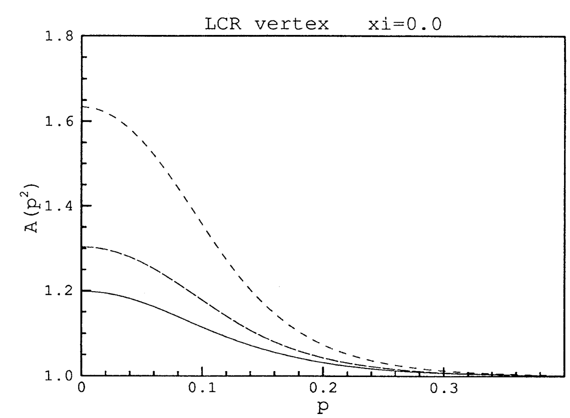

FIG. 2. Plot of $A(p^2)$ for various values of a obtained in Landau gauge with the light-cone regular vertex: $a=0.0$, solid curve; $a=0.5$, long dashed curve; $a=1.0$, short dashed curve. One observes that in contrast to the light-cone singular case $A(p^2 \sim 0)$ is nonzero even in Landau gauge when the light-cone regular vertex is used. A plot with the same qualitative features is obtained for $B(p^2)$. The sensitivity of the solution functions to ξ is what ensures that at $a=0.53$ $\langle \bar{\psi}\psi \rangle$ is $\approx$independent ξ.

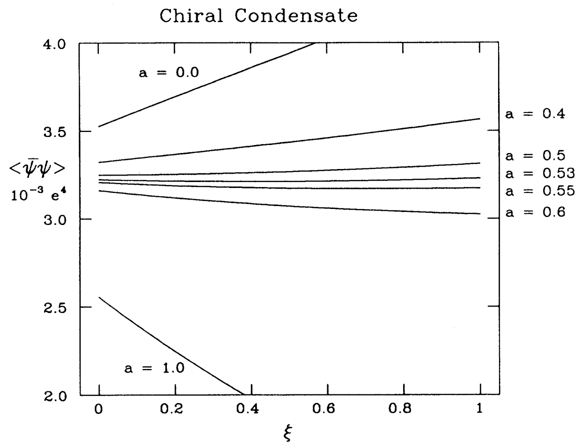

FIG. 3. Plot of $\langle \bar\psi\psi \rangle$, obtained from the solutions of the SDE using the light-cone regular vertex, as a function of the gauge parameter ξ for various values of a, the parameter that varies the transverse part of the vertex. A value of $a=0.53$ minimizes the sensitivity of $\langle \bar\psi\psi \rangle$ to ξ.

photon-fermion vertex is symmetric $[\Gamma_\mu(p,q)=\Gamma_\mu(q,p)]$ for $a=0.5$ and the fact that herein the gauge sensitivity of $\langle \bar\psi\psi \rangle$ is minimized

$$\frac{d}{d\xi}\langle \bar\psi\psi \rangle(\xi)\approx 0 , \tag{3.15}$$

for $a\approx 0.53$, is suggestive that, with a more sophisticated photon-fermion vertex ansatz, the symmetric vertex may in fact minimize the gauge dependence of $\langle \bar\psi\psi \rangle$. An important observation can be made by comparing Figs. 1 and 3: we see that by optimally choosing a the gauge parameter dependence of $\langle \bar\psi\psi \rangle$ can be reduced more when the LCR vertex is used in the SDE than when the LCS vertex is used. This is another positive feature of the LCR vertex.

It has been suggested [31] in the context of QCD that the asymptotic behavior of the fermion self-mass provides a stringent constraint on the structure of the gauge-boson–fermion vertex. With this in mind we studied the asymptotic behavior of $B(p^2)$. Our results are presented in Table II. We observe that the inclusion of the light-cone regular vertex does not alter the fact that

$$B(p^2\to\infty)=\frac{\lambda}{p^2} \tag{3.16}$$

where λ is a constant, which is consistent with one of the constraints of Ref. [31]. However, the simple relationship between λ and $\langle \bar\psi\psi \rangle$ in (3.4) is not preserved, being violated most strongly in gauges other than Landau gauge. This is not surprising: the complex interplay between the elements of the vertex and the propagator must obviously lead to a complex dependence of λ on ξ otherwise it would not be possible for $\langle \bar\psi\psi \rangle$ to be approximately gauge independent for $a\approx 0.53$. The fact that $\lambda=\lambda(\xi)$ was anticipated in Ref. [31]. The reason why (3.4) is almost satisfied in Landau gauge is easy to see from (2.16) and (2.17): with this gauge choice the SDE

TABLE II. (a) Comparison of the asymptotic form of the fermion self-mass obtained from a numerical solution of the SDE using the light-cone regular vertex ($a=0.0$) with $\langle \bar\psi\psi \rangle$. (b) Comparison of the asymptotic form of the fermion self-mass obtained from a numerical solution of the SDE using the light-cone regular vertex ($a=0.5$) with $\langle \bar\psi\psi \rangle$.

ξ	p (units of e^2)	$\frac{4}{2+\xi}p^2B(p^2)$ (units of 10^{-3} e^4)	$-\langle \bar\psi\psi \rangle$ (units of 10^{-3} e^4)
		(a)	
0.0	234	3.662	3.526
	548	3.660	
	1000	3.658	
0.5	234	4.495	3.941
	548	4.492	
	1000	4.490	
1.0	234	5.189	4.345
	548	5.185	
	1000	5.182	
		(b)	
0.0	234	3.342	3.248
	548	3.340	
	1000	3.339	
0.5	234	3.563	3.265
	548	3.561	
	1000	3.560	
1.0	234	3.728	3.312
	548	3.725	
	1000	3.724	

most closely resembles the *bare* equation in (3.1). We do not present tabulated results but the above statements are also true in connection with the LCS vertex. Clearly the transverse part of the vertex function also influences the ultraviolet behavior of the solution functions.

D. Comparison with lattice simulations

It is of some interest to compare our result for $\langle \bar\psi\psi \rangle$ with that obtained in lattice simulations of QED_3 [32–35]. In the lattice formulation

$$\langle \bar\psi\psi \rangle^{\rm lattice}\equiv\frac{\langle {\rm tr}M^{-1}(U)\rangle}{V} \tag{3.17}$$

where the angular brackets represent a Boltzmann weighted average over the gauge field U, $M^{-1}(U)$ is the inverse of the lattice Dirac operator in a given gauge-field configuration, and V is the number of lattice sites. Employing the formalism developed in Ref. [36] one obtains the following relationship between the lattice and continuum condensates:

$$-\langle \bar\psi\psi \rangle^{\rm continuum}=\frac{N_f}{2a^2}\langle \bar\psi\psi \rangle^{\rm lattice} \tag{3.18}$$

with N_f the number of fermion flavors which is one in the case we are considering here.

For a proper comparison we must look to the lattice results obtained in a scaling window where the results are thought to best represent continuum physics. One signal that a scaling window exists in the lattice theory is the observation that [33]

$$\beta^2\langle\bar{\psi}\psi\rangle^{\text{lattice}}=\mathcal{K}\ (\text{a constant}) \tag{3.19}$$

for large $\beta=1/(e^2a)$. In the scaling window it should be that

$$-\frac{1}{e^4}\langle\bar{\psi}\psi\rangle^{\text{continuum}}=\frac{N_f}{2}\mathcal{K}\ . \tag{3.20}$$

The simulations of Ref. [33] suggest that a scaling window exists for $\beta>1$.

We have neglected the fermion loop contribution to the photon polarization tensor and hence our calculations have been performed in what is described as the quenched approximation in the lattice formulation. This corresponds to the $N_f=0$ simulations of Ref. [33]. There exists now the possibility for confusion because of the N_f factor in (3.20). The quenched approximation really means that the factor $(\det M)^{1/2}$, present in $N_f=1$ simulations, is not included in the $N_f=0$ simulations. In this case the steps that led to (3.18) can be retraced and we find that we must compare

$$-\frac{1}{e^4}\langle\bar{\psi}\psi\rangle^{\text{continuum}}\ \text{to}\ \tfrac{1}{2}\mathcal{K}^{N_f=0}\ . \tag{3.21}$$

The simulations in Ref. [33] on an 8^3 lattice yield

$$\tfrac{1}{2}\mathcal{K}^{N_f=0}=0.006\pm0.001 \tag{3.22}$$

which is obtained by evaluating the lattice chiral condensate at values of $ma=0.050$, 0.025 and using linear extrapolation to estimate the value at $ma=0$. This value can be compared to our calculations which typically yield (for example: LCR vertex, $a\approx0.5$)

$$-\frac{1}{e^4}\langle\bar{\psi}\psi\rangle^{\text{continuum}}\approx0.003\ . \tag{3.23}$$

Given the fact that in the quenched approximation finite-size effects on the lattice are expected to be significant (because of the logarithmic behavior of the photon propagator) and that the linear extrapolation procedure employed in the lattice estimation may introduce a small error in the actual value of the condensate (the slopes of $\langle\bar{\psi}\psi\rangle$ versus m are considerable [33]) this is a good level of agreement. It must also be remembered that there remains some gauge ambiguity in our continuum result.

IV. CONCLUSIONS

We have analyzed a model SDE motivated by QED_3. We do not make the presumption of referring to the complex of equations (2.5) through to (2.10) as providing an approximate SDE for QED_3 because in the equations: (a) The vertex renormalization constant Z_1 is neglected. This aspect of our work is common to all analyses of the SDE and it is a shortcoming of the SDE continuum approach that is in need of refinement. (b) The free photon propagator (1.12) is used and in true QED_3 the fermion loop vacuum-polarization contribution is very important since it destroys the confining character of QED_3. Our model equations provide, however, an interesting context within which to explore the general features of a SDE whose kernel has a singularity characteristic of confinement and whose solutions depend on a gauge parameter.

We have made a significant improvement over previous studies of the model SDE by employing an ansatz for the dressed photon-fermion vertex that not only satisfies the Ward identity but is also consistent with calculations of the vertex in perturbation theory. Our vertex ansatz included a parameter that enabled us to study, albeit in a simple-minded fashion, the dependence of the solution on the transverse part of the vertex which is *not* constrained by the Ward identity. We found that the transverse part modifies both the infrared and ultraviolet behavior of the fermion propagator without destroying the asymptotic

$$\frac{1}{p^2}$$

dependence expected of the fermion mass function in QED_3.

We also demonstrated the intimate connection between gauge invariance and the transverse part of the vertex illustrating that the transverse part is necessary to ensure that the fermion condensate is independent of the gauge-fixing parameter. In this connection it is obvious that in modeling the photon-fermion vertex an important part of the field theory has been neglected in the past. Unless the transverse part can be constrained so as to preserve the gauge covariance of the model SDE the condensate obtained is essentially meaningless since by changing the gauge parameter any value of $\langle\bar{\psi}\psi\rangle$ can be obtained. This is one reason why we believe some effort needs to be directed toward understanding the integral equation for the vertex itself.

It is useful to summarize the understanding that our calculations provide. It is clear *a priori* that the equation for the vertex function depends explicitly on the gauge parameter ξ; hence, one would naively expect Γ to depend explicitly on ξ. We have constructed a vertex that has no explicit dependence on ξ, depending instead implicitly on ξ through the functions that determine the fermion propagator. This is sufficient to produce a fermion condensate that is approximately independent of the gauge parameter, at least on the restricted domain $0\le\xi\le1$. (This property is lost when we leave this domain but since ours is an extremely simple vertex ansatz we are not concerned by this.) The transverse part of the vertex is arbitrary if the Ward identity is all that is used to constrain Γ. However, we have shown that it is plausible to suppose that a simple choice for this part of the vertex would ensure that the SDE leads to physical quantities which are independent of the gauge parameter.

This is an interesting and useful possibility which may or may not be particular to the covariant gauge fixing procedure.

Our main conclusion is that the study of field theories using simply the SDE with the bare photon-fermion vertex, or minor variants thereof, is inadequate. Any serious attempt at a continuum solution of the field theory must begin to directly address the photon-fermion vertex and the equation of which it is a solution. We believe that true progress lies in this direction.

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$$\int_{p-\epsilon}^{p+\epsilon} dq\, f(q) \ln|p-q| \approx 2f(p)\epsilon(\ln\epsilon - 1) \, .$$

The solutions obtained for $A(p^2)$ and $B(p^2)$ were insensitive within the required tolerance to values of ϵ in the range 0.05–0.001 times the grid spacing. To check these results we also used the International Mathematical and Scientific Library and Numerical Algorithms Group library quadrature routines Q2AGS and D01AHF, respectively, to evaluate the integrals. Our results were independent of the procedure within the 0.1% tolerance we allowed.

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p^2/q^2 as p goes to zero. Therefore when p is small almost all the contribution to the integral comes from a very narrow spike at $q=p$. The initially small roundoff errors obtained in integrating over this spike gradually grow to dominate the iterative substitution procedure and large fluctuations develop in $A(p^2)$. To overcome this we smoothed $A(p^2)$ using a least-squares fit to a quadratic function in p^2

$$A(p^2)=a_0+a_1p^2+a_2(p^2)^2$$

over the first few grid points ($p<p_m\sim 0.1$) before each resubstitution. The effect of smoothing was to allow us to find a stable solution within the 0.1% tolerance we allowed. This solution was independent of the upper bound on the smoothing domain (p_m) within a given finite range, typically $0.1\leq p_m\leq 0.3$. We again checked our results using the library routines mentioned above.

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