Nonperturbative fermion propagator in massless quenched QED

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The Schwinger-Dyson equation for the massless fermion propagator in quenched four-dimensional QED is solved for different forms of the fermion-photon vertex. A nonperturbative, analytic solution that is multiplicatively renormalizable and self-consistently related to the vertex by the Ward identity is obtained. Its behavior is quantitatively different from that found in the simple ladder approximation, for instance.

I. INTRODUCTION

Even a well-known gauge theory, such as QED, is really only understood in the perturbative regime. Indeed, it may have a quite surprising nonperturbative behavior if the coupling is much larger than $\frac{1}{137}$ [1,2]. To learn whether such results are real consequences of such a four-dimensional field theory requires a systematic investigation of the approximations needed to make such nonperturbative studies tractable. For lattice computations this means a careful study of finite-size effects and of fermion mass extrapolations as well as the more obvious dependence on the number of lattice sites. In the continuum calculations we consider here, this involves different questions: in particular, the effect of the truncations and renormalizability of the Schwinger-Dyson equations.

Here we study the massless fermion propagator in four-dimensional quenched QED (QED_4). This is, in principle, one of the simplest problems one can look at since the photon propagator is assumed bare and consequently the coupling e^2 is unrenormalized. Nevertheless, the full fermion propagator is nontrivial. The Schwinger-Dyson equation for the fermion propagator $S_F(p)$, illustrated in Fig. 1, is given by

$$iS_F^{-1}(p)=iS_F^{0\,-1}(p)-\frac{e^2}{(2\pi)^4}\int d^4k\;\gamma^\mu S_F(k)\Gamma^\nu(k,p)\Delta_{\mu\nu}(q)\;, \tag{1}$$

where $S_F^0(p)$ is the bare fermion propagator, $\Gamma^\nu(k,p)$ the full fermion-boson vertex and, with $q=k-p$, the photon propagator is

$$\Delta^{\mu\nu}(q)=\frac{1}{q^2}\left[g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right]+\xi\frac{q^\mu q^\nu}{q^4}\;,$$

ξ being the covariant gauge parameter. We render the loop integral in Eq. (1) finite by introducing an ultraviolet cutoff Λ.

In general, Eq. (1) involves two fermion functions: its mass and wave function, which can, in principle, be determined self-consistently provided the 3-point vertex Γ^ν is known. Of course, the full vertex satisfies its own Schwinger-Dyson equation which relates it in turn to a 4-point function and we have the start of an infinite hierarchy of nested equations. The truncation of this system at the level of the fermion propagator requires an *Ansatz* for the vertex in terms of the known functions of the fermion propagator.

If such *Ansätze* for Γ^ν involve only odd numbers of γ matrices, then Eq. (1) can have massless solutions and it is these we study for simplicity. Then with

$$S_F(p)=\frac{\mathcal{F}(p^2)}{\not p}\;, \tag{2}$$

where the bare propagator has $\mathcal{F}(p^2)=1$, Eq. (1) is an integral equation for the fermion function $\mathcal{F}(p^2)$ which depends on the *Ansatz* assumed for the vertex. We now proceed to consider different forms for Γ^ν in Sec. II and discuss our results in Sec. III.

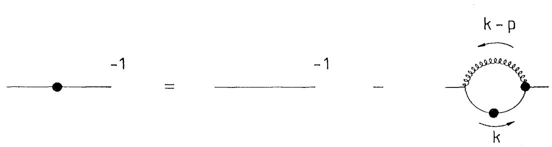

FIG. 1. Schwinger-Dyson equation for the fermion propagator in quenched QED. The straight lines represent fermions and the wavy lines the photon. The solid dots indicate full, as opposed to bare, quantities.

II. THE FERMION-BOSON VERTEX

Let us first consider the most commonly used approximation $\Gamma^\nu=\gamma^\nu$ [2,3]. Equation (1) is then

$$\frac{1}{\mathcal{F}(x)}=1+\frac{\alpha_0\xi}{4\pi x}\int_0^1 dy\;\mathcal{F}(y)\left[\theta(x-y)\frac{y}{x}+\theta(y-x)\frac{x}{y}\right] \tag{3}$$

where $x=p^2/\Lambda^2$, $y=k^2/\Lambda^2$, and $\alpha_0=e^2/4\pi$. Self-consistent solutions of this equation have been found numerically by an iterative procedure. The results are

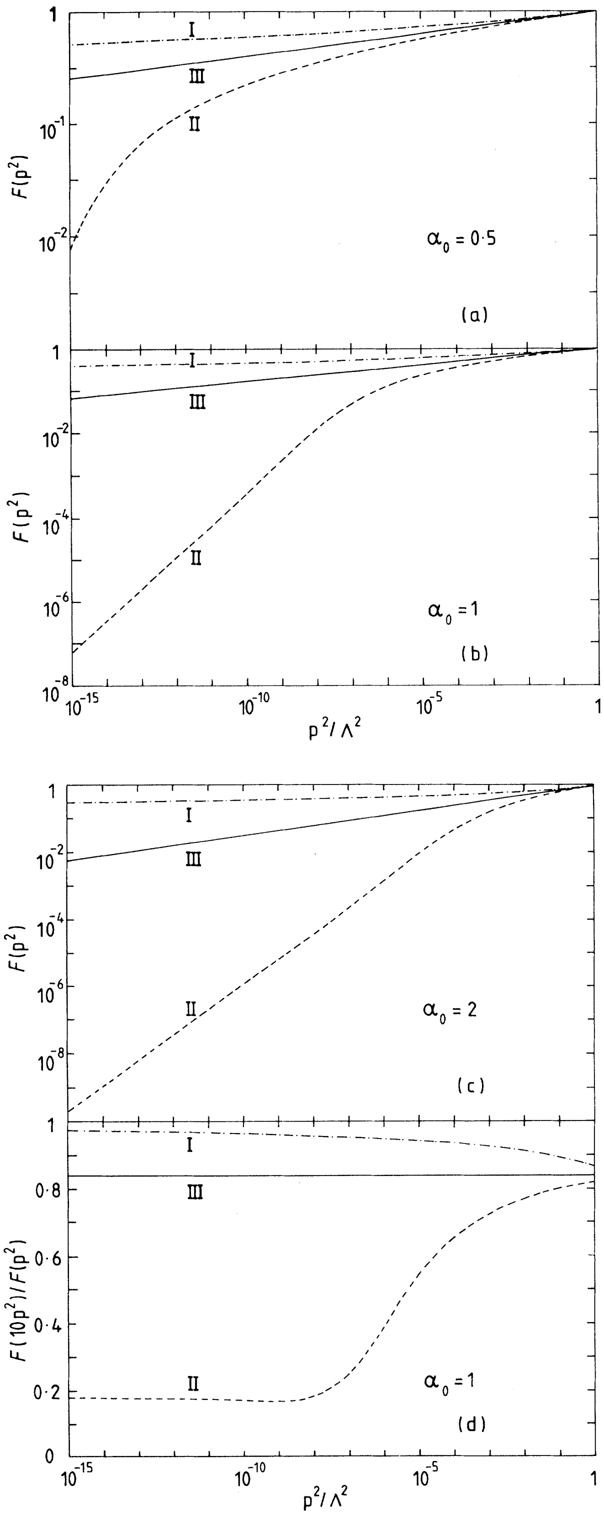

FIG. 2. The solutions of the Schwinger-Dyson equation for the fermion function $\mathcal{F}$ as a function of p^2/Λ^2 for the three vertex approximations. Curve I is for the bare vertex, curve II for the Ball-Chiu vertex, and curve III for the full vertex for three different couplings α_0 in the Feynman gauge ($\xi=1$). (a)–(c) have $\alpha_0=0.5$, 1, and 2, respectively. (d) shows the ratio $\mathcal{F}(p'^2)/\mathcal{F}(p^2)$ with $p'^2=10p^2$ for the case $\alpha_0=1$.

shown in curves I in Figs. 2(a)–2(c) as functions of momentum for different couplings in the Feynman gauge, $\xi=1$ as an illustration. In the Landau gauge $\xi=0$, as is well known, the loop integral vanishes [Eq. (3)] and so $\mathcal{F}(p^2)=1$ at all momenta. In all other gauges for momenta close to the ultraviolet cutoff Λ^2, the form of $\mathcal{F}(p^2)$ inevitably agrees with the lowest order in perturbation theory. However, it is quite different as the ratio p^2/Λ^2 decreases.

Though we have found nonperturbative solutions for $\mathcal{F}(p^2)$, these violate two fundamental principles. The first is a basic property of gauge theories: namely, the Ward-Takahashi identity. This requires that the vertex and fermion propagator are related by

$$q^\mu\Gamma_\mu(k,p)=S_F^{-1}(k)-S_F^{-1}(p)$$

$$=\frac{\not k}{\mathcal{F}(k^2)}-\frac{\not p}{\mathcal{F}(p^2)} . \tag{4}$$

The second is multiplicative renormalizability that requires that there exists a factor $Z_2^{-1}(\mu^2/\Lambda^2)$ that makes $\mathcal{F}(k^2/\Lambda^2)$ independent of Λ^2, to give the renormalized fermion function

$$\mathcal{F}_R(p^2/\mu^2)=Z_2^{-1}(\Lambda^2/\mu^2)\mathcal{F}(p^2/\Lambda^2) , \tag{5}$$

where μ^2 is the renormalization scale. This implies a "renormalization-group flow" requiring

$$\frac{\mathcal{F}(p'^2/\Lambda^2)}{\mathcal{F}(p^2/\Lambda^2)}=r(p'^2/p^2) . \tag{6}$$

On both counts the vertex $\Gamma^\mu=\gamma^\mu$ violates these conditions, except trivially in the Landau gauge when we recall $\mathcal{F}(p^2)=1$. For instance, the ratio of Eq. (6) with $p'^2=10p^2$ is shown in Fig. 2(d) as the curve I for $\alpha_0=1$. It is not independent of p^2. Clearly, such a simple vertex is *unacceptable.*

To go beyond this we note that the Ward identity Eq. (4) provides us with a powerful nonperturbative constraint on the full vertex. This specifies part of the vertex called the longitudinal component Γ_L^μ, while leaving the transverse part, defined by $q^\mu\Gamma_T^\mu=0$, unconstrained. The requirement of no kinematic singularities uniquely fixes Γ_L^μ to the Ball-Chiu form [4]:

$$\Gamma_L^\mu(k,p)=\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}+\frac{1}{\mathcal{F}(p^2)}\right]\gamma^\mu$$

$$+\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]\frac{(k+p)^\mu(\not k+\not p)}{k^2-p^2} \tag{7}$$

with $\Gamma_T^\mu(p,p)=0$. Simply setting $\Gamma_T^\mu=0$, so $\Gamma^\mu=\Gamma_L^\mu$, Eq. (1) becomes

$$\frac{1}{\mathcal{F}(x)}=1+\frac{\alpha_0}{4\pi x}\int_0^1 dy\left[\theta(x-y)\xi\frac{y}{x}+\theta(y-x)\xi\frac{x}{y}\frac{\mathcal{F}(y)}{\mathcal{F}(x)}+\frac{3}{4}\left[\frac{\mathcal{F}(y)}{\mathcal{F}(x)}-1\right]\frac{x+y}{x-y}\left[\theta(x-y)\frac{y}{x}+\theta(y-x)\frac{y}{x}\right]\right] . \quad (8)$$

The solutions of this equation for a range of couplings are shown as curves II in Figs. 2(a)–2(c), once again for the Feynman gauge, $\xi=1$. Though now the vertex satisfies the Ward identity, the function $\mathcal{F}(p^2/\Lambda^2)$ is not yet multiplicative renormalizable. In Fig. 2(d) is shown the ratio of Eq. (6) again as curve II. At low momenta this ratio is seen to be flat as required by multiplicative renormalizability. This happens because the solution is power behaved there. However, this is not matched by the behavior at larger momenta where the solution is again perturbative.

It has recently been shown that the transverse part of the vertex is crucial for multiplicative renormalizability [5,6]. In Ref. [6] it was proposed that this should have a simple form:

$$\Gamma_T^\mu(k,p)=\frac{1}{2}\left[\frac{1}{\mathcal{F}(k^2)}-\frac{1}{\mathcal{F}(p^2)}\right]C(k,p)[\gamma^\mu(k^2-p^2)-(k+p)^\mu(\not k-\not p)] \quad (9)$$

where $C(k,p)=(k^2+p^2)/(k^2-p^2)^2$ is the massless limit ensuring multiplicative renormalizability in the perturbative region for both leading and next-to-leading logarithms. Treating this Γ_T^μ as a non-perturbative *Ansatz* and adding this to Γ_L^μ of Eq. (7) to give the full vertex in Eq. (1) yields

$$\frac{1}{\mathcal{F}(x)}=1+\frac{\alpha_0}{4\pi x}\int_0^1 dy\left[\theta(x-y)\xi\frac{y}{x}+\theta(y-x)\xi\frac{x}{y}\frac{\mathcal{F}(y)}{\mathcal{F}(x)}\right.$$
$$\left.+\frac{3}{4}\left[\frac{\mathcal{F}(y)}{\mathcal{F}(x)}-1\right]\frac{x+y}{x-y}\left[1-c\frac{(x-y)^2}{x+y}\right]\left[\theta(x-y)\frac{y}{x}+\theta(y-x)\frac{y}{x}\right]\right] . \quad (10)$$

With $c\equiv C\Lambda^2=(x+y)/(x-y)^2$ of Eq. (9), the second set of terms in Eq. (11) cancels and this equation greatly simplifies to just

$$\frac{1}{\mathcal{F}(p^2)}=1+\frac{\alpha_0\xi}{4\pi}\left[\frac{1}{2}+\int_{p^2}^{\Lambda^2}\frac{dk^2}{k^2}\frac{\mathcal{F}(k^2)}{\mathcal{F}(p^2)}\right] . \quad (11)$$

The solutions of this equation are shown in Figs. 2(a)–2(c) as the solid lines III for a range of couplings with $\xi=1$. Remarkably, the result is seen to be multiplicatively renormalizable for *all* momenta, not just in the perturbative region, as is illustrated in Fig. 2(d). Though the results for $\mathcal{F}(p^2/\Lambda^2)$ for the three forms of the vertex we consider are qualitatively similar, all vanishing as $p^2\to 0$, their quantitative difference is quite dramatic, as would be well illustrated on a plot with linear scale. In particular, the final form III falls to zero more quickly at small momenta than the ladder approximation result.

The fact that vertex III gives an exactly multiplicatively renormalizable fermion function means that Eq. (11) must be soluble analytically. This is readily checked to be

$$\mathcal{F}(p^2/\Lambda^2)=\frac{1}{1+\alpha_0\xi/8\pi}\left[\frac{p^2}{\Lambda^2}\right]^\gamma , \quad (12)$$

where

$$\gamma=\frac{\alpha_0\xi/4\pi}{1+\alpha_0\xi/8\pi}$$

for the unrenormalized fermion function. Simply replacing Λ^2 by μ^2 gives the renormalized form as multiplicative renormalizability requires:

$$\mathcal{F}_R(p^2/\mu^2)=\mathcal{F}_R(1)\left[\frac{p^2}{\mu^2}\right]^\gamma , \quad (13)$$

where $\mathcal{F}_R(1)$ is fixed by the renormalization prescription. Of course, for $\alpha_0\ll 1$ our solution reproduces the well-known renormalization-group-improved perturbative result with $\mathcal{F}_R(1)=1$:

$$\mathcal{F}_R(p^2/\mu^2)=\left[\frac{p^2}{\mu^2}\right]^{\alpha_0\xi/4\pi} .$$

Equation (13) is however a genuinely nonperturbative answer valid at all momenta.

III. DISCUSSION

This calculation focuses sharply on the important role the form of the vertex plays in determining the fermion propagator beyond the perturbative regime. Remarkably, we have found a self-consistent, analytic solution to the coupled problem of the fermion propagator and its interaction in massless quenched QED_4. The results are nonperturbative, the vertex and propagator exactly related by the Ward identity, and the fermion function is multiplicatively renormalizable. This solution is quantitatively different from those found that violate these important constraints. Though this result is for a massless fermion, the success of this approach bodes well for future attempts to find a nonperturbative solution for the complete fermion propagator in four-dimensional gauge theories.

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