Nonrelativistic Chern-Simons vortex solitons in external magnetic field

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We study 2+1-dimensional Chern-Simons gauge theories with external magnetic field B and the self-interaction $|\psi|^4$ of the matter field. It is shown that the system has three phases depending on the strength g of the self-interaction. They are the symmetry-preserving phase for $g > g_c$, and the symmetry-breaking phase for $g_c > g > -g_c$, with $\pm g_c$ the critical coupling constants. When $g_c > g > 0$ the system has an excitation with gap; when $0 > g > -g_c$ its spectrum develops the absolute minimum at a nonzero momentum. The corresponding excitation becomes gapless at $g = -g_c$, and the system is unstable for $g < -g_c$. We then analyze vortex solitons which are anyons. Nontopological (topological) vortices are relevant in the symmetry-preserving (-breaking) phase. The charge, spin, and mass of these vortices are calculated. These vortices can be analyzed analytically at the critical points $g = \pm g_c$. The static energy of self-dual topological vortices is obtained explicitly, and is expressed as a spin-magnetic interaction. We also present analytic time-dependent solutions of nontopological vortices, which describe vortex solitons moving along the cyclotron orbit in the external magnetic field.

I. INTRODUCTION

In two-dimensional space the homotopy of paths is nontrivial; i.e., all the paths are classified in terms of the homotopy class by using a braid group. Correspondingly, there can be exotic statistics which are different from either bosonic or fermionic. The particle obeying such exotic statistics is called an anyon.[1]

For instance, the matter fields are anyons in (2+1)-dimensional Abelian gauge field theories with a Chern-Simons (CS) term. The effect of the CS term is to attach a flux on the matter field. In general, objects carrying both the charge Q and the flux Φ behave as anyons. This is because, when interchanging the positions of two such objects, the phase factor $e^{i\alpha}$ with

$$\alpha = \frac{Q\Phi}{2} \tag{1.1}$$

is produced through the Aharonov-Böhm (AB) effect. Then, the object is bosonic when $\alpha = 2\pi n$ and fermionic when $\alpha = \pi(2n+1)$, n being an integer, while we get anyonic statistics for other values of α. In these theories there are vortex solitons which are also anyons.[2–5] Their non-Abelian generalizations have also been considered.[6]

These anyonic objects have recently attracted much interest. In particular, the significance of anyonic vortices has been recognized in its application to the fractional quantum Hall effect[7] (FQHE). See also Ref. 8. The FQHE occurs in certain materials placed in a uniform external magnetic field. Therefore, the analysis of anyonic vortices in the magnetic field, which is yet to be explored, is important not only theoretically but also phenomenologically. In this paper, we reveal some new features specific to CS vortices in the external magnetic field.

This paper is organized as follows.

In Sec. II we define our model. It is the (2+1)-dimensional nonrelativistic CS gauge theory with self-interaction of the matter field, where a uniform external magnetic field is applied.

In Sec. III we first analyze the problem of the existence of the uniform background solution breaking the gauge symmetry, since vortex solitons appear on such a background. In this analysis we find the following unexpected property of the system. Whatever choice of the matter-field potential we make, there are no uniform background solutions breaking the symmetry if the external magnetic field does not exist. On the other hand, there is such a solution in the presence of the magnetic field even if no potential exists. It is not the matter potential but the external magnetic field that induces the symmetry breakdown. In this way, the external magnetic field plays a crucial role for the symmetry breakdown. We then give the uniform background solutions of the system explicitly (both symmetry-preserving and -breaking ones), and quantize small fluctuations around them. We get the following results in the model with the self-interaction $g|\psi|^4$. The system has three phases with the critical coupling constants being $\pm g_c$ ($g_c > 0$). When $g > g_c$, the ground state is the symmetry-preserving phase where anyons (the matter quantums) are in the cyclotron motion repelling each other. When $g_c > g > -g_c$, it is the symmetry-breaking phase where the anyons are interpreted to be condensed. Furthermore, when $g > 0$, the dispersion relation has the absolute minimum at zero momentum with a gap. When $g < 0$, this spectrum changes and there appears a new minimum at a nonzero momentum. This excitation may be identified with the *magnetoroton*.[11] The magnetoroton becomes gapless at the critical coupling constant $g = -g_c$, and the whole system is unstable for $g < -g_c$.

In Sec. IV we analyze topological and nontopological vortices. It is shown that their charge, spin, and mass are determined only by the asymptotic behavior at infinity. In particular, these quantities are quantized for the topological vortex. We also point out that the spin-statistics relation holds for the topological vortex but not for the nontopological vortex. We then give a result of our numerical calculations for these vortex solutions. It is interesting to see some types of critical behavior of the solution at $g = \pm g_c$; when $g > g_c$ the tail of the solution damps exponentially; when $g_c > g > -g_c$ it damps exponentially but with an undulation; when $-g_c > g$ it damps only in powers.

In Sec. V, generalizing the arguments given in Ref. 4, we discuss topological and nontopological self-dual vortices. These self-dual vortices may appear at the critical coupling constants $g = \pm g_c$. For the topological vortex we calculate the static energy analytically, and interpret it as an interaction between the spin of the vortex and the external magnetic field.

In Sec. VI, we present analytic time-dependent solutions of nontopological vortices. The first set are breather solutions with the center of mass at rest. The second set are breather solutions with the center of mass making a cyclotron motion. Both of these solutions are obtained from the Jackiw-Pi self-dual solutions[4] by making certain transformations.

II. ANYON SYSTEM

We consider a (2+1)-dimensional nonrelativistic CS gauge theory with uniform external magnetic field B. The system is composed of the CS gauge field a_μ and the bosonic complex matter field ψ coupled with it. We assume there is a self-interaction of ψ through the potential $\mathcal{V}(|\psi|^2)$. The Lagrangian density is given by

$$\mathcal{L} = \psi^\dagger iD_0\psi - \frac{1}{2m}|D_k\psi|^2 - \mathcal{V}(|\psi|^2) - \frac{1}{4\alpha}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda , \tag{2.1}$$

with $iD_\mu = i\partial_\mu - eA_\mu + a_\mu$; the Latin indices run over 1,2 and the Greek indices over 0,1,2. Here, A_μ is the external magnetic potential such that

$$A_0 = 0, \quad A_k = -\frac{B}{2}\varepsilon_{kl}x^l = \frac{B}{2}r^2\partial_k\theta , \tag{2.2}$$

with θ being the azimuthal angle. The form of the potential $\mathcal{V}$ is arbitrary in most parts of this paper. However, to have a definite idea, it is convenient to choose the potential

$$\mathcal{V} = \frac{g}{2}(|\psi|^2 - v)^2 . \tag{2.3}$$

Recall that such a potential has been postulated in the Landau-Ginzburg theory of the FQHE.[9] In nonrelativistic theory, the term linear in $|\psi|^2$ may be removed by the transformation $\psi \to \exp(-igvt)\psi$. When the external magnetic field is switched off, and when we choose $v = 0$ in (2.3),

$$\mathcal{V} = \frac{g}{2}|\psi|^4 , \tag{2.4}$$

the system is manifestly invariant under the SO(2,1) transformation.[5] We will come back to this point later.

There are two types of gauge symmetries in the system. The first one is the electromagnetic gauge symmetry

$$\psi \to \psi e^{if} , \qquad eA_\mu \to eA_\mu - \partial_\mu f . \tag{2.5}$$

The other is the statistical gauge symmetry

$$\psi \to \psi e^{if} , \qquad a_\mu \to a_\mu + \partial_\mu f . \tag{2.6}$$

The existence of topological or nontopological vortices are closely related with the breakdown of these symmetries.

The Hamiltonian density of this system is given by

$$\mathcal{H} = \frac{1}{2m}|D_k\psi|^2 + \mathcal{V}(|\psi|^2) . \tag{2.7}$$

Its Euler equations are

$$\frac{1}{2\alpha}a_{12} = |\psi|^2 , \tag{2.8}$$

$$\frac{1}{2\alpha}a_{k0} = \varepsilon_{kl}\frac{i}{2m}[\psi^\dagger D_l\psi - (D_l\psi)^\dagger\psi] , \tag{2.9}$$

$$iD_0\psi = -\frac{1}{2m}D_k^2\psi + \mathcal{V}'(|\psi|^2)\psi , \tag{2.10}$$

where $a_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ and $\mathcal{V}'(z) = \partial_z\mathcal{V}(z)$.

Equation (2.8) is a basic equation of the system and corresponds to the Gauss law in the Maxwell theory. However, its charge density is equal not to the divergence of the "electric" field but to the "magnetic" field. This is a typical nature of the CS gauge field theory. By virtue of this, the "magnetic" flux is automatically attached to the matter field ψ. This leads to anyonic character of the matter field. The current density is also equal not to the derivative of the field strength as in the Maxwell theory but to the "electric" field strength as in Eq. (2.9). Equation (2.10) is the Schrödinger equation with a nonlinear interaction.

We first notice that a_μ can be solved in terms of ψ. Fixing the gauge by the condition that $\sum_{k=1}^2 \partial_k a_k = 0$, we get from (2.8) that

$$a_k = -\frac{\alpha}{\pi}\varepsilon_{kl}\int d^2y\,\frac{(x-y)^l}{|x-y|^2}\,|\psi(y)|^2 . \tag{2.11}$$

By substituting this into (2.9), a_0 is also solved:

$$a_0 = \frac{\alpha}{2\pi m}\varepsilon_{kl}\int d^2y\,\frac{(x-y)^k}{|x-y|^2}\,\psi(y)^\dagger i\overleftrightarrow{D}_l\psi(y) . \tag{2.12}$$

Thus, all a_μ are dependent variables of ψ, and not dynamical.

Though a_μ is not dynamical, they play a crucial role. It is well known that they give ψ quantum exotic statistics through the AB effect. For a single particle, the relation

$$\int d^2x\,|\psi|^2 = 1 \tag{2.13}$$

holds by definition. This means that ψ quantum has unit statistical charge. Moreover, the ψ quantum has the statistical flux such that

$$\int d^2x\,a_{12} = 2\alpha\int d^2x\,|\psi|^2 = 2\alpha . \tag{2.14}$$

Therefore, through the AB effect, the phase $e^{i\alpha}$ is produced when the positions of two ψ quantums interchange. The quantum is a boson for $\alpha = 2\pi n$, or a fermion for $\alpha = \pi(2n+1)$. For other values of α, the quantum has exotic statistics and becomes an anyon.[1] This heuristic argument is supplemented by a path-integral formulation with a linking number.[10]

We can derive the spin-statistics relation of the ψ quantum. The total angular momentum J in this model is defined by

$$J = \int d^2x\,\varepsilon_{ij}x_i\mathcal{P}_j , \tag{2.15}$$

with

$$\mathcal{P}_j = \frac{1}{2i}[\psi^\dagger(\partial_j - ia_j)\psi - (\partial_j + ia_j)\psi^\dagger\psi] . \tag{2.16}$$

It is decomposed into the orbital part L and the spin part S. In our gauge condition for the external magnetic field, i.e., (2.2), they are explicitly given by

$$J = L + S \tag{2.17}$$

and

$$L = \frac{1}{2i}\varepsilon_{ij}\int d^2x\,x_i(\psi^\dagger\partial_j\psi - \partial_j\psi^\dagger\psi) , \tag{2.18}$$

$$S = -\varepsilon_{ij}\int d^2x\,x_ia_j|\psi|^2 . \tag{2.19}$$

Using (2.11), we find

$$\begin{aligned} S &= -\frac{\alpha}{\pi}\int d^2x\,d^2y\,\frac{x(x-y)}{|x-y|^2}\,|\psi(x)|^2|\psi(y)|^2 \\ &= -\frac{\alpha}{2\pi}\int d^2x\,d^2y\,\frac{|x-y|^2+|x|^2-|y|^2}{|x-y|^2} \\ &\qquad\times|\psi(x)|^2|\psi(y)|^2 \\ &= -\frac{\alpha}{2\pi}\left(\int d^2x|\psi(x)|^2\right)^2 . \end{aligned} \tag{2.20}$$

Hence, we get the relation between the spin (S) and the statistics (α) of the ψ quantum:

$$S = -\frac{\alpha}{2\pi} . \tag{2.21}$$

This relation is well known for the cases of bosons and fermions. It implies that in the anyonic case the spin can take an arbitrary value. Thus, one may say that the anyon interpolates not only the statistics but also the spin between bosons and fermions.

In this way ψ behaves as an anyon through the AB effect. There are other anyonic objects in this model. These anyons are the CS vortices, which we analyze later.

III. GROUND-STATE SOLUTIONS AND MAGNETOROTONS

Before analyzing CS vortices we need to find the ground state of the Hamiltonian and its property. Hence, we seek the static and spatially uniform classical solution of the Euler equations (2.8), (2.9), and (2.10). It is a remarkable property of the nonrelativistic CS gauge theory (2.1) that the symmetry-breaking solution exists only if the external magnetic field is present. This might be rather mysterious since it is our common knowledge that the gauge symmetry could be spontaneously broken by making a judicious choice of the potential.

Let us assume that there is no external magnetic field and that there is a static uniform solution such that $\psi = \widetilde{\psi} = \text{const}$. Then, from (2.8), we obtain

$$a_i = -\alpha\,\varepsilon_{ij}|\widetilde{\psi}|^2x^j . \tag{3.1}$$

Substituting these into (2.9), we get

$$a_0 = \frac{\alpha^2}{m}|\widetilde{\psi}|^4(x^2+y^2) + \text{const.} \tag{3.2}$$

Then, substituting this into (2.10), we find

$$\frac{\alpha^2}{2m}|\widetilde{\psi}|^4(x^2+y^2) = \mathcal{V}'(|\widetilde{\psi}|^2) + \text{const.} \tag{3.3}$$

Because this equality must hold at arbitrary points x and y, we get

$$\widetilde{\psi} = 0 . \tag{3.4}$$

Consequently, only the symmetry-preserving background solution is obtained in the absence of the external magnetic field B. (This conclusion is achieved even if we start with a slightly general assumption that $|\psi| = \text{const.}$) Note that this is not the case in relativistic CS gauge theory, where the cooperation of the particle and antiparticle degrees of freedom makes the symmetry-breaking solution possible even without the external magnetic field.

When there is the external magnetic field, it is easy to obtain the uniform background solutions explicitly. First, a symmetry-preserving solution always exists:

$$\widetilde{\psi} = 0 , \qquad \widetilde{a}_0 = \text{const}, \qquad \widetilde{a}_k = 0 . \tag{3.5}$$

If and only if $\alpha e B > 0$, a symmetry-breaking solution is also possible:

$$\widetilde{\psi} = \sqrt{\rho}\,, \qquad \widetilde{a}_0 = \mathcal{V}'(\rho)\,, \qquad \widetilde{a}_k = eA_k\,, \tag{3.6}$$

with

$$\rho = \frac{eB}{2\alpha}\,. \tag{3.7}$$

All the uniform background solutions are generated from these solutions by the gauge transformation (2.6). In (3.6), the nonzero value of $\widetilde{\psi}$ is determined by B. Thus, these solutions are generated not by the potential but by the external magnetic field, as we have mentioned.

Our immediate question is what the ground state of this system is. Let us evaluate the energies of these uniform solutions. To simplify discussions, we choose the potential (2.3). Substituting them into the Hamiltonian density (2.7) we get

$$\mathcal{E}_{\rm sym} = \mathcal{V}(0) = \frac{g}{2}v^2\,, \tag{3.8}$$

for the symmetry-preserving one, and

$$\mathcal{E}_{\rm non} = \mathcal{V}(\rho) = \frac{g}{2}(\rho - v)^2\,, \tag{3.9}$$

for the symmetry-breaking one.

In the case of relativistic field theory, they are the energies of the possible vacuums, and the solution having the smallest energy describes the true classical vacuum. However, this is not the case in the nonrelativistic theory because the number of particles is conserved. Hence, we have to find the lowest-energy state (ground state) in a sector with the particle number fixed. Hereafter we restrict ourselves to a sector with the average density of the particle number ρ, which is assumed to be equal to $eB/2\alpha$. This assumption is satisfied in the phenomenologically interesting case such as the FQHE. Then, the background configuration (3.6) is understood to describe a condensed state of anyons with a constant number density ρ.

To evaluate the energy of the system with anyon density ρ in the unbroken phase (3.5), it is necessary to quantize small fluctuations around the classical solution, i.e., $\psi = \delta\psi$. Substituting it into the Hamiltonian density, and taking the quadratic terms in $\delta\psi$ and $\delta\psi^\dagger$, we get

$$\mathcal{H} = \frac{1}{2m}|(\partial_k + ieA_k)\delta\psi|^2 + \mathcal{V}'(0)|\delta\psi|^2 + \mathcal{V}(0)\,. \tag{3.10}$$

Therefore, the excitations are in the cyclotron motion. By accommodating all excitations in the lowest Landau level, the energy density of the system is

$$\mathcal{E}_{\rm sym}(\rho) = \left(\left|\frac{eB}{2m}\right| - gv\right)\rho + \frac{g}{2}v^2\,. \tag{3.11}$$

We may rewrite this as

$$\mathcal{E}_{\rm sym}(\rho) - \mathcal{E}_{\rm non} = \tfrac{1}{2}(g_c - g)\rho^2\,, \tag{3.12}$$

with

$$g_c = \frac{2|\alpha|}{m}\,. \tag{3.13}$$

We conclude that the symmetry-preserving solution (3.5) is relevant for $g > g_c$, while the symmetry-breaking one (3.6) for $g_c > g$, with g_c being a critical coupling constant.

We next analyze the excitation modes when the symmetry-breaking solution is chosen as the ground state. We consider small fluctuations around the classical solution, i.e.,

$$\psi = \sqrt{\rho} + \delta\psi\,. \tag{3.14}$$

We substitute it into the Hamiltonian density, and take the quadratic terms in $\delta\psi$ and $\delta\psi^\dagger$. In so doing, we eliminate the gauge potentials by the constraint equation (2.8), or

$$a_i - eA_i = -\frac{\alpha}{\pi}\varepsilon_{ij}\int d^2y\,\frac{(x-y)_j}{(x-y)^2}(\psi^\dagger\psi - \rho)\,. \tag{3.15}$$

We set

$$\delta\psi = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}\neq 0} a_p e^{i\mathbf{p}\cdot\mathbf{x}}\,, \tag{3.16}$$

with

$$[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = \delta_{\mathbf{p}\cdot\mathbf{q}}\,, \tag{3.17}$$

and V being the volume of the system. Then, the Hamiltonian density is diagonalized as

$$\mathcal{H} = \sum_{\mathbf{p}\neq 0}\frac{\varepsilon_{\mathbf{p}}}{2}\left(\sqrt{1 + 4U_{\mathbf{p}}/\varepsilon_{\mathbf{p}}} - 1\right) + \sum_{\mathbf{p}\neq 0}\sqrt{\varepsilon_{\mathbf{p}} + 4\varepsilon_{\mathbf{p}}U_{\mathbf{p}}}\,b^\dagger_{\mathbf{p}}b_{\mathbf{p}}\,, \tag{3.18}$$

with

$$[b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}] = \delta_{\mathbf{p}\cdot\mathbf{q}}\,, \tag{3.19}$$

and

$$\varepsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m}\,, \qquad U_{\mathbf{p}} = \frac{2\pi^2\rho^2}{m\mathbf{p}^2}\left(\frac{\alpha}{\pi}\right)^2 + \frac{g\rho}{2}\,. \tag{3.20}$$

Here, operators $b_{\mathbf{p}}$ and $b^\dagger_{\mathbf{p}}$ are related to $a_{\mathbf{p}}$ and $a^\dagger_{\mathbf{p}}$ by a Bogolubov transformation:

$$a_{\mathbf{p}} = g_{\mathbf{p}}b_{\mathbf{p}} - h_{\mathbf{p}}b^\dagger_{-\mathbf{p}}\,, \tag{3.21}$$

with $h^2_{\mathbf{p}} = g^2_{\mathbf{p}} - 1$ and

$$g^2_{\mathbf{p}} = \frac{1}{2}\left(\frac{\varepsilon_{\mathbf{p}} + 2U_{\mathbf{p}}}{\sqrt{\varepsilon^2_{\mathbf{p}} + 4\varepsilon_{\mathbf{p}}U_{\mathbf{p}}}} + 1\right)\,. \tag{3.22}$$

The ground state $|0\rangle$ is defined by $b_{\mathbf{p}}|0\rangle = 0$, and $b^\dagger_{\mathbf{p}}$ is the creation operator of the excitations with momentum p. The energy density of the ground state diverges, as is usual in quantum field theory, and is to be renormalized appropriately.

From (3.18) and (3.20), the dispersion relation reads

$$E^2(p) = \left(\frac{eB}{m}\right)^2 + \frac{p^2}{2m}\left(\frac{p^2}{2m} + g\frac{eB}{\alpha}\right), \tag{3.23}$$

where $p = |\mathbf{p}|$. First, we note that $E(p) \to eB/m$ as $p \to 0$. Thus, there is an energy gap at $p = 0$. This energy gap is generated by the Anderson-Higgs mechanism. When the gauge symmetry is spontaneously broken, the gapless mode appears by the Goldstone mechanism; however, it is absorbed by the gauge field a_μ. As a result the excitation acquires an energy gap.

The form of the dispersion relation is different typically whether $g > 0$ or $g < 0$; see Fig. 1. In the first case ($g > 0$), the function $E(p)$ has only one stationary point at $p = 0$, which is the absolute minimum. Near $p = 0$, the dispersion relation reads

$$E(p) \simeq E_m + \frac{p^2}{2M_m}, \tag{3.24}$$

with E_r and M_r being the gap energy and the mass of the excitation:

$$E_m = \left|\frac{eB}{m}\right|, \quad M_m = \left|\frac{2\alpha}{g}\right|. \tag{3.25}$$

In the second case ($g < 0$), it has two stationary points at $p = 0$ (a local maximum) and $p = p_r$ (the absolute minimum) with

$$p_r = \sqrt{2m|g|\rho}. \tag{3.26}$$

Now, it is possible to make a wave packet at $p = p_r$ and regard it as a local excitation because its group velocity dE/dp is zero. Such an excitation is called a *magnetoroton*.[11] Near $p = p_r$ the dispersion relation is

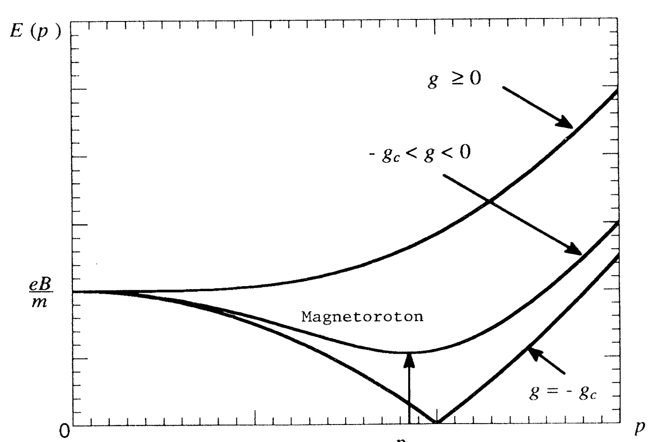

FIG. 1. Dispersion relation $E(p)$ for the excitations around the symmetry-breaking background. When $g \geq 0$ there is the minimum at $p = 0$. When $0 > g > -g_c$ there appears the new minimum at $p = p_r$. The magnetoroton becomes gapless at $g = -g_c$.

approximately written as

$$E(p) \simeq E_r + \frac{(p - p_r)^2}{2M_r}, \tag{3.27}$$

with E_r and M_r being the gap energy and the mass of the roton:

$$\begin{aligned} E_r &= \left|\frac{eB}{m}\right| \sqrt{1 - \left(\frac{g}{g_c}\right)^2}, \\ M_r &= \left|\frac{\alpha}{g}\right| \sqrt{1 - \left(\frac{g}{g_c}\right)^2}, \end{aligned} \tag{3.28}$$

where g_c is given by (3.13). It should be noticed that when $g = -g_c$ the roton becomes gapless at nonzero momentum and causes an irregularity in the system. Futhermore, the system is unstable for $g < -g_c$ because the rotons are produced unlimitedly.

It is well known that the system is unstable for $g < 0$ in ordinary relativistic gauge theory since the potential is unbounded from below. However, this is not the case for nonrelativistic Chern-Simons gauge theory, because the system does not contain antiparticles and conserves the particle number. Moreover, the kinetic energy is not independent from the potential energy due to the constraint condition (2.8). Indeed, we have shown that the system is stable at least perturbatively even for attractive potential, i.e., $0 > g > -g_c$.

IV. TOPOLOGICAL AND NONTOPOLOGICAL VORTICES

A. Boundary conditions

The topological vortex is stable for its conservation of the winding number. It is characterized by the asymptotic behavior of the matter field at infinity such that

$$\psi \simeq \tilde{\psi} e^{in\theta}, \tag{4.1}$$

where $\tilde{\psi}$ is a real constant and θ is the azimuthal angle. This vortex is built upon the uniform background solution $\psi = \tilde{\psi}$. When $\tilde{\psi} \neq 0$, it is impossible to deform the vortex configuration continuously to the uniform background configuration (3.6) while keeping the vortex energy finite. Thus, for the existence of topological vortex we need a uniform nonzero background which breaks the gauge symmetry (2.6). This is not the case for the nontopological vortex. In general, we expect to have nontopological vortices if the uniform background solution is vanishing, i.e., given by (3.5).

To obtain vortex solutions, topological or nontopological, we make the following *Ansatz* on the field variables:

$$\psi = \hat{\psi}(r)\, e^{in\theta}, \tag{4.2}$$

$$a_0 = a_0(r), \tag{4.3}$$

$$a_k = a(r)\partial_k\theta, \tag{4.4}$$

where (r, θ) is the polar coordinate. Here, the variables in the *Ansatz* must satisfy several conditions as discussed below.

First of all, the field ψ should be a single-valued function of the coordinate. This forces the winding number n to take an integer value. Furthermore, $\hat{\psi}(r)$ should vanish at the origin, $\hat{\psi}(0) = 0$, because otherwise the field is singular there. Also the field a_μ must be regular, from which we get

$$\lim_{r\to 0} \frac{a(r)}{r} = 0 . \tag{4.5}$$

Moreover, the vortex solution must approach to the background configuration at infinity. This condition can be expressed separately for topological and nontopological vortices.

The boundary conditions for nontopological vortex solutions are

$$\lim_{r\to\infty} \hat{\psi}(r) = 0 , \tag{4.6}$$

$$\lim_{r\to\infty} a_0(r) = \text{const}, \tag{4.7}$$

$$\lim_{r\to\infty} a(r) = 0 . \tag{4.8}$$

On the other hand, the boundary conditions for topological vortex solutions read

$$\lim_{r\to\infty} \hat{\psi}(r) = \sqrt{\rho} , \tag{4.9}$$

$$\lim_{r\to\infty} a_0 = \mathcal{V}'(\rho) , \tag{4.10}$$

$$\lim_{r\to\infty} \frac{a(r)}{r^2} = \frac{eB}{2} , \tag{4.11}$$

up to gauge ambiguities. It is also important to check the finiteness of vortex energy. The energy is represented with this *Ansatz* as

$$E = 2\pi \int_0^\infty \frac{dr}{r} \Big[\hat{\psi}(r)^2 \Big(a(r) - \frac{eB}{2} r^2 - n\Big)^2 + \Big(r\frac{d\hat{\psi}(r)}{dr}\Big)^2 + r^2[\mathcal{V}(\hat{\psi}(r)^2) - \mathcal{V}_{\rm BG}]\Big] , \tag{4.12}$$

where $\mathcal{V}_{\rm BG} = \mathcal{V}(0)$ or $\mathcal{V}(\rho)$, by subtracting the background energy. For the topological vortex, requiring that the energy contribution from infinity does not diverge, we get an asymptotic behavior such that

$$\lim_{r\to\infty} a(r) = \frac{eB}{2} r^2 + n . \tag{4.13}$$

These equations are closely related to topological quantization of the charge, spin, and mass of vortex solitons.

B. Charge, spin, and mass

We discuss the charge, spin, and mass of CS vortices. We show that all these quantities are determined solely in terms of its statistical charge Q:

$$Q = \int d^2x \,(\psi^\dagger\psi - |\tilde{\psi}|^2) . \tag{4.14}$$

Here, we have subtracted the background contribution $|\tilde{\psi}|^2$, which is nonzero for the topological vortex, (3.6), and is zero for the nontopological vortex, (3.5). The statistical charge is uniquely given by the statistical flux of the vortex, since we have

$$Q = \frac{1}{2\alpha} \int d^2x \,(a_{12} - \tilde{a}_{12}) = \frac{1}{2\alpha} \oint dx^k \,(a_k - \tilde{a}_k) = \frac{\Phi}{2\alpha} . \tag{4.15}$$

The surface integral is carried out at infinity. Therefore, the charge, mass, and spin of the CS vortex are characterized by the asymptotic behavior of the fields at infinity.

The electric charge density is obtained by differentiating the action with respect to A_μ, and is given by $e\psi^\dagger\psi$. Now, the electric charge of the vortex is defined by subtracting the background contribution, if it exists, and hence is given by

$$Q_{\rm EM} = eQ , \tag{4.16}$$

where Q is the statistical charge of the vortex.

On the other hand, the total angular momentum of the system is obtained by transforming the action rotationally. When there is a single vortex configuration, we should identify the total angular momentum J as the spin of the vortex S_v. The S part of J is given by (2.20), while the L part now reads

$$L = n \int d^2x \,|\psi|^2 . \tag{4.17}$$

However, a caution is needed for the topological vortex since there is a nonzero background. The background contribution is not $\tilde{\psi}$ but $\tilde{\psi}e^{in\theta}$ with an infinitesimal hole removed around the vortex center. Thus, defining $S_v \equiv J(\text{vortex}) - J(\text{background})$, we obtain

$$S_v = nQ - \frac{\alpha}{2\pi} Q^2 , \tag{4.18}$$

with (4.14).

What about the mass of the vortex ? It is well known that the mass of a vortex in a relativistic model is given by substituting its static configuration into the Hamiltonian of the system and subtracting the vacuum energy. This mass $M^{(\rm rel)}$ so defined comes out in the relativistic dispersion relation

$$E = M^{(\rm rel)} + \frac{\mathbf{p}^2}{2M^{(\rm rel)}} + O(\mathbf{p}^4) . \tag{4.19}$$

However, in the nonrelativistic model we are using, the dispersion relation needs not be (4.19) but rather is

$$E = E_v + \frac{\mathbf{p}^2}{2M} , \tag{4.20}$$

where E_v is the static energy of the vortex. The energy of a static vortex configuration is not the mass. To get the mass M from the static vortex configuration, it is necessary to perform a Galilean boost with speed $\mathbf{v}$:

$$\begin{aligned}
\psi &\to \psi' = \exp\Big[-i\Big(mv_k x_k + \frac{m\mathbf{v}^2}{2}t\Big)\Big]\psi(\mathbf{x}+\mathbf{v}t)\,,\\
a_0 &\to a_0' = a_0(\mathbf{x}+\mathbf{v}t) + v_k a_k(\mathbf{x}+\mathbf{v}t)\,,\\
a_k &\to a_k' = a_k(\mathbf{x}+\mathbf{v}t)\,,\\
A_0 &\to A_0' = v_k A_k(\mathbf{x}+\mathbf{v}t)\,,\\
A_k &\to A_k' = A_k(\mathbf{x}+\mathbf{v}t)\,.
\end{aligned} \tag{4.21}$$

When we substitute these into the Hamiltonian, we find

$$H' = \int d^2x\,\Big(\mathcal{H} + \frac{m\mathbf{v}^2}{2}\psi^\dagger\psi + v_k\psi^\dagger iD_k\psi\Big)\,. \tag{4.22}$$

The term linearly depending on v_k vanishes for the spherical symmetric vortex configuration. Hence,

$$E = E_0 + \frac{m\mathbf{v}^2}{2}\int d^2x\,\psi^\dagger\psi\,, \tag{4.23}$$

in accordance with (4.20). This formula implies that the soliton is made up of the ψ quanta, whose number is $\int d^2x\,\psi^\dagger\psi$. Hence, in the case of the topological vortex which is constructed on the nonzero background, the number $\int d^2x\,\psi^\dagger\psi$ contains also the ψ quanta condensed in the background state. By the Galilean transformation all those condensed quanta acquire the momentum $\mathbf{p} = m\mathbf{v}$. Therefore, the resulting energy is not the vortex energy. What we need is the energy when the vortex moves in the background ψ quanta which are at rest. In order to get this energy we calculate the number of the ψ quanta contained in the vortex, which is given by (4.14). Then, if $Q > 0$, the vortex is composed of a positive number of the ψ quanta, but if $Q < 0$, it is composed of a negative number of them. Namely, when $Q > 0$, the vortex is formed on top of the background ψ quanta, and the mass is given by $M = mQ$. On the other hand, when $Q < 0$, it is formed as a hole in the background ψ quanta. Since the move of a hole is equivalent to the move of $-Q$ of the background quanta in the opposite direction, the vortex mass is given by $M = -mQ > 0$. In any case, we obtain

$$M = m|Q|\,, \tag{4.24}$$

with (4.14). This formula holds obviously also for the nontopological vortex where $\widetilde{\psi} = 0$. It is a remarkable feature of the nonrelativistic CS gauge theory that the mass of the vortex is determined rigorously without any detailed calculations.

In the case of topological vortex, the charge, mass, and spin are explicitly calculable from the boundary condition (4.13). Substituting (3.6) and (4.13) into (4.15), we get

$$Q = \frac{1}{2\alpha}\oint dx^k\,(a_k - eA_k) = \frac{\pi}{\alpha}n\,. \tag{4.25}$$

The statistical charge of the topological vortex is quantized by its winding number n. Consequently, the electric charge (4.16), mass (4.24), and spin (4.18) are also quantized:

$$Q_{\rm EM} = e\frac{\pi}{\alpha}n\,, \tag{4.26}$$

$$M = m\,\Big|\frac{\pi}{\alpha}n\Big|\,, \tag{4.27}$$

$$S_v = \frac{\pi}{2\alpha}n^2\,. \tag{4.28}$$

The topological quantization prevents these quantities from depending on continuous parameter B or ρ.

In this way, the charge, spin, and mass of the topological vortex are quantized. However, these are not true, of course, for the nontopological vortex.

We comment on the statistics of the topological vortex. The statistical charge of the topological vortex is given by (4.25), while the statistical flux is $2\pi n$. Hence, from (1.1), we expect that the statistical parameter of the vortex is $\alpha_v = \frac{\pi^2}{\alpha}n^2$. Actually, this argument is too crude to determine the overall sign. A detail analysis shows that[10]

$$\alpha_v = -\frac{\pi^2}{\alpha}n^2\,. \tag{4.29}$$

Consequently, from (4.28) and (4.29), the spin-statistics relation holds in the topological vortex case:

$$S_v = -\frac{\alpha_v}{2\pi}\,. \tag{4.30}$$

Note that this relation is not valid in the nontopological case. In this sense it would be difficult to consider the nontopological vortex as a particle.

C. Numerical solutions and undulations

In order to make the Euler equations dimensionless, we rescale the variables as follows.

$$\psi = \sqrt{|\rho|}\,F(z)\,e^{in\theta}\,, \tag{4.31}$$

$$a_0 = \Big|\frac{\alpha\rho}{m}\Big|\,G(z)\,, \tag{4.32}$$

$$a_k = eA_k + n\partial_k\theta + \frac{\alpha}{|\alpha|}\sqrt{|\alpha\rho|}\,\varepsilon_{kj}\frac{x_j}{r}R(z)\,, \tag{4.33}$$

and

$$r = \frac{z}{\sqrt{|\alpha\rho|}}\,, \tag{4.34}$$

where ρ is defined by (3.7). The functions F, G, and R are dimensionless.

Substituting these *Ansätze* into the Euler equations, we obtain

$$\frac{dR}{dz} = -\frac{R}{z} - 2F^2 + 2\frac{eB\alpha}{|eB\alpha|}\,, \tag{4.35}$$

$$\frac{dG}{dz} = -2F^2R\,, \tag{4.36}$$

$$\begin{aligned}\frac{d^2F}{dz^2} &= -\frac{1}{z}\frac{dF}{dz} - 2GF + R^2F\\ &\quad + \Big|\frac{4m}{eB}\Big|\,\mathcal{V}'(|\rho|F^2)F\,.\end{aligned} \tag{4.37}$$

At the vertex center the boundary conditions are

$$\lim_{z\to 0} F(z) = 0\,, \qquad \lim_{z\to 0} R(z) = \frac{N}{z}\,, \tag{4.38}$$

with $N = n\alpha/|\alpha|$, while at infinity they are

$$\lim_{z\to\infty} F(z) = 1\,, \qquad \lim_{z\to\infty} R(z) = 0\,, \tag{4.39}$$

for the topological vortex and

$$\lim_{z\to\infty} F(z) = 0\,, \qquad \lim_{z\to\infty} \frac{R(z)}{z} = \frac{eB\alpha}{|eB\alpha|}\,, \tag{4.40}$$

for the nontopological vortex.

In both the topological and nontopological vortex cases, the asymptotic solutions near the vortex core are easily found:

$$R(z) \simeq \frac{N}{z} + z\frac{eB\alpha}{|eB\alpha|} - \frac{a^2}{|N|+1} z^{2|N|+1}\,, \tag{4.41}$$

$$G(z) \simeq b - a^2 \frac{N}{|N|} z^{2|N|}\,, \tag{4.42}$$

$$F(z) \simeq a\, z^{|N|}\,, \tag{4.43}$$

where a and b are undetermined constants.

Our numerical analysis reads as follows. We integrate the set of equations (4.35), (4.36), and (4.37) by assuming appropriate values for the initial data a and b in the asymptotic solutions (4.41), (4.42), and (4.43) at the vortex center, and examine whether or not an obtained solution satisfies the boundary condition (4.39) or (4.40) at infinity. We repeat this process by changing the initial data. Here, for simplicity, we take the model without the potential $\mathcal{V}$, and give a numerical result for topological vortices with $n = 1$ and $n = -1$. We find that $a \simeq 2.089\,11$, $b \simeq 3.766\,18$ for $n = 1$, and that $a \simeq 0.871\,57$, $b \simeq 0.311\,30$ for $n = -1$. See Fig. 2. It is interesting to note that these numerical solutions have asymptotic tails with undulation; See Fig. 2(d).

In order to investigate the undulation in the tail of topological vortices, we solve the Euler equations to find the asymptotic solution at infinity. Let us decompose the variables into the background part and the fluctuations around it:

$$F = 1 + \Delta F\,, \tag{4.44}$$

$$G = \left|\frac{2m}{eB}\right| \mathcal{V}'(\rho) + \Delta G\,, \tag{4.45}$$

$$R = \Delta R\,. \tag{4.46}$$

Substituting these into the Euler equations (4.35), (4.36), (4.37), and linearizing the equations of motion, we get

$$\left(\frac{d}{dz} + \frac{1}{z}\right)\Delta R = -4\Delta F\,, \tag{4.47}$$

$$\frac{d}{dz}\Delta G = -2\Delta R\,, \tag{4.48}$$

$$\left(\frac{d^2}{dz^2} + \frac{d}{zdz}\right)\Delta F = \frac{4mg}{|\alpha|}\Delta F - 2\Delta G\,, \tag{4.49}$$

where $g = \mathcal{V}''(\rho)$. Then, the asymptotic solution at infinity is obtained as

$$\Delta F \simeq \frac{\varepsilon}{\sqrt{z}} e^{-\beta_\pm z}\,, \tag{4.50}$$

$$\Delta G \simeq \frac{8\varepsilon}{\beta_\pm^2}\frac{1}{\sqrt{z}} e^{-\beta_\pm z}\,, \tag{4.51}$$

$$\Delta R \simeq \frac{4\varepsilon}{\beta_\pm}\frac{1}{\sqrt{z}} e^{-\beta_\pm z}\,, \tag{4.52}$$

where ε is an unknown constant and

$$\beta_\pm^2 = \frac{2mg}{|\alpha|}\left[1 \pm \sqrt{1-\left(\frac{2\alpha}{mg}\right)^2}\right] = \frac{4g}{g_c}\left[1 \pm \sqrt{1-\left(\frac{g_c}{g}\right)^2}\right]\,, \tag{4.53}$$

where g_c is defined by (3.13). Hence, if $g \geq g_c$, $\beta_\pm$ is real and the tail is monotonically and exponentially damping to the background value.

However, if $g_c > g > -g_c$, $\beta_\pm$ becomes complex:

$$\beta_\pm = \sqrt{2}\left(\sqrt{1+\frac{g}{g_c}} \pm i\sqrt{1-\frac{g}{g_c}}\right)\,. \tag{4.54}$$

Therefore, its tail is exponentially damping to the background value but with undulation. This undulation, found also in the numerical solution [Fig. 2(d)], is obviously related to the dispersion relation (3.23) of the excitations around the background solution (3.6).

Furthermore, if $g \leq -g_c$, $\beta_\pm$ is pure imaginary and the tail dumps only as $\frac{1}{\sqrt{z}}$. The energy of the vortex diverges in this case. The power damping is attributed to the fact that the magnetorotons become gapless at $g = -g_c$, and the ground state (3.6) is unstable for $g < -g_c$.

D. Static energy

As we have mentioned, the energy of the static vortex gives the gap energy E_v and not the mass M of the vortex. In general, it will be necessary to perform numerical calculations to investigate this quantity. (As we discuss in Sec. V, it is obtained analytically for the self-dual topological vortex.) Here, we discuss how the energy E_v depends on the external magnetic field B when the potential (2.4) is assumed.

When the external magnetic field is switched off, there is an SO(2,1) symmetry in this system.[5] The generators are the Hamiltonian H, the dilation generator D, and the special conformal generator K. When the external magnetic field B is switched on, only the Hamiltonian defines a good symmetry. The dependence of the static energy on B is determined by examining how the dilation symmetry is broken.

It is trivial to see that, when $B = 0$, the action $S =$

$\int d^3x\,\mathcal{L}$ with the Lagrangian density (2.1) is invariant under the dilation:

$$x \to \Omega x\,, \quad t \to \Omega^2 t\,, \qquad \psi \to \frac{1}{\Omega}\psi\,, \quad a_k \to \frac{1}{\Omega}a_k\,, \quad a_0 \to \frac{1}{\Omega^2}a_0\,. \tag{4.55}$$

The invariance is generalized to the case where $B \neq 0$ if we also change the external field as

$$A_k \to \frac{1}{\Omega}A_k\,. \tag{4.56}$$

This implies that $S(B) = S(B/\Omega^2)$, or

$$E_v(B) = \frac{B}{B'}E_v(B')\,, \tag{4.57}$$

after subtracting the background energy which also obeys the same scaling law. Hence, the static energy E_v depends on B linearly.

The linear dependence can also be verified as follows. When the potential is given by (2.4), the Euler equation (4.37) is rewritten as

$$\frac{d^2F}{dz^2} = -\frac{1}{z}\frac{dF}{dz} - 2GF + R^2F + \frac{2mg}{|\alpha|}F^3\,. \tag{4.58}$$

Thus, all variables F, G, and R are independent of the value of the external field B. The static energy is given by

$$E_v = \frac{eB\pi}{m\alpha}\int dz\,z\left(GF^2 - \frac{mg}{2|\alpha|}F^4\right), \tag{4.59}$$

for nontopological vortices, and by

$$E_v = \left|\frac{eB\pi}{m\alpha}\right|\int dz\,z\left(GF^2 - \frac{mg}{2|\alpha|}F^4 - \frac{mg}{2|\alpha|}\right), \tag{4.60}$$

for topological vortices after subtracting the background energy (3.9). In the previous subsection we have performed a numerical calculation for topological vortices for the simplified model with $g = 0$. Using the numerical results we find that $E_v \simeq 1.2(eB\pi/m\alpha)$ for the vortex with $n = 1$ and $E_v \simeq 0.25(eB\pi/m\alpha)$ for the vortex with $n = -1$.

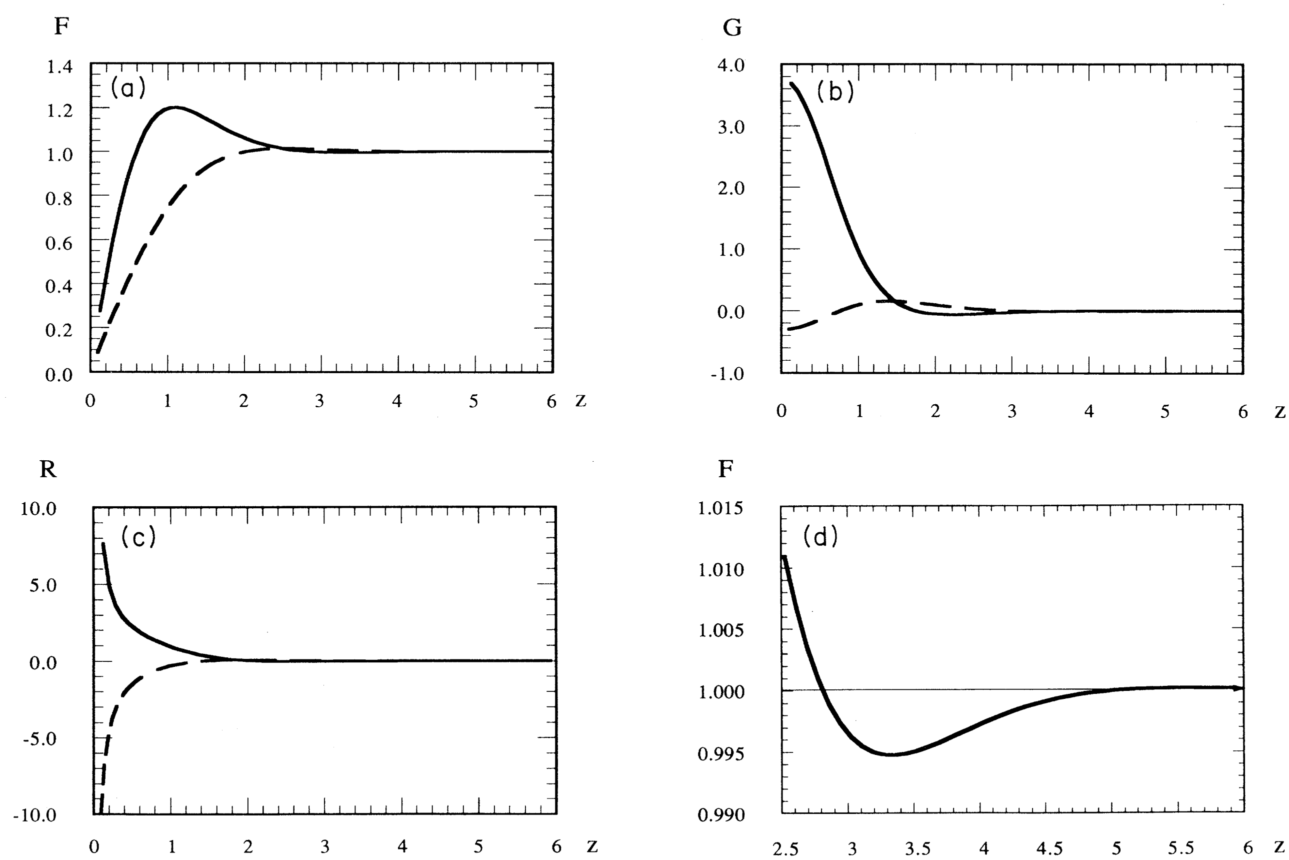

FIG. 2. Numerical solutions for topological vortices with $n = 1$ (solid line) and with $n = -1$ (dashed line); (a) for function F, (b) for G, (c) for R, and (d) for an enlarged portion of curve F where an undulation can be seen.

V. SELF-DUAL VORTICES IN MAGNETIC FIELD

It is known that there are self-dual nontopological vortices [4] in the absence of the external magnetic field B. It is straightforward to generalize the analysis so as to include the external magnetic field. We then find self-dual topological and nontopological vortices. In this section, for simplicity, we assume that the potential $\mathcal{V}$ is given by (2.4), although the case of the potential (2.3) can also be discussed easily.

We use the Bogomol'nyi decomposition:

$$|D_k\psi|^2 = |(D_1 \pm iD_2)\psi|^2 \pm \varepsilon_{ij}\partial_i(\psi^\dagger iD_j\psi) \pm \omega_{12}|\psi|^2, \tag{5.1}$$

where $D_k = \partial_k - i\omega_k$, $\omega_{12} = \partial_1\omega_2 - \partial_2\omega_1$, and $\omega_k = a_k - eA_k$. The energy reads

$$E = \int d^2x \left[\frac{1}{2m} |(D_1 \pm iD_2)\psi|^2 \mp \frac{eB}{2m}|\psi|^2 + \frac{1}{2}\left(g \pm \frac{2\alpha}{m}\right)|\psi|^4 \right]. \tag{5.2}$$

Now, if $g = \mp 2\alpha/m$, the coefficient of $|\psi|^4$ vanishes. Furthermore, the term $\int d^2x|\psi|^2$ is the number of particles and is assumed to be a constant. Therefore, the variation of the energy is reduced to

$$\delta E = \frac{1}{2m} \int d^2x\, \delta|(D_1 \pm iD_2)\psi|^2 . \tag{5.3}$$

In general, the static solutions are given by solving $\delta E = 0$. Because of the positive semidefiniteness of the integrand, self-dual vortices are obtained by solving the self-dual equation

$$(D_1 \pm iD_2)\psi = 0 . \tag{5.4}$$

It is interesting that there are self-dual vortices when $g = \pm g_c$ with g_c being the critical coupling given by (3.13).

The existence of self-dual vortex solutions can be seen as follows. Instead of (4.31) we parametrize the matter field as

$$\psi = \sqrt{\rho}\, e^{u(r)+in\theta} . \tag{5.5}$$

Substituting this *Ansatz* into the self-duality equation (5.4), we find that

$$\omega_k = a_k - eA_k = n\partial_k\theta \pm \varepsilon_{kl}\partial_l u(r) , \tag{5.6}$$

We then substitute (5.5) and (5.6) into the constraint equation (2.8):

$$\frac{d^2u}{dz^2} = -\frac{1}{z}\frac{du}{dz} - \frac{\partial U}{\partial u} , \tag{5.7}$$

where

$$U = -\frac{g}{|g|}\left(e^{2u} - \frac{\alpha eB}{|\alpha eB|}2u\right) , \tag{5.8}$$

and z is the rescaled radius defined by (4.34). This equation is the same one as that of the classical particle moving with a frictional force in the potential U, where z and u play the role of the time and the particle position. This analogy is very useful in establishing the existence of vortex solutions.

The boundary condition at the vortex center reads

$$\lim_{z\to 0} u(z) = -\infty , \qquad \lim_{z\to 0} u'(z) = -\frac{g\alpha}{|g\alpha|}\frac{n}{z} , \tag{5.9}$$

both for the topological and nontopological vortex, while at infinity it is

$$\lim_{z\to\infty} u(z) = 0 , \tag{5.10}$$

for the topological vortex, and

$$\lim_{z\to\infty} u(z) = -\infty , \tag{5.11}$$

for the nontopological vortex. Thus, the analog classical particle motion is as follows: At an initial time ($z = 0$), the particle is placed at the infinite remote position ($u \to -\infty$). At the final time ($z \to \infty$), the particle comes back to the initial position ($u \to -\infty$) for the nontopological vortex, or it approaches to a stationary point ($u \to 0$) for the topological vortex. This analogy implies the following conclusions. See Fig. 3.

When $g = g_c$ and $\alpha eB > 0$, both topological and nontopological self-dual vortices may exist. This is because U has a maximum at $u = 0$. The possible solution, approaching to this maximum as $z \to \infty$, is topological. On the other hand, the solutions, leaving away from the maximum and coming back to $-\infty$ as $z \to \infty$ because of the energy lack, are nontopological.

When $g = g_c$and $\alpha eB < 0$, U is a monotonically decreasing function of u; consequently, u diverges when z goes to infinity. So, there are no self-dual vortices.

When $g = -g_c$ and $\alpha eB > 0$, U has the minimum. So the topological vortex may be considered, but its energy diverges as we have mentioned before. Moreover, the nontopological vortex is impossible.

When $g = -g_c$ and $\alpha eB < 0$, U is a monotonically increasing function of u. So, only nontopological vortices are possible. These nontopological vortices are reduced smoothly to the Jackiw-Pi self-dual solutions.

We comment here that not all of vorticity n are realized in the self-dual vortex. Near the vortex center $z = 0$, u is approximately solved as follows:

$$u \simeq -\frac{g\alpha eB}{|g\alpha eB|}\frac{z^2}{2} - \frac{g\alpha}{|g\alpha|}n \ln\frac{z}{z_0} . \tag{5.12}$$

Because $u = -\infty$ at $z = 0$, it is necessary that $\alpha gn < 0$. Therefore, if $\alpha g > 0$, only antivortices ($n < 0$) are allowed in the self-dual equation (5.4). On the other hand, if $\alpha g < 0$, only vortices ($n > 0$) are allowed.

Finally, we show that the static energy E_v of the self-dual topological vortex may be calculated analytically. The kinetic energy vanishes for self-dual vortices. Subtracting the ground-state energy (3.9) from (5.2) we find that

$$E_v = \frac{eB}{2m}\frac{\pi}{|\alpha|}n < 0. \tag{5.13}$$

Then, using (4.26), (4.27), and (4.28), we may rewrite this

$$E_v = g_n \frac{Q_{\rm EM}}{2M} BS_v , \tag{5.14}$$

where

$$g_n = \left|\frac{2}{n}\right| . \tag{5.15}$$

We may interpret $Q_{\rm EM}/2M$ as the vortex magneton and g_n as the vortex g factor. It is interesting that the g factor is 2 for $n = \pm 1$.

VI. TIME-DEPENDENT ANALYTIC SOLUTIONS

In the previous section we discussed static self-dual vortex solutions in the presence of the external magnetic field B. However, although we have shown their existence, no analytic solutions have been found. In this section we present analytic solutions of *time-dependent* vortices.

We start with the Jackiw-Pi analytic solutions[4] of self-dual nontopological vortices in the system without external magnetic field B:

$$\psi(r,\theta) = \exp\left(i\frac{\alpha}{|\alpha|}(N-1)\theta\right) \times\sqrt{\frac{2}{|\alpha|}\frac{N}{r}\left[\left(\frac{r}{r_0}\right)^N + \left(\frac{r_0}{r}\right)^N\right]^{-1}} , \tag{6.1}$$

where

$$\int d^2x\, |\psi|^2 = \frac{2\pi}{|\alpha|}N , \tag{6.2}$$

and $N = 2, 3, \ldots$. Here, r_0 is a free parameter. It is remarkable that the static energy (5.2) of all solutions described by (6.1) is vanishing. Thus, there is an infinite degeneracy with respect to r_0. This degeneracy is a result of the symmetry breakdown of the dilation (4.56). The vorticity of this solution is given by $n = (\alpha/|\alpha|)(N-1)$; compare (6.1) with (5.5). When we switch on the external magnetic field B adiabatically, this solution is expected to become the static self-dual nontopological vortex we analyzed; see case (iv) of Fig. 3. On the other hand, if we switch on B suddenly, say at $t = 0$, the above static solution will begin an oscillation. We now construct exact solutions corresponding to such oscillating vortices.

The easiest way to construct such a solution is to employ a method developed in Ref. 12, by way of which we may relate the Schrödinger system with B to the same system without B. Let us consider the following set of transformations:

$$x'^0 = \frac{1}{\omega}\tan(\omega x^0) ,$$
$$\mathbf{x}' = \begin{pmatrix} 1 & \tan(\omega x^0) \\ -\tan(\omega x^0) & 1 \end{pmatrix}\mathbf{x} ,$$
$$\psi'(x'^0,\mathbf{x}') = \cos(\omega t)\exp\left(i\frac{m\omega}{2}r^2\tan(\omega t)\right)\psi(x^0,\mathbf{x}) ,$$
$$a'_\mu(x'^0,\mathbf{x}') = \frac{\partial x^\nu}{\partial x'^\mu}a_\nu(x^0,\mathbf{x}) , \tag{6.3}$$

where

$$\omega = \frac{eB}{2m} . \tag{6.4}$$

Note that this is a unitary transformation since $\int d^2x\, |\psi|^2$ is invariant. It is straightforward to show that the action $S \equiv \int d^3x\, \mathcal{L}$ with the Chern-Simons Lagrangian (2.1) is transformed into the equivalent action $S \equiv \int d^3x' \mathcal{L}'$ but with

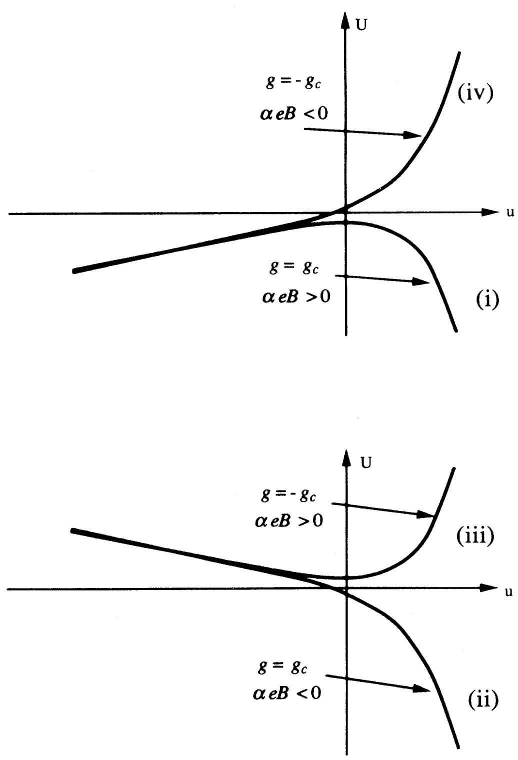

FIG. 3. Potential U. From this potential we can draw the following conclusions for self-dual vortices. (i) When $g = g_c$ and $\alpha eB > 0$, both topological and nontopological vortices exist. (ii) When $g = g_c$ and $\alpha eB < 0$, neither topological nor nontopological vortices exist. (iii) When $g = -g_c$ and $\alpha eB > 0$, topological vortices exist (but with an infinite energy) but nontopological vortices do not exist. (iv) When $g = -g_c$ and $\alpha eB < 0$, topological vortices do not exist but nontopological vortices exist, which is smoothly reduced to the Jackiw-Pi solution as $B \to 0$.

$$\mathcal{L}' = \psi'^\dagger(i\partial_0' + a_0')\psi' - \frac{1}{2m}|(i\partial_k' + a_k')\psi'|^2 - \mathcal{V}(|\psi'|^2) - \frac{1}{4\alpha}\varepsilon^{\mu\nu\lambda}a_\mu'\partial_\nu' a_\lambda' . \quad (6.5)$$

This is the CS system without B. Consequently, we are able to create solutions of the CS system with B from those without B by using the above transformation.

In particular, when the potential $\mathcal{V}$ is given by (2.4) with the critical coupling constant $g = -g_c$, there are analytic solutions (6.1). From these self-dual solutions we get

$$\psi(t,r,\theta) = \exp\Big[-i\Big(\frac{m\omega}{2}r^2\tan(\omega t) + \frac{\alpha}{|\alpha|}(N-1)\omega t\Big)\Big]\exp\Big(i\frac{\alpha}{|\alpha|}(N-1)\theta\Big) \times \sqrt{\frac{2}{|\alpha|}}\frac{N}{r}\Big[\Big(\frac{r}{r_0\cos(\omega t)}\Big)^N + \Big(\frac{r_0\cos(\omega t)}{r}\Big)^N\Big]^{-1} . \quad (6.6)$$

They are solutions of the CS system (2.1) with B. Note that they are reduced to the static solutions (6.1) at $t = 0$. The energy is calculated as

$$E = -\frac{eB}{m}\frac{\pi}{\alpha}N + \frac{e^2B^2\pi^2 r_0^2}{4|\alpha|m\sin(\pi/N)} . \quad (6.7)$$

The density $|\psi|^2$ oscillates in the radial direction with the frequency 2ω. It is interesting to notice that the degeneracy with respect to the length scale r_0 in the original self-dual solution (6.1) is removed when B is switched on. This is due to an explicit breakdown of the dilation symmetry (4.55).

Since the vortex soliton carries an electric charge, it is expected to make a cyclotron motion. The corresponding analytic solution is also obtainable from the self-dual solution (6.1) after making the Galilean boost (4.22):

$$\psi(t,r,\theta) = \exp\Big[-i\Big(\frac{m\omega}{2}r^2\tan(\omega t) + \frac{\alpha}{|\alpha|}(N-1)\omega t\Big)\Big]\exp\Big[i\Big(\frac{2m\omega}{\sin(2\omega t)}\mathbf{x}\mathbf{R}(t) - \frac{m\mathbf{v}^2}{2\omega}\tan(\omega t)\Big)\Big] \times \exp\Big(i\frac{\alpha}{|\alpha|}(N-1)\theta(\mathbf{x}-\mathbf{R}(t))\Big)\sqrt{\frac{2}{|\alpha|}}\frac{N}{r(t)}\Big[\Big(\frac{r(t)}{r_0\cos(\omega t)}\Big)^N + \Big(\frac{r_0\cos(\omega t)}{r(t)}\Big)^N\Big]^{-1} , \quad (6.8)$$

with $r(t) = |\mathbf{x} - \mathbf{R}(t)|$, $\theta(\mathbf{x}) = \arctan^{-1}(x^2/x^1)$, and

$$\mathbf{R}(t) = \frac{\sin(\omega t)}{\omega}\begin{pmatrix}\cos(\omega t) & -\sin(\omega t)\\ \sin(\omega t) & \cos(\omega t)\end{pmatrix}\mathbf{v} , \quad (6.9)$$

where $\mathbf{v}$ is the velocity of the vortex and is a free parameter. A detailed analysis of time-dependent vortex solitons in the external magnetic field will be published elsewhere.[13]

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