Study of sphaleron transitions by means of the real-time Langevin equation

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Microscopic Langevin evolution is introduced in Abelian gauge field theory. It is studied numerically in the $(1+1)$-dimensional Abelian Higgs model at finite temperature. The Brownian behavior of the Chern-Simons number is established. Sharp transitions between the classical vacua with different Chern-Simons numbers are observed. The rate of these transitions depends exponentially on the temperature, indicating the relevance of the thermal activation theory involving the sphaleron saddle point.

INTRODUCTION

Anomalous fermion-number nonconservation in theories with a nontrivial vacuum structure is receiving much attention nowadays.[1] The most famous example is the standard electroweak theory where baryonic and leptonic currents are not conserved due to the triangle anomaly which relates the evolution of the fermionic charge to the fluctuations of the gauge fields with a change in the Chern-Simons number.[2] Anomalous fermionic-number nonconservation, while being severely suppressed in the vacuum,[2] becomes appreciable in the high-temperature plasma.[3,4] This effect is based essentially on the fact that classical vacua with different values of the Chern-Simons number $N_{\rm CS}$ are separated by a static energy barrier of nonzero minimal height $E_{\rm st}$.[5] Therefore at temperatures less than $E_{\rm st}$ the thermal activation theory becomes relevant. It predicts the exponential dependence of the nonconservation rate on the inverse temperature.

The static energy barrier separating two neighboring vacua cannot be represented by an effective potential, but rather by some functional. The saddle point of that functional is called a sphaleron.[5] The energy $E_{\rm sph}$ of this static solution of the Weinberg-Salam equations of motion is just the minimal height of the static energy barrier $E_{\rm st}=E_{\rm sph}$. At temperatures $T<E_{\rm sph}$ anomalous fermion number nonconservation is calculated semiclassically. It occurs due to real-time fluctuations of the bosonic fields, which go through the vicinity of the sphaleron. This naive physical picture suggests a way to evaluate the rate of nonconservation directly by looking at the classical evolution of the system.[6]

The considerations above are fully applicable to the $(1+1)$-dimensional Abelian Higgs model[6,7] whose lattice version was studied in Ref. 8. The strategy of Ref. 8 contained two steps. First prepare an initial field configuration corresponding to a given temperature T_0. Then evolve the starting configuration according to the classical equations of motion. This is an isolated system. Its energy is conserved and determines the temperature: $E=E(T_0)$. The average over this time evolution coincides with the statistical average over microcanonical distribution due to ergodicity. During the microcanonical evolution sphalerons were observed and counted "by hand." The rate of anomalous fermion number nonconservation was extracted from the average sphaleron density.

In this paper an alternative approach is developed. We study the system at a fixed temperature, that is, one determined by the canonical distribution. Since we are interested in the effects of the temperature on the semiclassical fluctuations, we basically deal with a Hamiltonian system interacting with a thermal bath. Such a system is known to be described by the Langevin equation. Our use of the Langevin equation is twofold. On one hand the Langevin evolution is a formal technical trick to measure canonical Gibbs averages of various static quantities by means of the ergodicity relation.[9] On the other hand we introduce Langevin evolution in terms of independent physical variables. It incorporates the principal effect of the interaction with a heat bath during the sphaleron formation in real time. This second point of view allows one to identify the Langevin evolution with the relevant real-time evolution. It means that we identify the rate of anomalous fermion number nonconservation with the diffusion rate of the Chern-Simons variable.

This sort of real-time Langevin approach was previously used by us for the evaluation of kink-antikink pair production at finite temperature.[10] The study of sphaleron transitions in the Abelian gauge theory that we consider is somewhat more involved. It needs formulation of the real-time Langevin equation in terms of the original microscopic variables that would respect gauge invariance.

In Sec. I we specify the model that we deal with, namely, the Abelian Higgs theory in $1+1$ dimensions. Then (Sec. II) we construct the Langevin equation for the Abelian gauge theory. The white noise is introduced in such a way as not to destroy conservation of the electric current, that is, for gauge-invariant variables only. Gauge-variant variables evolve microcanonically. At first sight the whole set of the equations of motion looks peculiar. Langevin evolution leads to a canonical distribution while the Hamiltonian evolution leads to a microcanonical one. What is the resulting distribution of our set of equations? The answer is simple: the final distribution is the canonical one $\sim\exp(-H/T)$, but with the constraint that physical states be gauge invariant.[11] The microcanonical part of the evolution guarantees that Gauss's law (in particular, vanishing of the total charge) is satisfied by the allowed fluctuations. We do not introduce a random force for the Chern-Simons variable itself. It may be done in principle but it is not important since it affects only one degree of freedom. Section III contains the results of the numerical solution of our set of equations. The ground state of the theory under consideration possesses a Θ-vacuum structure. On the classical level there are many degenerate vacuum field configurations with different integer values of the Chern-Simons variable. At finite temperature we observe explicitly transitions between the topologically distinct vacua. Moreover the direct measurement of the average displacement squared as a function of real time reveals the Brownian behavior of the Chern-Simons number. We study the low-temperature domain: $T < E_{\rm sph}$, where the semiclassical expansion is valid. The diffusion rate exhibits exponential dependence on the inverse temperature according to expectation. The behavior thus obtained proves the existence of a finite energy barrier between the different classical vacua with definite Chern-Simons numbers, associated with the sphaleron saddle point. In Sec. IV we explore the possibility of calculating the rate of the anomalous fermion-number nonconservation beyond the semiclassical approximation, in particular in the high-temperature domain. We present an analytical expression for the rate suitable for Monte Carlo measurements. We also study numerically the correlator of the canonical momentum conjugated to the Chern-Simons variable and reveal the problem in extracting the rate from the correlator of the operator $F_{\mu\nu}\tilde{F}_{\mu\nu}$ in the real electroweak theory.

I. THE MODEL

We study the (1+1)-dimensional Abelian Higgs model defined by the Lagrangian of the form

$$L = -\tfrac{1}{4}F_{\mu\nu}^2 + |D_\nu\varphi|^2 - V(\varphi) + i\bar{\psi}\gamma_\mu(\partial_\mu - i\gamma_5 g A_\mu)\psi \,, \qquad V(\varphi) = \lambda(|\varphi|^2 - v^2/2)^2 \,, \tag{1.1}$$

where φ, ψ, and A_μ are scalar, spinor, and vector fields, respectively, and $D_\nu = \partial_\nu + igA_\nu$. The spectrum consists of a massive vector particle ($m_W = gv$) and a Higgs boson of mass $m_H = \sqrt{2\lambda}\,v$. The gauge-invariant fermionic current is not conserved due to the anomaly

$$\partial_\nu I_\nu = \frac{g}{4\pi}\varepsilon_{\mu\nu}F_{\mu\nu} \,. \tag{1.2}$$

Equation (1.2) implies that any fluctuation of the fermionic charge is always accompanied by the proper fluctuation of the Chern-Simons number of the gauge field. Therefore only the bosonic part of Lagrangian (1.1) is essential. Hence it is considered below. In order to observe fermion number nonconservation a second ingredient is necessary: namely, the Θ-vacuum structure of the ground state. To see that it is indeed a property of this theory consider the classical vacuum configurations. Let us look at the Hamiltonian of the system (see Ref. 7). A rescaling of the variables is suitable:

$$x \to \sqrt{\lambda}\,vx, \quad A_\mu \to (g/\sqrt{\lambda}\,v)A_\mu, \quad \varphi \to v\varphi \,. \tag{1.3}$$

Then the bosonic action is of the form

$$S = v^2 \int d^2x \left[-\frac{1}{4\xi}F_{\mu\nu}^2 + |\partial_\nu\varphi + iA_\nu\varphi|^2 + (|\varphi|^2 - \tfrac{1}{2})^2 \right] . \tag{1.4}$$

The Lagrangian in (1.4) contains only one free parameter $\xi = g^2/\lambda$, which is not renormalized. The scalar condensate v^2 enters in front of the Planck coupling constant, so that v^{-2} is the loop-expansion parameter. One may choose various dynamical variables related to each other by canonical transformations. We find polar coordinates particularly useful: $\varphi = \rho e^{i\alpha}$. On the classical level (when v is unobservable) the action is

$$S = \int d^2x \left[\frac{1}{2\xi}E^2 + (\partial_\nu\rho)^2 + (\partial_\nu\alpha + A_\nu)^2 - (\rho^2 - \tfrac{1}{2})^2 \right] \tag{1.5}$$

where the electric field $E = F_{01}$ serves as the canonical momentum p_A of the vector field: $p_A = E/\xi$. The other canonical momenta are

$$p_\rho = 2\dot{\rho}, \quad p_\alpha = 2\rho^2(\dot{\alpha} + A_0) \,. \tag{1.6}$$

The canonical momentum of the vector field satisfies the gauge constraint

$$\partial_x p_A = -p_\alpha \,. \tag{1.7}$$

So the momentum conjugated to the phase of the scalar field is just the density of electric charge. We choose the Coulomb gauge $\partial_1 A_1 = 0$ and resolve the constraint (1.7) explicitly using the decomposition

$$p_A \equiv p_N - \frac{1}{\xi}\partial A_0 \,, \tag{1.8}$$

where p_N is constant in space while A_0 determines the longitudinal component of the momentum p_A. Equation (1.7) then yields

$$A_0(x) = \xi \int dy\, \Delta(x-y) p_\alpha(y) \,, \tag{1.9}$$

where the Coulomb force Δ in a finite volume of size L is

$$\Delta(x-y) = \tfrac{1}{2}|x-y| + (x-\kappa)(y-\kappa)/L \,, \tag{1.10}$$

where κ is arbitrary constant.[12] In the physical Coulomb gauge the only independent degree of freedom left from the gauge field is a constant in space which in the finite volume L we denote as $A_1 \equiv N/L$, so that p_N in (1.8) is the canonical momentum conjugated to N. From Eq. (1.2) one can see that this physical degree of freedom is nothing but the Chern-Simons variable $N = 2\pi N_{\rm CS}$. The Legendre transformation of Lagrangian (1.5) leads to a Hamiltonian of the form

$$H = \frac{\xi L}{2}p_N^2 + \frac{1}{4}\int dx\, p_\rho^2 + H^{\rm st} \,, \tag{1.11}$$

where the static Hamiltonian (which gives the energy of an arbitrary static configuration) is

$$H^{st}=\int dx\,[\tfrac{1}{4}p_\alpha^2/\rho^2+(\partial_y\rho)^2 +\rho^2(\partial_y\alpha+N/L)^2+(\rho^2-\tfrac{1}{2})^2]+H^c\ , \quad (1.12)$$

$$H^c=-\frac{\xi}{2}\int dx\,dy\,p_\alpha(x)\Delta(x-y)p_\alpha(y)\ . \quad (1.13)$$

From the static Hamiltonian (1.12) one immediately obtains the form of the classical vacuum field configurations.[7] The density of electric charge vanishes in the vacuum: $p_\alpha=0$. The absolute value of the field ρ must be constant in space equal $1/\sqrt{2}$ for the second and fourth terms in (1.12) to vanish. The gradient of the phase α must be constant in space and equal $-N/L$ for the third term in (1.12) to vanish. Now the periodicity of the fields on the interval $[0,L]$ requires that N should be integer modulo 2π. Thus the classical vacuum field configurations are specified by

$$p_\alpha=0,\quad \rho=1/\sqrt{2},\quad \alpha=-2\pi N_{CS}x/L\ ,\qquad N_{CS}=0,\pm1,\pm2,\ldots\ . \quad (1.14)$$

Topologically distinct vacua (1.14) are separated by the static energy barrier. The field configuration, corresponding to the top of that energy barrier is called a sphaleron.[5] The explicit form of this static solution of the equations of motion is[7]

$$N_{CS}=\tfrac{1}{2}\ ,\qquad \varphi_{sph}=\frac{e^{i\pi x/L}}{\sqrt{2}}\tanh(x/\sqrt{2})\ ,\qquad E_{sph}=\frac{2\sqrt{2}}{3}v^2\ . \quad (1.15)$$

The Hamiltonian equations of motion are relatively simple in polar coordinates:

$$\dot\rho=\tfrac{1}{2}p_\rho+\tfrac{1}{2}p_\alpha^2/\rho^3\ ,\qquad \dot p_\rho=2\partial^2\rho-2\rho(\partial\alpha+N/L)^2-4\rho(\rho^2-\tfrac{1}{2})\ , \quad (1.16a)$$

$$\dot\alpha=\tfrac{1}{2}p_\alpha/\rho^2-A_0,\quad \dot p_\alpha=2\partial[\rho^2(\partial\alpha+N/L)]\ , \quad (1.16b)$$

$$\dot N=\xi L p_N,\quad \dot p_N=-\frac{2}{L}\int dx\,\rho^2(\partial\alpha+N/L)\ . \quad (1.16c)$$

From Eqs. (1.16) one deduces that in the gauge theory a static configuration (with all time derivatives vanishing) may have a nonzero momentum p_α if the configuration has a nonzero density of electric charge. An arbitrary static configuration is defined by the set $(\rho,\alpha,p_\alpha,N_{CS})$. Now instead of the Hamiltonian evolution (1.16) which allows one to average over the energy surface we would like to introduce Langevin evolution to compute Gibbs averages.

II. LANGEVIN EQUATION FOR THE ABELIAN GAUGE THEORY

The general Langevin evolution of a scalar field $\varphi(x)$ obeys the phenomenological equation of the form

$$\varphi(x,t+\varepsilon)-\varphi(x,t)=-\frac{\varepsilon}{\eta}f^{reg}+f_x^{ran}(t,\varepsilon)\ . \quad (2.1)$$

Here the regular force comes from the static Hamiltonian $f^{reg}=-\partial H^{st}/\partial\varphi$. The friction coefficient η determines how fast the system reaches thermal equilibrium. The random force, which is essentially a white noise, is a sort of external field so that the solution of Eq. (2.1) is a functional with argument $f_x^{ran}(t)$. The function $f_x^{ran}(t)$ is not, in fact, specified explicitly. Only the probability distribution of the magnitude f^{ran} at each moment t is known. It is a Gaussian one with the variance

$$\langle f_x^{ran}(t,\varepsilon)f_y^{ran}(t,\varepsilon)\rangle=2A\,\delta(x-y)\varepsilon\ . \quad (2.2)$$

The variance (2.2) implies that (2.1) is not a conventional differential equation, since the random force scales as $\sqrt{\varepsilon}$. Therefore only a discrete version of the Langevin equation such as Eq. (2.1) may be considered in practice.

The probability distribution for $\varphi(x,t)$ satisfies the Fokker-Planck equation associated with (2.1). That equation allows one to establish Einstein's relation between the coefficients η and A: $\eta A=T$, where T is the temperature of the system. Thus the discrete version of the Langevin equation for the lattice regularized theory takes the form

$$\varphi_n(t+\varepsilon)-\varphi_n(t)=-\frac{\varepsilon}{\eta}f_n^{reg}+\left[\frac{\varepsilon T}{\eta a}\right]^{1/2}\xi_n(t)\ , \quad (2.3)$$

where a is the lattice spacing, $n=0,1,\ldots,L/a$, and $\xi_n(t)$ is a Gaussian random number. Any physical magnitude obtained from the solution of Eq. (2.3) is a function of ε or, to be more precise, a function of the ratio (ε/η). Then the final answer which is physically meaningful is given by the limit $(\varepsilon/\eta)\to0$. So we set $\eta=1$ below.

Now let us address the case of the gauge theory described in Sec. I. According to Sec. I the set of variables that specifies any static configuration includes $\rho,\alpha,p_\alpha,N_{CS}$. The crucial step in proceeding from the Hamiltonian evolution to the Langevin one is the introduction of a random force in whose presence the canonical momentum is not defined anymore, while the canonical variable evolves according to Eq. (2.1). This can be easily done for the gauge-invariant (or "colorless" according to the terminology of Ref. 13) ρ variable introduced in the previous section. Instead of Eq. (1.16a) one obtains

$$\rho_n(t+\varepsilon)-\rho_n(t)=[\partial^2\rho_n-\rho_n(\partial\alpha+N/L)^2 -2\rho_n(\rho_n^2-\tfrac{1}{2})]\varepsilon a+\left[\frac{\varepsilon T}{a}\right]^{1/2}\xi_n(t)\ , \quad (2.4)$$

where ∂ stands for the discrete version of the spatial derivative, for instance, $\partial\alpha=(\alpha_{n+1}-\alpha_n)/a$. However there is no way to introduce white noise to the gauge-variant variable α. Indeed, the canonical momentum p_α is just the density of electric charge. One can see that the appearance of any additional term ξ on the right-hand side (RHS) of Eq. (1.16b) implies nonconservation of the electric current J_ν: $\partial_\nu J_\nu = \xi$. Physically it means that the random force corresponding to the gauge-invariant variable α simply cannot emerge in a system with gauge-invariant interactions. Therefore we leave Eqs. (1.16b) intact. The resulting evolution respects Gauss's constraint and conserves electric current. In particular, the total electric charge vanishes as it should, being a spatial integral of the total derivative [see Eqs. (1.7) and (1.16b)].

The Chern-Simons variable is gauge invariant (the operator $N_{\rm CS}$ commutes with the constraint). Therefore, in general, the equations of motion for $N_{\rm CS}$ (1.16c) can include a random force but the effect of this force must be negligible in the limit of infinite volume, so we find it gratifying to observe thermalization of the $N_{\rm CS}$ variable without doing it explicitly through the white noise in the equation of motion of $N_{\rm CS}$.

Thus the set of equations that determines the Langevin evolution of our system contains Eqs. (2.4), (1.16b), and (1.16c). They have been solved numerically.

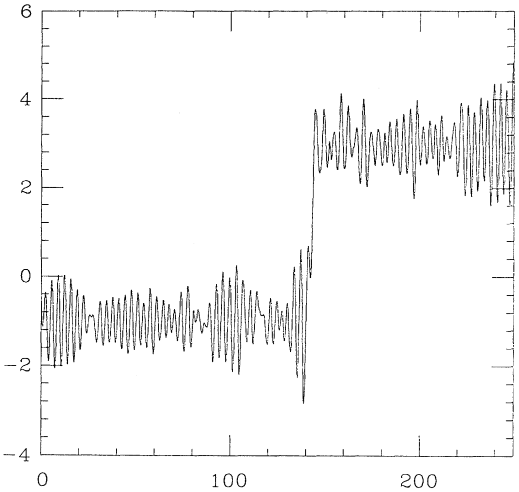

FIG. 1. Time dependence of the Chern-Simons variable for $a=1$, $L=200$, $\xi=4$, $\beta=18$.

III. NUMERICAL RESULTS

Basically we have considered a system of size $L=200$, lattice spacing $a=1$ using Langevin step $\varepsilon=0.02$ in the temperature range $\beta \approx 6-14$, for gauge coupling constants corresponding to $\xi=4$, 10, 28. There is a problem in the implementation of the numerical algorithm associated with our choice of polar coordinates. They are singular at the origin which is incompatible with the discretization of the Hamiltonian. Sometimes when the ρ variable gets close to zero the forces on the RHS of Eqs. (1.16) diverge, which is an artifact of discretization. To eliminate this problem we use an adaptive Langevin step; namely, we take a smaller value of ε each time the displacement (defined as the product of the force and time step ε) of the canonical coordinate ρ, α, or $N_{\rm CS}$ exceeds some threshold. Such a procedure is somewhat time consuming but it guarantees that we do not affect the dynamics while avoiding a problem at $\rho \to 0$.

The time dependence of the Chern-Simons variable is shown in Fig. 1. High-frequency oscillations are a normal mode which is easily traceable from Eq. (1.16c) for ρ close to the vacuum value (1.14). The frequency of the oscillations is $\sqrt{\xi}$. The beats which are also clearly seen come from the interaction of this mode with the Higgs field fluctuation around the vacuum. One can see that at this rather low temperature $T=\frac{1}{18}$ the system fluctuates most of the time near one classical vacuum with a definite value of the Chern-Simons number ($N_{\rm CS}=-1$). Then it jumps to another vacuum with $N_{\rm CS}=+3$. This is a so-called "sphaleron transition" in the sense that during the jump the field configuration goes near the sphaleron saddle point.[7] Figure 2 shows the real-time behavior of the Chern-Simons variable on a larger time scale. Many sphaleron transitions are visible. It is obviously difficult to calculate the density of transitions "by hand." So the question arises of how to determine the transition rate. Figure 2 looks like a Brownian walk along different topological sectors. To verify this conjecture we measure the quantity $\Delta_{\rm CS}(t)$:

$$\Delta_{\rm CS}(t) = \langle (N_{\rm CS}(t) - N_{\rm CS}(0))^2 \rangle \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau dt' [N_{\rm CS}(t+t') - N_{\rm CS}(t')]^2 \; . \quad (3.1)$$

Some typical examples of our numerical evaluation of the function $\Delta_{\rm CS}(t)$ are shown in Figs. 3 and 4. The time dependence of the Brownian variable $\Delta_{\rm CS}$ contains three different parts. For small values of $\Delta_{\rm CS} \lesssim 1$ one can see perfect straight lines in Figs. 3 and 4 which imply the power law

$$\Delta_{\rm CS}(t) = (t/t_0)^\sigma \; . \quad (3.2)$$

The exponent σ extracted from Figs. 3 and 4 is $\sigma=1.7$. It depends neither on the temperature nor on the gauge coupling ξ. The explanation is obvious. For short macroscopic time intervals (when $\Delta_{\rm CS} \lesssim 1$) the Chern-Simons

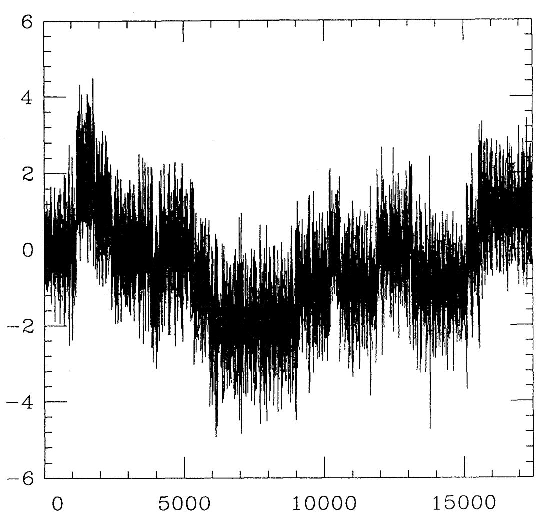

FIG. 2. Same as in Fig. 1 but on a larger time scale.

variable moves approximately like a free particle with some average velocity,

$$\Delta N = \xi L \langle p_N \rangle \Delta t \ , \tag{3.3}$$

which predicts the deterministic power law $\Delta_{CS} \sim t^2$, with an exponent close to that obtained numerically. Evaluating the thermal average $\langle p_N \rangle$ [see Eq. (3.5) below] one obtains

$$t_0 = \frac{2\pi}{\sqrt{\xi L T}} \ . \tag{3.4}$$

Numerical data presented in Figs. 3 and 4 yields $t_0(\xi=10)=0.55$, $t_0(\xi=28)=0.22$, in perfect agreement with (3.4).

For large $\Delta_{CS} > 1$ one can see other straight lines which also indicate a power law such as in (3.2). This reveals the Brownian behavior of the Chern-Simons variable. It diffuses along the topologically distinct classical vacua. The three lines in Fig. 3 corresponding to different temperatures are parallel to each other. Therefore the exponent σ does not depend on temperature while the parameter t_0 fixing the diffusion rate changes with T.

At large time Δ_{CS} reaches asymptotics (see Figs. 3 and 4), which is a finite-size effect. There is a finite number of the topologically distinct classical vacua in the finite box, since the gradient of the phase α which is adjusted to the Chern-Simons variable in the ground state is bounded from above on the finite lattice due to (1.14). Therefore a big lattice is necessary to have a long and clear straight line which allows one to extract the Brownian exponent and the diffusion rate precisely.

The observation of the diffusion of the Chern-Simons variable provides us with information about the effective potential $V(N_{CS})$. For a potential that grows to infinity at large N_{CS}, as in the case of an harmonic oscillator, for instance, there can be no Brownian behavior (3.2). We observe this obvious fact in our study. The ρ variable obeys the ordinary Langevin equation (2.4) so that one could expect Brownian behavior. However the effective potential $V(\rho)$ is unbounded at $\rho \to \infty$ and we obtain numerically a fixed value of $\langle \rho^2 \rangle$ whose temperature dependence is shown on Fig. 5. One can see, by the way, that the scalar condensate decreases as the temperature grows, which is a typical effect of Higgs's theory. The power law (3.2) indicates that the static energy barrier between different topological sectors is finite and the effective potential $V(N_{CS})$ is bounded at $N_{CS} \to \infty$. So it looks like $\sin(N_{CS})$.

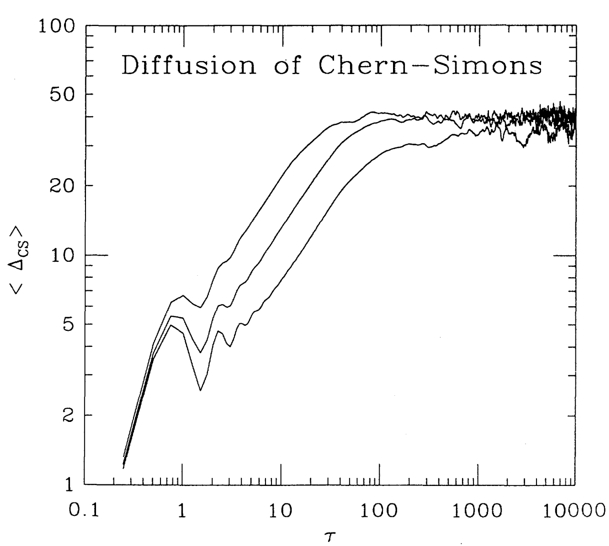

FIG. 3. Brownian behavior of the Chern-Simons variable $\Delta_{CS}(t)$ at various temperatures: $\beta=10$, 12, 14; foı the gauge coupling $\xi=28$; $a=1$, $L=200$.

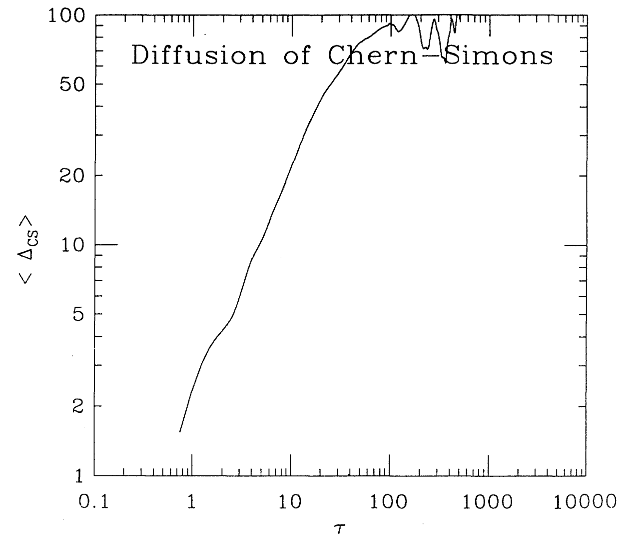

FIG. 4. Same as in Fig. 3 for $\beta=10$, $\xi=10$, $L=100$, and small lattice spacing $a=0.25$. The exponent $\sigma \approx 1$.

The Brownian behavior (3.2) implies also that the Chern-Simons variable is thermalized despite the fact that we have not introduced white noise in equations of motion (1.16c) of the variable N_{CS}. As a double check one can measure directly, so to say, "the temperature of the Chern-Simons variable." Note that the Chern-Simons canonical momentum p_N enters only in one term of the Hamiltonian (1.11). This implies that one can evaluate the Gibbs average of the Chern-Simons momentum squared exactly and explicitly:

$$\langle p_N^2 \rangle = \frac{T}{\xi L} \ . \tag{3.5}$$

The LHS of Eq. (3.5) is easily calculated as the Langevin average. We obtain numerically that the temperature from Eq. (3.5) follows the one introduced in the original equation (2.4), being 25% bigger (see Fig. 6). This gives us an idea of how big the systematic errors of our approach are. The fact that the actual temperature of the system exceeds the Langevin one is a well-known artifact of the discrete version of the Langevin equation that is considered.[14] The discrepancy disappears in the limit $\varepsilon \to 0$.

The exponent σ in Eq. (3.2) does not depend on tem-

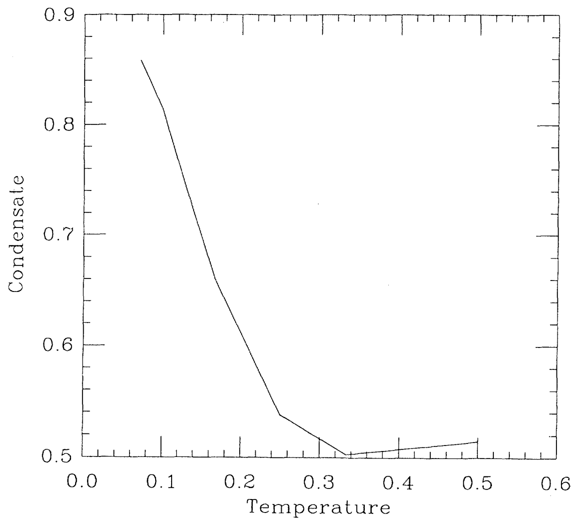

FIG. 5. Temperature dependence of the scalar condensate $\langle \varphi^\dagger \varphi \rangle$ obtained as an average over the Langevin evolution.

perature for all three values of the gauge coupling constant studied, $\xi=4$, 10, 28, but it exhibits a lattice spacing dependence as measured for $\xi=10$. One can see from Table I that in the limit of small lattice spacing the exponent σ approaches one. Already at $a=0.25$ one has $\sigma \approx 1$. No finite-size effects in physical quantities have been observed in a run with $L=400$.

The characteristic time t_0 determines the rate of the transitions between the different classical vacua. In the limit of small lattice spacing we obtain the conventional Brownian behavior

$$\Delta_{CS}(t)=\Gamma t \; , \tag{3.6}$$

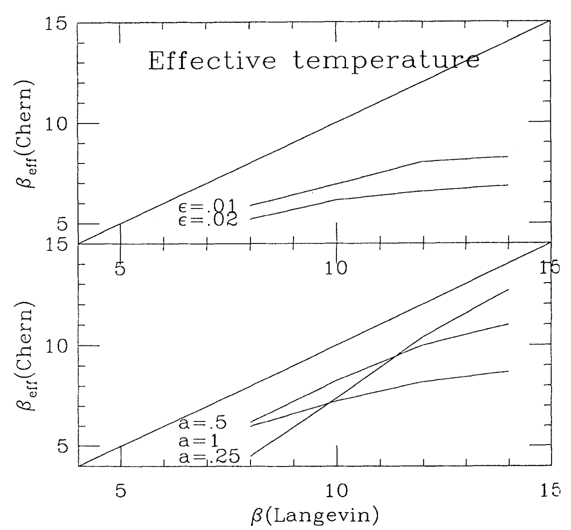

FIG. 6. The "temperature of the Chern-Simons variable" measured according to Eq. (3.5) vs the temperature inserted originally into the Langevin equation (2.4). The straight line represents an ideal case where the two temperatures coincide. Top: for $\xi=28$, two lines correspond to $\varepsilon=(0.02,0.01)$; bottom: for $\xi=10$, three lines below correspond to $a=\{1,0.5,0.25\}$.

TABLE I. Dependence of the Brownian exponent σ on the lattice spacing a for $\xi=10$, $L=200$.

a	σ
1	0.66
0.5	0.83
0.25	1

so that $t_0=1/\Gamma$. Large values of the lattice spacing introduce systematic errors in the determination of Γ. We extrapolate our data to the limit of vanishing lattice spacing and Langevin step size.

Constant Γ in Eq. (3.6) determines the diffusion rate of the Chern-Simons variable which we identify with the rate of anomalous fermion number nonconservation. In the low-temperature domain $T < E_{sph}$ under consideration, the rate Γ was estimated semiclassically[7,15,16] in ac-

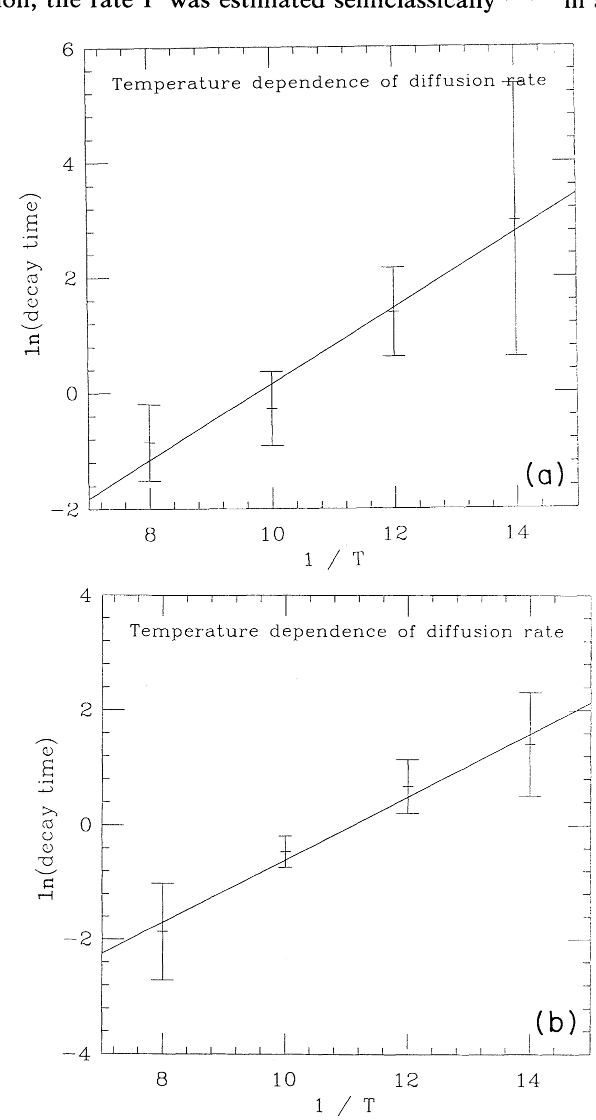

FIG. 7. Temperature dependence of the diffusion rate for different gauge coupling constants: (a) for $\xi=10$; (b) for $\xi=28$. Solid lines are fits.

cordance with the definition suggested in Ref. 17. The expected temperature dependence is of the form

$$\Gamma(T)=C\left[\frac{E_{\rm sph}}{T}\right]^{1/2}\exp\left[-\frac{E_{\rm sph}}{T}\right], \tag{3.7}$$

where the constant C depends on the gauge coupling but not on the temperature. To analyze our numerical data we first transfer the preexponential factor in (3.7) to the LHS and see whether our numerical data exhibits an exponential dependence on the inverse temperature. Figure 7 shows that there is a rather wide interval of temperature where the dots representing our numerical results follow precisely the exponential behavior. The systematic errors depicted in Fig. 7 come from the measurements for various values of the lattice spacing and Langevin step size. The slope of the straight lines in Fig. 7 determines the sphaleron mass which is supposed to be 0.94 in our dimensionless units (1.15). The numerical data are consistent with this value of the sphaleron mass within the uncertainties due to the variation of a,ε as well as to the deviation of the real temperature from that introduced in the normalization of the white noise.

We consider the exponential dependence of the diffusion rate on the inverse temperature as an unambiguous indication of the relevance of the thermal activation theory involving the sphaleron saddle point.

IV. BEYOND THE SEMICLASSICAL APPROXIMATION

The evaluation of the rate Γ of the anomalous fermion number nonconservation in the high-temperature domain (for temperatures greater than the sphaleron mass) is a challenging problem. Semiclassical expansion is of no help. So is the direct observation of sphalerons in the classical numerical simulations. We discuss here two ways to deal with the problem.

The first one is based on Affleck's definition of the rate which originates from the simple quantum-mechanical example.[17] We now use the Hamiltonian formulation of the $(1+1)$-dimensional Abelian Higgs theory presented in Sec. I. The rate Γ measures the average Chern-Simons momentum in one direction at the fixed value of the Chern-Simons variable $N_{\rm CS}=\frac{1}{2}$. This particular value of $N_{\rm CS}$ corresponds to the hypersurface separating topologically distinct vacua.[18] To be more precise it defines a manifold of the finite-energy static field configurations in the background of which there is a normalizable fermionic zero eigenmode which gets occupied leading to the observable anomalous fermion number nonconservation. This consideration is not based on the semiclassical approximation and therefore can be used in the high-temperature domain. So, by definition rate Γ is given by the functional integral

$$\Gamma=\frac{\rm num}{\rm den}\ ,\quad {\rm num}=\int\frac{dN\,dp_N}{2\pi}D\varphi\,D\varphi^*\,Dp\,Dp^*\,p_N\theta(p_N)\delta(N-\pi)\exp(-\beta H)\ , \tag{4.1}$$

where D stands for the measure of the functional integration, den is a partition function given by the same functional integral as for the numerator num but without a preexponential factor and the Hamiltonian H is given by Eq. (1.12). Consider the possibility of calculating the integral (4.1) by means of Monte Carlo simulations. For this end we need to recast the preexponential factor in (4.1) in a less singular form.

An essential point is that integration over the Chern-Simons degree of freedom in (4.1) may be done explicitly:

$${\rm num}=\frac{1}{\pi\beta\xi L}\int D\varphi\,D\varphi^*\,Dp\,Dp^*\exp[-\beta H(\varphi,p,p_N=0\ ,N_{\rm CS}=\tfrac{1}{2})]\ . \tag{4.2}$$

The partition function den can be also represented by the functional integral over the scalar field and its canonical conjugated momentum only. This is because the integral over Chern-Simons variables N,p_N is Gaussian [see Eq. (1.12)]. One obtains the following form for the partition function:

$${\rm den}=\int D\varphi\,D\varphi^*\,Dp\,Dp^*(2\beta\xi\|\varphi\|)^{-1/2}\exp[-\beta H_{\rm eff}(\varphi,p)]\ , \tag{4.3}$$

where $\|\varphi\|\equiv\int dx\,\varphi^\dagger\varphi/L$. The rate Γ is recast in the form suitable for Monte Carlo measurements:

$$\Gamma={\rm den}^{-1}\int D\varphi\,D\varphi^*\,Dp\,Dp^*\,O\exp[-\beta H_{\rm eff}(\varphi,p)]\ , \tag{4.4}$$

$$O=\frac{1}{\pi L}\left[\frac{2\|\varphi\|}{\beta\xi}\right]^{1/2}\exp\left[-\beta L\|\varphi\|\left[\langle\langle\partial\alpha\rangle\rangle+\frac{\pi}{L}\right]^2\right]\ , \tag{4.5}$$

where α is a phase of the scalar field and $\langle\langle\partial\alpha\rangle\rangle\equiv\int dx\,\varphi^\dagger\varphi\partial\alpha/\int dx\,\varphi^\dagger\varphi$. One can also restore integration over the Chern-Simons degree of freedom and represent the rate as a thermal average of the operator O with the original integration measure introduced in (4.1).

Thus we conclude that the rate of the anomalous fermion under nonconservation in the $(1+1)$-dimensional Abelian Higgs model may be, in principle, computed by means of the Metropolis algorithm. However the exponential form of the operator O basically implies large uncertainties in such a computation.

The second way to study the high-temperature domain is to measure the diffusion rate of the Chern-Simons number as it was performed in the preceding section in the case of low temperatures. Our numerical simulations for $\beta=6,4$ show that the average displacement squared Δ_{CS} of the Chern-Simons variable reaches quickly the region of the finite-size effect that one can see in Figs. 3 and 4. The very big lattice which would allow for the larger values of the Chern-Simons number is necessary to study the diffusion rate in the high-temperature domain. This makes it time consuming to deal with even $(1+1)$-dimensional gauge theory. Meanwhile our study of some quantum-mechanical toy model[19] with a periodic potential shows excellent agreement with expectations: namely, the thermal activation behavior of the rate at low temperatures and power law $\Gamma \sim T$ (see Refs. 4 and 18) in the high-temperature domain.

To reduce systematic errors one can try another way to determine the diffusion rate. Differentiating expression (3.1) with respect to time and using the first of Eqs. (1.16c) one obtains

$$\Gamma \equiv \lim_{t\to\infty} \dot{\Delta}_{CS}(t) = 2\xi^2 L^2 \lim_{t\to\infty} \int_0^t d\tau \langle p_N(t) p_N(\tau) \rangle \ , \tag{4.6}$$

where the correlator of the Chern-Simons canonical momentum may be evaluated as a Langevin average such as Eq. (3.1). For instance if it goes like

$$\langle p_N(t) p_N(0) \rangle = \frac{T}{\xi L} \exp(-\gamma t) \ , \tag{4.7}$$

with some γ for large t, one immediately obtains $\Gamma = 2\xi T/\gamma$. The result of our numerical evaluation of the correlator (4.7) presents however a new point, namely, high-frequency oscillations (see Fig. 8). Therefore the integral in Eq. (4.6) is a small number originating from the cancellation of large ones. Unfortunately this implies a large uncertainty in extracting the diffusion rate from the correlator of the Chern-Simons canonical momentum. This conclusion is in fact a very general one. Our study of the correlator (4.7) demonstrates the difficulties one meets in extracting the rate Γ from the correlator of the operator $F_{\mu\nu}\tilde{F}_{\mu\nu}$ in the real case of the standard electroweak theory.

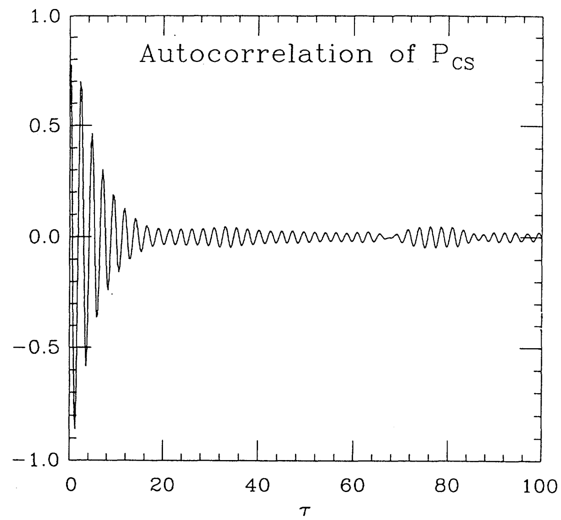

FIG. 8. Real-time correlator of the canonical momentum conjugated to the Chern-Simons variable for $\xi=10$, $\beta=14$, $a=0.5$, $L=200$.

SUMMARY

In this paper we have introduced real-time Langevin evolution in the Abelian gauge theory. It is based on the explicit separation of the gauge-invariant and gauge-variant degrees of freedom. By construction gauge-invariant variables obey the conventional Langevin equation while gauge-variant variables evolve microcanonically so that the resulting distribution is canonical with the constraint that the physical states be gauge invariant. We identify Monte Carlo time of such an evolution with physical real time which enters as an argument of the average squared displacement that characterizes a Brownian walk. In the $(1+1)$-dimensional Abelian Higgs model our numerical simulations reveal sharp transitions between topologically distinct vacua with different integer values of the Chern-Simons number. The Chern-Simons variable is shown to exhibit a Brownian behavior. Its diffusion rate depends exponentially on the inverse temperature in the low-temperature domain that we considered. This indicates the relevance of the thermal activation theory involving the sphaleron saddle point.

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