Anyon model on a cylinder

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The anyon model of superconductivity is investigated on a cylinder. The nonintegrable phase of the Wilson-line integral plays an essential role to maintain the translation invariance of the system. The response functions are evaluated in a closed form for both neutral and charged anyon gases.

I. INTRODUCTION

Recently much attention has been given to an anyon model [1,2], or more specifically, a nonrelativistic electron system with Chern-Simons interactions, which may describe newly discovered high-T_c superconductors [3-27]. Its ground state has completely filled Landau levels with respect to the Chern-Simons magnetic field. The basic physics involves the quantum Hall effect [28] and de Haas–van Alphen effect [29]. It has been shown that the model exhibits a (partial) Meissner effect, and therefore a superconductivity at both zero [3–7] and finite temperature [14,16].

It is well known that in the presence of a uniform magnetic field two translation operators do not commute with each other [30]. Also, if the system is defined on a multiply connected manifold such as a torus, two unitary operators which generate large gauge transformations along two distinct noncontractible loops do not commute with each other [31]. Here two kinds of invariances are at issue, one under spatial translations and the other under large gauge transformations.

These are intertwined with each other. In this paper we analyze the anyon model on a cylinder ($S^1\times R^1$). It will be shown that both the translation and gauge invariance of the ground state can be maintained only if the nonintegrable phase of the Wilson-line integral along S^1 is properly taken into account [32].

After establishing the ground-state wave function, we develop a perturbation theory in the presence of small gauge field fluctuations to determine the response functions for neutral and charged anyon gases. As an application, the Meissner effect is examined in detail as the circumference of the cylinder varies.

II. MODEL

The Lagrangian density of the model [3,4,6] is

$$\begin{aligned}\mathcal{L}=&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{e^2\nu_0}{2\pi}\varepsilon^{\lambda\nu\rho}a_\lambda\partial_\nu a_\rho+en_e A_0\\&+i\psi^\dagger D_0\psi-\frac{1}{2m}|D_k\psi|^2\ ,\\ D_0=&\ \partial_0+ie(A_0+a_0),\quad D_k=\partial_k-ie(A^k+a^k)\ .\end{aligned}\tag{2.1}$$

Here A_μ and a_μ are electromagnetic and Chern-Simons gauge fields, respectively. ψ is a two-component nonrelativistic electron (hole) field. We have excluded magnetic-moment interactions. ν_0 is an integer parameter. In the following discussions we take $e,\nu_0>0$ for definiteness. If $\psi(x)$ is a spinless fermion, then the model reduces to the holon model [33].

Coordinates of a cylinder $S^1\times R^1$ are (x_1,x_2): $x_1\in[0,L]$ and $x_2\in(-\infty,+\infty)$. x_1 and x_1+L are identified. Although the fields on the cylinder need to be single valued only up to gauge transformations

$$\begin{aligned}&A_\mu(x_1+L)=A_\mu(x)+\frac{1}{e}\partial_\mu\beta_{\rm EM}(x)\ ,\\&a_\mu(x_1+L)=a_\mu(x)+\frac{1}{e}\partial_\mu\beta_{\rm CS}(x)\ ,\\&\psi(x_1+L)=e^{-i\beta_{\rm E.M.}(x)-i\beta_{\rm CS}(x)}\psi(x)\ ,\end{aligned}\tag{2.2}$$

we assume in this paper that $\beta_{\rm EM}(x)=\beta_{\rm CS}(x)=0$, i.e., that all fields are single valued.

Under gauge transformations

$$\begin{aligned}&A'_\mu=A_\mu+\frac{1}{e}\partial_\mu\Omega(x)\ ,\\&a'_\mu=a_\mu+\frac{1}{e}\partial_\mu\omega(x)\ ,\\&\psi'=e^{-i(\omega+\Omega)}\psi\ .\end{aligned}\tag{2.3}$$

Within the boundary condition (2.2) the most general ω and Ω are

$$\begin{aligned}&\omega(x)=\frac{2\pi m_1x_1}{L}+\overline{\omega}(x)\ ,\\&\Omega(x)=\frac{2\pi m_2x_1}{L}+\overline{\Omega}(x)\ ,\end{aligned}\tag{2.4}$$

where m_1 and m_2 are integers, and $\overline{\omega}(x)$ and $\overline{\Omega}(x)$ are arbitrary periodic functions.

The $(t,\mathbf{x})$-independent parts of a_1 and A_1 are physical degrees of freedom on the cylinder, and play an important role. We define

$$\begin{aligned}&a^1(x)=\frac{1}{eL}\phi_{\rm CS}+(x\text{-dependent terms})\ ,\\&A^1(x)=\frac{1}{eL}\phi_{\rm EM}+(x\text{-dependent terms})\ .\end{aligned}\tag{2.5}$$

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Then gauge-invariant Wilson line integrals along a noncontractible loop ($x_2=$const) on the cylinder become

$$\begin{aligned} W_{\rm CS}[a_1;t,x_2]&=\exp\left[ie\int_0^L dx_1 a^1\right] \\ &=e^{i\phi_{\rm CS}}W'_{\rm CS}[f_{\mu\nu};t,x_2]\ , \\ W_{\rm EM}[A_1;t,x_2]&=\exp\left[ie\int_0^L dx_1 A^1\right] \\ &=e^{i\phi_{\rm EM}}W'_{\rm EM}[F_{\mu\nu};t,x_2]\ . \end{aligned} \tag{2.6}$$

$\phi_{\rm CS}$ and $\phi_{\rm EM}$ are nonintegrable phases of the Wilson-line integrals, which are undetermined by the fields strengths $f_{\mu\nu}$ and $F_{\mu\nu}$. As is shown below, the wave function of the system has nontrivial dependence on $\phi_{\rm CS}+\phi_{\rm EM}$, without which the translation invariance cannot be maintained.

Note that, under a gauge transformation (2.4),

$$\begin{aligned} &\phi_{\rm CS}\to\phi_{\rm CS}-2\pi m_1\ , \\ &\phi_{\rm EM}\to\phi_{\rm EM}-2\pi m_2\ , \end{aligned} \tag{2.7}$$

that is, physics must be periodic in $\phi_{\rm CS}$ and $\phi_{\rm EM}$. Transformations with m_1 or $m_2\neq 0$ are called large gauge transformations.

The Euler equations for the gauge fields are given by

$$\begin{aligned} &-\frac{e^2\nu_0}{2\pi}\varepsilon^{\lambda\nu\rho}f_{\nu\rho}=j^\lambda\ , \\ &\partial_\nu F^{\nu\lambda}+en_e\delta^{\lambda 0}=j^\lambda\ . \end{aligned} \tag{2.8}$$

Here

$$\begin{aligned} &j^0=e\psi^\dagger\psi\ , \\ &j^k=-\frac{ie}{2m}[\psi^\dagger D_k\psi-(D_k\psi)^\dagger\psi]\ . \end{aligned} \tag{2.9}$$

The electron field obeys

$$i\partial_0\psi=\left[-\frac{1}{2m}D_k^2+e(A_0+a_0)\right]\psi\ . \tag{2.10}$$

We solve these equations in the self-consistent field approximation in which the currents in Eq. (2.8) are replaced by the expectation value

$$\begin{aligned} &-\frac{e^2\nu_0}{2\pi}\varepsilon^{\lambda\nu\rho}f_{\nu\rho}=\langle j^\lambda\rangle\ , \\ &\partial_\nu F^{\nu\lambda}+en_e\delta^{\lambda 0}=\langle j^\lambda\rangle\ , \\ &\langle j^\lambda\rangle=\langle\Psi(f,F)|j^\lambda|\Psi(f,F)\rangle\ . \end{aligned} \tag{2.11}$$

Here $|\Psi(f,F)\rangle$ is the wave function in the electron (ψ), $\phi_{\rm CS}$, and $\phi_{\rm EM}$ sectors with given $f_{\mu\nu}$ and $F_{\mu\nu}$.

III. GROUND STATE

We shall seek the wave function $|\Psi(f,F)\rangle$ with given $f_{\mu\nu}$ and $F_{\mu\nu}$ in the form

$$\begin{aligned} &|\Psi(f,F)\rangle=\int_0^{2\pi}d\phi\, f(\phi)|\phi\rangle\otimes|\Psi_e(\phi)\rangle\ , \\ &f(\phi+2\pi)=f(\phi),\quad |\Psi_e(\phi+2\pi)\rangle=|\Psi_e(\phi)\rangle\ . \end{aligned} \tag{3.1}$$

Here $|\phi\rangle$ is an eigenstate of the Wilson-line phase $(\phi_{\rm CS}+\phi_{\rm EM})|\phi\rangle=\phi|\phi\rangle$, and $|\Psi_e(\phi)\rangle$ is a state in the electron sector with given $f_{\mu\nu}$, $F_{\mu\nu}$, and $\phi_{\rm CS}+\phi_{\rm EM}=\phi$. The value of $\phi_{\rm CS}-\phi_{\rm EM}$ is arbitrary, since electrons see only the sum ϕ. $|\Psi(f,F)\rangle$ is invariant under large gauge transformations.

In this section we determine the wave function of the ground state in which there is neither macroscopic current nor any electromagnetic fields: $\langle j^0\rangle=en_e$, $\langle \mathbf{j}\rangle=0$, and $F_{\mu\nu}=0$. In the following sections we shall look at states with $\langle \mathbf{j}\rangle\neq 0$ and $B\neq 0$. Equation (2.11) implies that $-ef_{12}=\pi n_e/\nu_0\equiv l^{-2}$, $f_{0k}=0$. We choose the Landau gauge $e(a^1+A^1)=(\phi/L)-(x_2/l^2)$, which satisfies the periodic boundary condition in the x_1 direction. The corresponding one-particle Schrödinger equation is

$$\begin{aligned} &H_0(\phi)\psi_{np\sigma}(\mathbf{x};\phi)=\epsilon_n\psi_{np\sigma}(\mathbf{x};\phi)\ , \\ &H_0(\phi)=-\frac{1}{2m}\left[\left(\partial_1-i\frac{\phi}{L}+i\frac{x_2}{l^2}\right)^2+\partial_2^2\right]\ , \\ &\epsilon_n=(n+\tfrac{1}{2})\frac{1}{ml^2}\ , \\ &\psi_{np\sigma}(\mathbf{x};\phi)=\frac{1}{\sqrt{lL}}e^{-2\pi ipx_1/L}u_n[\xi(x_2,p,\phi)]v_\sigma\ , \\ &\xi(x_2,p,\phi)=\frac{x_2}{l}-\frac{(2\pi p+\phi)l}{L}\ , \end{aligned} \tag{3.2}$$

where $n=0,1,2,\ldots,p\in\mathbb{Z}$, and

$$\begin{aligned} &u_n(z)=\frac{(-1)^n}{2^{n/2}(n!)^{1/2}\pi^{1/4}}e^{z^2/2}\frac{d^n}{dz^n}e^{-z^2}\ , \\ &\int_{-\infty}^{\infty}dz\, u_n(z)u_m(z)=\delta_{nm}\ . \end{aligned} \tag{3.3}$$

v_σ is a spin-wave function ($v_\sigma^\dagger v_\rho=\delta_{\sigma\rho}$).

It is important to recognize that the structure of the Landau levels [34] does not depend on the value of ϕ, which implies that all ϕ's are equally likely. This is very special to the particular Chern-Simons model (2.1). In general gauge theory models the effective potential has nontrivial dependence on nonintegrable phases of Wilson-line integrals [35–40]. Since the energy spectrum does not depend on ϕ, the one-loop effective potential for ϕ is completely flat. Hence we choose, in (3.1),

$$f(\phi)=\text{const}=\frac{1}{\sqrt{2\pi}}\ . \tag{3.4}$$

We expand the electron field operator ψ as

$$\begin{aligned} &\psi(t,\mathbf{x})=\sum_{np\sigma}a_{np\sigma}^{(0)}(\phi)\psi_{np\sigma}(\mathbf{x};\phi)e^{-i\epsilon_n t}\ , \\ &\{a_{np\sigma}^{(0)},a_{mq\rho}^{(0)\dagger}\}=\delta_{nm}\delta_{pq}\delta_{\sigma\rho}\ . \end{aligned} \tag{3.5}$$

We define $|\Psi_e^{(0)}(\phi)\rangle$ as a state with the lowest ν_0 Landau levels completely filled:

$$|\Psi_e^{(0)}(\phi)\rangle=\prod_{n=0}^{\nu_0-1}\prod_{p,\sigma}a_{np\sigma}^{(0)\dagger}(\phi)|0\rangle\ . \tag{3.6}$$

Then the ground state is

$$|\Psi_g\rangle=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}d\phi|\phi\rangle\otimes|\Psi_e^{(0)}(\phi)\rangle \ . \tag{3.7}$$

We now demonstrate that the state $|\Psi_e^{(0)}(\phi)\rangle$ itself is not translation invariant, but the ground state $|\Psi_g\rangle$ is. Indeed,

$$\begin{aligned}\langle\Psi_e^{(0)}(\phi)|j^0(x)|\Psi_e^{(0)}(\phi)\rangle\\ &=e\sum_{n=0}^{\nu_0-1}\sum_{p,\sigma}\psi_{np\sigma}^\dagger(\mathbf{x};\phi)\psi_{np\sigma}(\mathbf{x};\phi)\\ &=e\sum_{n=0}^{\nu_0-1}\frac{2}{lL}\sum_p u_n[\xi(x_2,p,\phi)]^2 \ ,\end{aligned} \tag{3.8}$$

which has nontrivial x_2 dependence, but

$$\begin{aligned}\langle\Psi_g|j^0(x)|\Psi_g\rangle&=\frac{1}{2\pi}\int_0^{2\pi}d\phi\langle\Psi_e^{(0)}(\phi)|j^0(x)|\Psi_e^{(0)}(\phi)\rangle\\ &=\sum_{n=0}^{\nu_0-1}\frac{e}{\pi lL}\int_0^{2\pi}d\phi\sum_p u_n[\xi(x_2,p,\phi)]^2\\ &=\sum_{n=0}^{\nu_0-1}\frac{e}{\pi l^2}\int_{-\infty}^{\infty}dz\ u_n(z)^2\\ &=\frac{e\nu_0}{\pi l^2}=en_e \ .\end{aligned} \tag{3.9}$$

It is easy to understand why the nonintegrable phase of the Wilson-line integral in the x_1 direction, ϕ, is important to maintain the translation invariance in the x_2 direction. In the ground state $e(a^1+A^1)=(\phi/L)-(x_2/l^2)$ so that a translation $T_2(d;x_2\to x_2-d)$ transforms ϕ into $\phi+(Ld/l^2)$. In particular,

$$\begin{aligned}T_2(d)|\phi\rangle&=\left|\phi+\frac{Ld}{l^2}\right\rangle \ ,\\ T_2(d)|\Psi_e^{(0)}(\phi)\rangle&=\left|\Psi_e^{(0)}\left[\phi+\frac{Ld}{l^2}\right]\right\rangle \ .\end{aligned} \tag{3.10}$$

The latter relation may be explicitly checked from the definition (3.6). Clearly $|\Psi_g\rangle$ is invariant under $T_2(d)$.

Similarly,

$$\begin{aligned}\langle\Psi_g|j^1(x)|\Psi_g\rangle&=\sum_{n=0}^{\nu_0-1}\frac{e}{\pi m l^2L}\int_0^{2\pi}d\phi\sum_p\xi(x_2,p,\phi)\\ &\qquad\times u_n[\xi(x_2,p,\phi)]^2\\ &=\sum_{n=0}^{\nu_0-1}\frac{e}{\pi m l^3}\int_{-\infty}^{\infty}dz\ zu_n(z)^2=0 \ ,\\ \langle\Psi_g|j^2(x)|\Psi_g\rangle&=0 \ .\end{aligned} \tag{3.11}$$

With (3.9) and (3.11) inserted, Eq. (2.11) is consistently solved by the configuration $-ef_{12}=l^{-2}, f_{0k}=F_{\mu\nu}=0$.

In this section we have shown that the state (3.7) gives a translation-invariant, self-consistent ground state. The Wilson-line phase ϕ in the x_1 direction is essential to maintain the translation invariance in the x_2 direction.

IV. HALL CURRENTS IN A NEUTRAL ANYON GAS

Next we consider static states in which there is nonvanishing current $\langle\mathbf{j}\rangle\neq 0$. At this stage one has to distinguish two theories without and with electromagnetic fields, namely, theories of neutral and charged anyon gases. If electromagnetic fields are absent, one can have a uniform constant current $\langle j^1\rangle$. To see this, consider a Chern-Simons field configuration

$$\begin{aligned}&a_0=-\overline{\mathscr{E}}x_2, \quad a^1=\frac{1}{eL}\phi-bx_2, \quad a^2=0 \ ,\\ &-f_{12}=b=\frac{\pi n_e}{e\nu_0}, \quad f_{01}=0, \quad f_{02}=\overline{\mathscr{E}} \ .\end{aligned} \tag{4.1}$$

The corresponding one-particle Schrödinger equation is

$$\begin{aligned}&\left\{-\frac{1}{2m}\left[\left[\partial_1-i\frac{\phi}{L}+i\frac{x_2}{l^2}\right]^2+\partial_2^2\right]-e\overline{\mathscr{E}}x_2\right\}\psi_{np\sigma}(\mathbf{x};\phi,\overline{\mathscr{E}})\\ &\qquad\qquad=\epsilon_{np}\psi_{np\sigma}(\mathbf{x};\phi,\overline{\mathscr{E}}) \ ,\\ &\epsilon_{np}=\epsilon_n-e\overline{\mathscr{E}}\frac{(2\pi p+\phi)l^2}{L}-\frac{1}{2}me^2\overline{\mathscr{E}}^2l^4 \ ,\\ &\psi_{np\sigma}(\mathbf{x};\phi,\overline{\mathscr{E}})=\psi_{np\sigma}(x_1,x_2-me\overline{\mathscr{E}}l^4;\phi) \ ,\end{aligned} \tag{4.2}$$

where $\psi_{np\sigma}(\mathbf{x};\phi)$ is defined in (3.2). The expansion of $\psi(t,\mathbf{x})$ in terms of $[\psi_{np\sigma}(\mathbf{x};\phi,\overline{\mathscr{E}})]$ defines annihilation operators $a_{np\sigma}(\phi;\overline{\mathscr{E}})$, from which we define an electron state

$$|\Psi_e(\phi;\overline{\mathscr{E}})\rangle=\prod_{n=0}^{\nu_0-1}\prod_{p,\sigma}a_{np\sigma}^\dagger(\phi;\overline{\mathscr{E}})|0\rangle \ .$$

Since the system is translation invariant in the x_2 direction, we expect that all values of ϕ are equally likely. Hence we are led to considering a state

$$|\Psi_1(\overline{\mathscr{E}})\rangle=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}d\phi|\phi\rangle\otimes|\Psi_e(\phi;\overline{\mathscr{E}})\rangle \ . \tag{4.3}$$

Evaluation of currents proceeds similarly as in (3.9) and (3.11), resulting in $\langle j^0(x)\rangle=en_e$ and $\langle j^2(x)\rangle=0$. However, for $\langle j^1(x)\rangle$,

$$\begin{aligned}\langle\Psi_1(\overline{\mathscr{E}})|j^1(x)|\Psi_1(\overline{\mathscr{E}})\rangle&=\sum_{n=0}^{\nu_0-1}\frac{e}{\pi m l^2L}\int_0^{2\pi}d\phi\sum_p\xi u_n(\xi-me\overline{\mathscr{E}}l^3)^2\\ &=\sum_{n=0}^{\nu_0-1}\frac{e}{\pi m l^3}\int_{-\infty}^{\infty}dz(z+me\overline{\mathscr{E}}l^3)u_n(z)^2=\frac{e^2\nu_0}{\pi}\overline{\mathscr{E}} \ .\end{aligned} \tag{4.4}$$

where $\xi=\xi(x_2,p,\phi)$. It can be seen that (4.1) and (4.4) solve the Chern-Simons equation (2.11). Equation (4.4) represents integer quantum Hall effect with respect to the Chern-Simons magnetic field $b=\pi n_e/e\nu_0$.

In the presence of electromagnetic fields a simple configuration with constant current and electric field does not solve the Maxwell equation. A constant current configuration $j^1(x)=c$ implies $B=cx_2+B_0$ so that the Landau-level picture itself breaks down.

V. INDUCED CURRENTS

As shown in Sec. III, $F_{\mu\nu}=f_{0k}=0$ but $b=-f_{12}=\pi n_e/e\nu_0$ in the ground state. When a small external field is applied, a self-consistent gauge field configuration becomes $a_\mu=a_\mu^{(0)}+a_\mu^{(1)}$ and $A_\mu\neq 0$. We would like to find out how the system responds to an external field.

Since the electron field equation (2.10) depends on the gauge fields only in the combination $a_\mu+A_\mu$, the electron wave function $|\Psi_e(\phi)\rangle$ is a function of $a_\mu^{(1)}+A_\mu$. Furthermore, if an external field is x_2 independent so that the system is translation invariant in the x_2 direction, the wave function (3.4) of the Wilson-line phase ϕ remains intact. Hence, for $a_\mu^{(1)}(x_1)+A_\mu(x_1)$,

$$|\Psi(a+A)\rangle=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}d\phi|\phi\rangle\otimes|\Psi_e(\phi)\rangle$$
$$=|\Psi_g\rangle+|\Psi^{(1)}(a^{(1)}+A)\rangle+\cdots \ . \quad (5.1)$$

Therefore the induced current

$$J^\mu_{\rm ind}(\mathbf{x})=\langle\Psi|J^\mu(\mathbf{x})|\Psi\rangle-\langle\Psi_g|J^\mu(\mathbf{x})|\Psi_g\rangle$$
$$=\langle\Psi|J^\mu(\mathbf{x})|\Psi\rangle-en_e\delta^{\mu0}\ , \quad (5.2)$$

being a function of $a_\mu^{(1)}(x_1)+A_\mu(x_1)$, is given to the leading order by [6,16]

$$J^\mu_{\rm ind}(x_1)=-\frac{1}{L}\int_0^L dy_1 K^\mu_\nu(x_1,y_1)[a^{(1)\nu}(y_1)+A^\nu(y_1)]\ . \quad (5.3)$$

In Fourier series representation,

$$f(x_1)=\sum_{q=-\infty}^{\infty}f(q)e^{2\pi iqx_1/L}\ ,$$
$$f(q)=\frac{1}{L}\int_0^L dx_1 f(x_1)e^{-2\pi iqx_1/L}\ .$$

Relation (5.3) becomes

$$J^\mu_{\rm ind}(q)=-K^\mu_\nu(q)[a^{(1)\nu}(q)+A^\nu(q)]\ ,$$
$$K^\mu_\nu(x_1,y_1)=\sum_q K^\mu_\nu(q)e^{2\pi iq(x_1-y_1)/L}\ . \quad (5.4)$$

To find $K^\mu_\nu(q)$, we notice that for a sufficiently small fluctuation $a^{(1)\nu}+A^\nu$, the corresponding electron state still has the structure of completely filled Landau levels, though wave function of each electron suffers a perturbation. In the radiation gauge $\text{div}\,\mathbf{a}=\text{div}\,\mathbf{A}=0$, $a^{(1)1}(x_1)+A^1(x_1)=0$. The one-particle Schrödinger equation becomes

$$H\psi(\mathbf{x})=\epsilon\psi(\mathbf{x})\ ,$$
$$H=H_0+V_1+V_2\ ,$$
$$V_1=\frac{ie}{m}(a^2+A^2)\partial_2+e(a^0+A^0)\ ,$$
$$V_2=\frac{e^2}{2m}(a^2+A^2)^2\ . \quad (5.5)$$

H_0 is given by (3.2).

We shall determine eigenstates $\psi(\mathbf{x})$ in perturbation theory. We recognize that the appropriate zeroth-order basis is not $[\psi_{np\sigma}(\mathbf{x};\phi)]$ in (3.2), but the corresponding θ basis

$$\psi^{(0)}_{n\theta\sigma}(\mathbf{x};\phi)\equiv\frac{1}{\sqrt{2\pi}}\sum_{p=-\infty}^{\infty}e^{ip\theta}\psi_{np\sigma}(\mathbf{x};\phi)\ . \quad (5.6)$$

Matrix elements of the perturbation V_1, though not diagonal in p, are diagonal in θ:

$$\langle n'\theta'\sigma'|V_1|n\theta\sigma\rangle=\delta_{\sigma'\sigma}\frac{1}{2\pi}\sum_{p,p'}e^{-ip'\theta'+ip\theta}\frac{1}{Ll}$$
$$\times\int_0^L dx_1\int_{-\infty}^{\infty}dx_2 e^{2\pi i(p'-p)x_1/L}u_{n'}(\xi')\left[\frac{ie}{m}(a^2+A^2)(x_1)\partial_2+e(a^0+A^0)(x_1)\right]u_n(\xi)\ ,$$

where $\xi=\xi(x_2,p,\phi)$ and $\xi'=\xi(x_2,p',\phi)$. Introducing $q=p-p'$ and converting the x_2 integral to the integral over $z=\xi(x_2,\frac{1}{2}(p+p'),\phi)$, one finds

$$\langle n'\theta'\sigma'|V_1|n\theta\sigma\rangle=\delta_{\sigma'\sigma}\delta(\theta-\theta')\langle n'|V_1|n\rangle_\theta\ ,$$
$$\langle n'|V_1|n\rangle_\theta=\sum_{q=-\infty}^{\infty}e^{iq\theta}\left[\frac{ie}{ml}(a^2+A^2)(q)C^{(1)}_{n'n}(\alpha_q)+e(a^0+A^0)(q)C^{(0)}_{n'n}(\alpha_q)\right]\ , \quad (5.7)$$
$$\alpha_q=\frac{\pi lq}{L},\quad C^{(p)}_{n'n}(\alpha)=\int_{-\infty}^{\infty}dz\,u_{n'}(z+a)\frac{d^p}{dz^p}u_n(z-a)\ .$$

Note that the matrix elements are independent of ϕ in the θ basis, and $\langle n'|V_1|n\rangle_\theta^* = \langle n|V_1|n'\rangle_\theta$. The integration in $C_{n'n}^{(p)}(a)$ can be performed in a closed form making use of recursion formulas for $u_n(z)$'s and [41]

$$\int_{-\infty}^{\infty} dz\, u_m(z+a)u_n(z-a) = 2^{(m-n)/2}\left[\frac{n!}{m!}\right]^{1/2} a^{m-n}e^{-a^2}L_n^{m-n}(2a^2)\,,$$
$$L_n^{m-n}(x) = \frac{1}{n!}x^{-m+n}e^{x}\frac{d^n}{dx^n}(x^m e^{-x}) \quad \text{for } m \ge n\,. \tag{5.8}$$

In the θ basis the Schrödinger equation reads

$$H\psi_{n\theta\sigma}(\mathbf{x}) = \epsilon_{n\theta}\psi_{n\theta\sigma}(\mathbf{x}),\quad \epsilon_{n\theta} = \epsilon_n + \epsilon_{n\theta}^{(1)} + \cdots\,,$$
$$\psi_{n\theta\sigma}(\mathbf{x}) = \psi_{n\theta\sigma}^{(0)}(\mathbf{x}) + \psi_{n\theta\sigma}^{(1)}(\mathbf{x}) + \cdots,\quad \psi_{n\theta\sigma}^{(1)}(\mathbf{x}) = \sum_{n'\neq n}\frac{\langle n'|V_1|n\rangle_\theta}{\epsilon_n - \epsilon_{n'}}\psi_{n'\theta\sigma}^{(0)}(\mathbf{x})\,. \tag{5.9}$$

With these eigenstates the electron wave function is given by

$$\psi(t,\mathbf{x}) = \sum_{n\theta\sigma} a_{n\theta\sigma}(\phi)\psi_{n\theta\sigma}(\mathbf{x};\phi)e^{-i\epsilon_{n\theta}t},\quad |\Psi_e(\phi)\rangle = \prod_{n=0}^{\nu_0-1}\prod_{\theta,\sigma} a_{n\theta\sigma}^\dagger(\phi)|0\rangle\,. \tag{5.10}$$

The computation of the induced current J_{ind}^μ, (5.2), is straightforward. J_{ind}^0 is

$$J_{\text{ind}}^0(\mathbf{x}) = \frac{e}{2\pi}\int_0^{2\pi}d\phi\int_0^{2\pi}d\theta\sum_{\sigma=\pm1}\sum_{n=0}^{\nu_0-1}(\psi_{n\theta\sigma}^{(0)*}\psi_{n\theta\sigma}^{(1)} + \text{c.c.})$$
$$= \frac{e}{2\pi}\int_0^{2\pi}d\phi\, d\theta\sum_\sigma\sum_{n=0}^{\nu_0-1}\sum_{n'\neq n}\frac{ml^2}{n-n'}\left[\langle n'|V_1|n\rangle_\theta\frac{1}{2\pi}\sum_p\sum_{p'}e^{-i(p-p')\theta}\psi_{np\sigma}^{(0)*}\psi_{n'p'\sigma}^{(0)} + \text{c.c.}\right]\,.$$

Insertion of (5.7) leads to

$$J_{\text{ind}}^0(\mathbf{x}) = \frac{2em}{\pi}\sum_{n=0}^{\nu_0-1}\sum_{n'\neq n}\frac{1}{n-n'}\sum_{q=-\infty}^{\infty}e^{2\pi iqx_1/L}C_{n'n}^{(0)}(\alpha_q)\left[\frac{ie}{ml}(a^2+A^2)(q)C_{n'n}^{(1)}(\alpha_q) + e(a^0+A^0)(q)C_{n'n}^{(0)}(\alpha_q)\right]\,. \tag{5.11}$$

For J_{ind}^k we have

$$J_{\text{ind}}^1(x_1) = -\frac{ie}{2m}\langle\psi^\dagger D_1\psi\rangle + \text{c.c.}$$
$$= -\frac{e}{2m}\int_0^{2\pi}d\phi\int_0^{2\pi}d\theta\sum_{\sigma=\pm1}\sum_{n=0}^{\nu_0-1}\{[\psi_{n\theta\sigma}^{(0)*}(i\partial_1 + ea^{1(0)})\psi_{n\theta\sigma}^{(1)} + \text{c.c.}] + [\psi_{n\theta\sigma}^{(1)*}(i\partial_1 + ea^{1(0)})\psi_{n\theta\sigma}^{(0)} + \text{c.c.}]\}$$
$$= \frac{e}{2\pi l}\sum_{n=0}^{\nu_0-1}\sum_{n'\neq n}\frac{1}{n-n'}\left[\sum_{q=-\infty}^{\infty}e^{i2\pi qx_1/L}\left[\frac{ie}{ml}[A^2(q)+a^2(q)]C_{n'n}^{(1)}(\alpha_q) + e[A^0(q)+a^0(q)]C_{n'n}^{(0)}(\alpha_q)\right]\right.$$
$$\times\left[\int_{-\infty}^{\infty}d\xi(\xi+\alpha_q)u_{n'}(\xi+\alpha_q)u_n(\xi-\alpha_q)\right.$$
$$\left.\left. + (-1)^{n+n'}\int_{-\infty}^{\infty}d\xi(\xi-\alpha_q)u_{n'}(\xi-\alpha_q)u_n(\xi+\alpha_q)\right] + (n\leftrightarrow n')\right] = 0 \tag{5.12}$$

and

$$J_{\text{ind}}^2(x_1) = -\frac{ie}{2m}\langle\psi^\dagger D_2\psi\rangle + \text{c.c.}$$
$$= -\frac{e^2}{m}\langle\psi^\dagger\psi\rangle(a^2+A^2) - \frac{e}{2m}\int_0^{2\pi}d\phi\int_0^{2\pi}d\theta\sum_{\sigma=\pm1}\sum_{n=0}^{\nu_0-1}[\psi_{n\theta\sigma}^{(0)*}(i\partial_2)\psi_{n\theta\sigma}^{(1)} + \psi_{n\theta\sigma}^{(1)*}(i\partial_2)\psi_{n\theta\sigma}^{(0)} + \text{c.c.}]$$
$$= -\frac{e^2 n_e}{m}(a^2+A^2)(x_1)$$
$$+ \frac{2ie}{\pi l}\sum_{n=0}^{\nu_0-1}\sum_{n'\neq n}\frac{1}{n-n'}\sum_{q=-\infty}^{\infty}e^{2\pi iqx_1/L}C_{n'n}^{(1)}(\alpha_q)\left[\frac{ie}{ml}(a^2+A^2)(q)C_{n'n}^{(1)}(\alpha_q) + e(a^0+A^0)(q)C_{n'n}^{(0)}(\alpha_q)\right]\,. \tag{5.13}$$

For the rest of this work we restrict ourselves to the semion (half-fermion) case ($\nu_0=1$), where the sum over n' can also be performed and will be written in terms of a function $S(x)$ which is closely related to the exponential integral. We define [41]

$$S(x)=e^{-x}\sum_{n=1}^{\infty}\frac{1}{n}\frac{x^n}{n!}=e^{-x}\int_0^x dy\frac{1}{y}(e^y-1)$$
$$\sim\begin{cases} x-\frac{3}{4}x^2+\frac{11}{36}x^3+\cdots & \text{for } x\ll 1, \\ \frac{1}{x}+\frac{1}{x^2}+\cdots+\frac{(n-1)!}{x^n}+\cdots & \text{for } x\gg 1. \end{cases} \tag{5.14}$$

The expression for $x \gg 1$ is an asymptotic expansion. Making use of (5.8) and (5.14), one can show

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}C_{n0}^{(0)}(a)^2=S(2a^2)\ ,\\
&\sum_{n=1}^{\infty}\frac{1}{n}C_{n0}^{(0)}(a)C_{n0}^{(1)}(a)=\frac{1}{2a}[2a^2S(2a^2)-1+e^{-2a^2}]\ ,\\
&\sum_{n=1}^{\infty}\frac{1}{n}C_{n0}^{(1)}(a)^2=a^2S(2a^2)-\tfrac{1}{2}+e^{-2a^2}\ .
\end{aligned} \tag{5.15}$$

Applying these formulas to (5.11) and (5.13), we find

$$J_{\mathrm{ind}}^{\mu}(q)=-K_{\nu}^{\mu}(q)[a^{(1)\nu}(q)+A^{\nu}(q)]$$

with

$$\begin{aligned}
&K_0^0(q)=\frac{2e^2m}{\pi}S(2\alpha_q^2)\ ,\\
&K_2^0(q)=K_0^2(q)=\frac{ie^2}{\pi l\alpha_q}[2\alpha_q^2S(2\alpha_q^2)-1+e^{-2\alpha_q^2}]\ ,\\
&K_2^2(q)=-\frac{2e^2}{\pi m l^2}[\alpha_q^2S(2\alpha_q^2)-1+e^{-2\alpha_q^2}],\ \text{all others}=0\ .
\end{aligned} \tag{5.16}$$

Chern-Simons fields a^{ν} can be eliminated making use of the field equations (2.8). The resulting equations relate currents to EM fields, replacing the London equations in the BCS theory. We shall find below that this form of the equations, (5.16), is more convenient and suggestive.

VI. RESPONSE FUNCTION

Response function on a plane has been previously evaluated in the random-phase approximation (RPA) [3,4,8] at $T=0$ and in the self-consistent field method [16] at both $T=0$ and $T\neq 0$. Both methods yield the same result at small momentum for a neutral anyon gas. It has also been noticed that there is an important difference between neutral and charged anyon gases [6,16].

In this paper we are examining the system on a cylinder, placing particular attention to topology and finite size of the cylinder. We have already shown that the Wilson-line phase associated with nontrivial topology of the manifold is crucial for the translation invariance of the system. In this section we examine the finite-size effect on response function.

In the $L\to\infty$ limit the $K_{\nu}^{\mu}(x_1,y_1)$'s, and therefore the response function, must reduce to those previously obtained on a plane [6]. It is easy to check this. It is convenient to introduce a momentum variable k by

$$k=\frac{2\pi q}{L}=\frac{2\alpha_q}{l}\ . \tag{6.1}$$

$K_{\nu}^{\mu}(k)$'s on a plane is given by (5.16) and (6.1).

In particular, for small momentum k one finds [recalling $l^2=(\pi n_e)^{-1}$]

$$\begin{aligned}
&K_0^0=\frac{e^2m}{\pi^2 n_e}\left[k^2-\frac{3}{8\pi n_e}k^4+\frac{11}{144\pi^2n_e^2}k^6+\cdots\right],\\
&K_2^0=K_0^2=-\frac{ie^2}{\pi}\left[k-\frac{3}{4\pi n_e}k^3+\frac{11}{48\pi^2n_e^2}k^5+\cdots\right],\\
&K_2^2=\frac{e^2}{\pi m}\left[k^2-\frac{1}{2\pi n_e}k^4+\frac{13}{96\pi^2n_e^2}k^6+\cdots\right].
\end{aligned} \tag{6.2}$$

Converting these expressions to the coordinate space by

$$k\to\frac{1}{i}\frac{\partial}{\partial x_1}$$

one finds

$$\begin{aligned} J^0_{\rm ind}(x_1) &= \frac{e^2 m}{\pi^2 n_e}\left[\frac{\partial^2}{\partial x_1^2}+\frac{3}{8\pi n_e}\frac{\partial^4}{\partial x_1^4}+\frac{11}{144\pi^2 n_e^2}\frac{\partial^6}{\partial x_1^6}\right][a^0(x_1)+A^0(x_1)] \\ &\quad +\frac{e^2}{\pi}\left[\frac{\partial}{\partial x_1}+\frac{3}{4\pi n_e}\frac{\partial^3}{\partial x_1^3}+\frac{11}{48\pi^2 n_e^2}\frac{\partial^5}{\partial x_1^5}\right][a^2(x_1)+A^2(x_1)] , \\ J^2_{\rm ind}(x_1) &= \frac{e^2}{\pi}\left[\frac{\partial}{\partial x_1}+\frac{3}{4\pi n_e}\frac{\partial^3}{\partial x_1^3}+\frac{11}{48\pi^2 n_e^2}\frac{\partial^5}{\partial x_1^5}\right][a^0(x_1)+A^0(x_1)] \\ &\quad +\frac{e^2}{\pi m}\left[\frac{\partial^2}{\partial x_1^2}+\frac{1}{2\pi n_e}\frac{\partial^4}{\partial x_1^4}+\frac{13}{96\pi^2 n_e^2}\frac{\partial^6}{\partial x_1^6}\right][a^2(x_1)+A^2(x_1)] . \end{aligned} \tag{6.3}$$

This agrees with and generalizes the result of Ref. [6].

We note that (6.3) is valid even for a finite but large L ($L \gg l$) so long as gauge configurations are smooth so that $a^{(1)\nu}(q)$ and $A^\nu(q)$ are negliblly small for large $|q|$.

In the opposite case $L \ll l$, the asymptotic expansion (5.14) of $S(x)$ may be employed:

$$\begin{aligned} K_0^0(q) &= \frac{4e^2 m}{\pi}\frac{1}{(2\alpha_q)^2}\left[1+\frac{2}{(2\alpha_q)^2}+\frac{8}{(2\alpha_q)^4}+\cdots\right] , \\ K_2^0(q) &= K_0^2(q)=\frac{4ie^2}{\pi l}\frac{1}{(2\alpha_q)^3}\left[1+\frac{4}{(2\alpha_q)^3}+\cdots\right] , \\ K_2^2(q) &= \frac{e^2 n_e}{m}\left[1-\frac{2}{(2\alpha_q)^2}-\frac{8}{(2\alpha_q)^4}+\cdots\right] . \end{aligned} \tag{6.4}$$

In the limit $L/l \to 0$, all components except for K_2^2 vanish. [One can assume that $a^{(1)\nu}(q)+A^\nu(q)$ vanishes for $q=0$.] $K_2^2(q)$ approaches $e^2 n_e/m$, the value representing the diamagnetism of an electron gas [42]. Hence we expect that the superconductivity disappears in this limit at least for x_2 independent configurations. The formula (6.4) with $2\alpha_q = kl$ is valid on a plane ($L \to \infty$) for $kl \gg 1$.

In the linear response theory we examine a response of the system to external currents of the δ function form [43]. Equations to be solved are

$$\begin{aligned} \partial_\nu F^{\nu\mu} &= J^\mu_{\rm ind} + J^\mu_{\rm EM,ext} , \\ -\frac{e^2}{2\pi}\varepsilon^{\mu\nu\rho} f_{\nu\rho} &= e n_e \delta^{\mu 0} + J^\mu_{\rm ind} + J^\mu_{\rm CS,ext} . \end{aligned} \tag{6.5}$$

External fields are defined by

$$\begin{aligned} \partial_\nu F^{\nu\mu}_{\rm ext} &= J^\mu_{\rm EM,ext} \\ -\frac{e^2}{2\pi}\varepsilon^{\mu\nu\rho} f^{\rm ext}_{\nu\rho} &= J^\mu_{\rm CS,ext} . \end{aligned} \tag{6.6}$$

Notice that $J^\mu_{\rm EM,ext}$ and $J^\mu_{\rm CS,ext}$ may be different from one another.

Solving Eqs. (5.16), (6.5), and (6.6) altogether, one finds a relation between $J^\mu_{\rm ind}$ and $(a^\mu_{\rm ext}, A^\mu_{\rm ext})$, which defines the response function. For a charged anyon gas we define $Q_c{}^\mu{}_\nu$ for x_2-independent gauge field configurations, by

$$J^\mu_{\rm ind}(q) = -k^2 Q_c{}^\mu{}_\nu(q)[a^\nu_{\rm ext}(q)+A^\nu_{\rm ext}(q)] . \tag{6.7}$$

It is checked that only the sum of $a^\nu_{\rm ext}$ and $A^\nu_{\rm ext}$ appears in the relation to $J^\mu_{\rm ind}$. For a neutral anyon gas, EM fields are absent and the coupling constant e is redundant. Analogous definition is

$$\frac{1}{e}J^\mu_{\rm ind}(q) = -\frac{k^2}{e^2} Q_n{}^\mu{}_\nu(q) e a^\nu_{\rm ext}(q) . \tag{6.8}$$

Q_n and Q_c are related to current-current correlation functions in neutral and charged anyon gases, respectively, so that they are symmetric in their indices. Q_n was first evaluated on a plane in Refs. [3], [4], and [8].

To determine Q_n and Q_c, we first write (6.5) in the radiation gauge in the form

$$\begin{aligned} &\text{neutral:}\quad L_n{}^\mu{}_\nu a^{(1)\nu} = k^2 a^\mu_{\rm ext} , \\ &\text{charged:}\quad L_c{}^\mu{}_\nu (a^{(1)\nu}+A^\nu) = k^2 (a^\mu_{\rm ext}+A^\mu_{\rm ext}) . \end{aligned} \tag{6.9}$$

For gauge field configurations under consideration the equations reduce to a 2×2 matrix involving the indices $\mu=0$ and 2. L_n and L_c are related by

$$L_c = L_n + K , \tag{6.10}$$

and given in the $(\mu=0,2)$ subspace by

$$L_n(q) = k^2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - i\frac{\pi}{e^2}k\begin{bmatrix}K_0^2 & K_2^2\\ K_0^0 & K_2^0\end{bmatrix} . \tag{6.11}$$

Q_n and Q_c are given by

$$\begin{aligned} Q_n &= K L_n^{-1} , \\ Q_c &= K L_c^{-1} = Q_n (1+Q_n)^{-1} . \end{aligned} \tag{6.12}$$

More explicitly

$$
\begin{aligned}
Q_n &= \frac{1}{\det L_n}\left[k^2 K + i\frac{\pi}{e^2}k\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\det K\right] ,\\
\det L_n &= k^4 - i\frac{\pi}{e^2}k^3(K_0^2+K_2^0)+\left[\frac{\pi}{e^2}\right]^2 k^2\det K ,\\
Q_c &= \frac{1}{\det L_c}\left\{k^2K+\left[i\frac{\pi}{e^2}k\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right]\det K\right\} ,\\
\det L_c &= \det L_n + \det K + k^2(K_0^0+K_2^2) .
\end{aligned} \tag{6.13}
$$

Approximate formulas for Q_n and Q_c for small and large $|\alpha_q|$ can be easily found from (6.2) and (6.4). To this end we notice that

$$
\det K = \frac{e^4}{\pi^2 l^2 \alpha_q^2}(1-e^{-2\alpha_q^2})^2 = \begin{cases} \dfrac{e^4}{\pi^2}\left[k^2 - \dfrac{1}{2\pi n_e}k^4\right] & \text{for small}|k| ,\\[2ex] \dfrac{e^4 n_e}{\pi\alpha_q^2} & \text{for large}|\alpha_q| . \end{cases} \tag{6.14}
$$

For small momentum k,

$$
\begin{aligned}
&\det L_n = \frac{1}{\pi n_e}k^6 + O(k^8), \quad \det L_c = \left[\frac{e^2}{\pi}\right]^2 k^2(1+\lambda_0^2k^2)+O(k^6) ,\\
&\lambda_0^2 = \frac{m}{e^2 n_e} - \frac{1}{2\pi n_e} + \frac{\pi}{e^2 m} \sim \frac{m}{e^2 n_e} = \lambda_L^2 .
\end{aligned} \tag{6.15}
$$

In the last expression we have retained only the numerically dominant term. λ_L is the London penetration depth. (See Ref. [16] for numerical values of parameters m, e^2, and n_e relevant to anyon superconductivity.) The response functions are given by

$$
Q_n = e^2\begin{bmatrix} \dfrac{m}{\pi}\dfrac{1}{k^2} & \dfrac{i}{4\pi}\dfrac{1}{k} \\[2ex] \dfrac{i}{4\pi}\dfrac{1}{k} & \dfrac{n_e}{m}\dfrac{1}{k^2} \end{bmatrix} \tag{6.16}
$$

and

$$
Q_c = \begin{bmatrix} 1-\dfrac{\pi}{e^2 m}\dfrac{k^2}{1+\lambda_0^2k^2} & \dfrac{i}{4e^2 n_e}\dfrac{k^3}{1+\lambda_0^2k^2} \\[2ex] \dfrac{i}{4e^2n_e}\dfrac{k^3}{1+\lambda_0^2k^2} & 1-\dfrac{m}{e^2n_e}\dfrac{k^2}{1+\lambda_0^2k^2} \end{bmatrix} \sim \begin{bmatrix} 1 & \dfrac{i}{4e^2 n_e}\dfrac{k^3}{1+\lambda_0^2k^2} \\[2ex] \dfrac{i}{4e^2n_e}\dfrac{k^3}{1+\lambda_0^2k^2} & \dfrac{1}{1+\lambda_0^2k^2} \end{bmatrix} . \tag{6.17}
$$

For a neutral anyon gas the response function is defined by $(k^2/e^2)Q_n$ in the literature, and the above result agrees with that of Refs. [4] and [8] obtained on a plane using RPA. [In the holon model $(k^2/e^2){Q_n}^0{}_0$ becomes $m/2\pi$. Other components remain the same as in (6.16).] Q_n and Q_c at finite temperature ($T\neq 0$) on a plane have been evaluated in Ref. [16]. Higher-order corrections to Q_n have been evaluated in Refs. [8] and [27].

There is a significant difference between neutral and charged anyon gases as demonstrated by Q_n and Q_c. In particular the pole at $k^2=0$ in ${Q_n}^2{}_2$ is shifted to $k^2=-\lambda_0^{-2}$ in ${Q_c}^2{}_2$, indicating a Meissner effect. We shall elaborate on this phenomenon in the next section. It is worth mentioning that the situation at $T\neq 0$ is more involved. It has been shown, for instance, that ${Q_c}^2{}_2$ at $T\neq 0$ develops two poles, one associated with the $T=0$ pole and the other at $k^2=0$ (Refs. [14] and a [16]). These facts suggest an important difference between excitation spectra in neutral and charged anyon gases. A more rigorous examination requires inclusion of time (or frequency) dependence in gauge field configurations.

We stress that the difference between Q_n and Q_c is not just in the location of a pole. The structure of the 00 and 02 elements of Q_n and Q_c is quite different. Q_n and Q_c are related by the simple formula (6.12) due to the fact that both Chern-Simons and EM fields have the same gauge interaction to the electron field.

For a very thin cylinder ($L/l \ll 1$),

$$\det L_n = \frac{(2\alpha_q)^4}{l^4}\left[1+\frac{12}{(2\alpha_q)^4}+\cdots\right]$$
$$\det L_c = \frac{(2\alpha_q)^4}{l^4}\left[1+\frac{e^2}{\pi m}\frac{1}{(2\alpha_q)^2} + \left[12+\frac{4e^2ml^2}{\pi}-\frac{2e^2}{\pi m}\right]\frac{1}{(2\alpha_q)^4} + \cdots\right] . \tag{6.18}$$

For Q_n and Q_c,

$$Q_n = \begin{bmatrix} \dfrac{4e^2m}{\pi^2 n_e (2\alpha_q)^4}\left[1+\dfrac{2}{(2\alpha_q)^2}\right] & \dfrac{8ie^2l}{\pi(2\alpha_q)^5}\left[1+\dfrac{2}{(2\alpha_q)^2}\right] \\ \dfrac{8ie^2l}{\pi(2\alpha_q)^5}\left[1+\dfrac{2}{(2\alpha_q)^2}\right] & \dfrac{e^2}{\pi m(2\alpha_q)^2}\left[1-\dfrac{2}{(2\alpha_q)^2}\right] \end{bmatrix} \tag{6.19}$$

and

$$Q_c = \begin{bmatrix} \dfrac{4e^2m}{\pi^2 n_e (2\alpha_q)^4}\left[1+\dfrac{2}{(2\alpha_q)^2}\right] & \dfrac{8ie^2l}{\pi(2\alpha_q)^5}\left[1+\left[2-\dfrac{e^2}{\pi m}\right]\dfrac{1}{(2\alpha_q)^2}\right] \\ \dfrac{8ie^2l}{\pi(2\alpha_q)^5}\left[1+\left[2-\dfrac{e^2}{\pi m}\right]\dfrac{1}{(2\alpha_q)^2}\right] & \dfrac{e^2}{\pi m(2\alpha_q)^2}\left[1-\left[2+\dfrac{e^2}{\pi m}\right]\dfrac{1}{(2\alpha_q)^2}\right] \end{bmatrix} . \tag{6.20}$$

Q_n and Q_c agree with each other to the leading order. A difference shows up to the next to the leading order, which is numerically negligible ($e^2/\pi m \sim 10^{-5}$).

VII. MEISSNER EFFECT

The Meissner effect is the expulsion of external magnetic fields, achieved in a system by annuling the applied magnetic field by currents generated in response. In this paper we are limiting ourselves to x_2-independent configurations. We introduce an external current which generates a constant external magnetic field:

$$J^2_{\mathrm{EM,ext}}(x_1) = -2B_0[\delta(x_1)-\delta(x_1-\tfrac{1}{2}L)] ,$$
$$J^2_{\mathrm{EM,ext}}(q) = -\frac{2B_0}{L}(1-e^{i\pi q}) , \tag{7.1}$$
$$J^0_{\mathrm{EM,ext}}(x_1) = J^1_{EM,ext}(x_1) = J^\mu_{\mathrm{CM,ext}}(x_1) = 0 .$$

This generates, through (6.6), an external field

$$B_{\mathrm{ext}} = -F^{\mathrm{ext}}_{12} = \begin{cases} +B_0 & \text{for } 0<x_1<\frac{1}{2}L, \\ -B_0 & \text{for } \frac{1}{2}L<x_1<L, \end{cases}$$
$$F^{\mathrm{ext}}_{0k} = f^{\mathrm{ext}}_{\mu\nu} = 0 . \tag{7.2}$$

In the radiation gauge only A^2_{ext} is nonvanishing.

Before examining a response of the system to (7.1) or (7.2), it is helpful to recall the mechanism of the Meissner effect in the BCS theory [43]. With a Cooper-pair condensate on a plane,

$$\text{BCS:}\quad J^k_{\mathrm{ind}}(k) = -K(k)A^k(k) . \tag{7.3}$$

In the London limit $K=1/\lambda_L^2$. Substitution of (7.3) into the Maxwell equations yields

$$\text{BCS:}\quad \begin{cases} J^k_{\mathrm{ind}}(k) = -\dfrac{1}{1+\lambda_L^2k^2}J^k_{\mathrm{EM,ext}}(k), \\ J^k_{\mathrm{EM,tot}}(k) = \dfrac{\lambda_L^2k^2}{1+\lambda_L^2k^2}J^k_{\mathrm{EM,ext}}(k), \\ B(k) = \dfrac{i\lambda_L^2k}{1+\lambda_L^2k^2}J^k_{\mathrm{EM,ext}}(k). \end{cases} \tag{7.4}$$

Hence, so long as $\lambda_L(T)$ is finite, J^k_{tot} $(k=0)=0$. That is, the external current is completely shielded (in the bulk) by the induced current. In particular, for $B_{\mathrm{ext}} = B_0\epsilon(x_1)$ and $J^2_{\mathrm{EM,ext}} = -2B_0\delta(x_1)$, one finds $B(x_1) = B_0\epsilon(x_1)e^{-|x_1|/\lambda_L}$, implying a complete Meissner effect.

In our case only $J^2_{\mathrm{EM,ext}}$ is nonvanishing in (7.1) so that Eq. (6.7) can be written as

$$J^{\mu}_{\rm ind}(q)=-Q_c{}^{\mu}{}_2(q)J^2_{\rm EM,ext}(q)\ . \tag{7.5}$$

In particular,

$$\begin{aligned} J^2_{\rm EM,tot}(q)&=[1-Q_c{}^2{}_2(q)]J^2_{\rm EM,ext}(q)\ ,\\ B(q)&=\frac{iL}{2\pi q}[1-Q_c{}^2{}_2(q)]J^2_{\rm EM,ext}(q)\ . \end{aligned} \tag{7.6}$$

A relevant quantity for a Meissner effect is $1-Q_c{}^2{}_2(q)$.

It is easy to see that in the $L\to\infty$ limit a charged anyon gas responds to an external magnetic field just as a BCS superconductor does. Indeed, in this limit $Q_c{}^2{}_2$ of (6.17) has exactly the same form as in BCS. However, as mentioned in the previous section, an important departure from BCS appears at finite temperature.

An analytic expression of $B(x_1)$ for a finite L is available in two cases $L\gg l$ and $L\ll l$ with an external field (7.1) (Ref. [41]). For a fat cylinder ($L\gg l$),

$$\begin{aligned} B(x_1)&=-\frac{8B_0\lambda_0^2}{L}\frac{\partial}{\partial x_1}\sum_{\substack{q>0\\ \rm odd}}^{\infty}\frac{1}{1+(2\pi q\lambda_0/L)^2}\cos\frac{2\pi q x_1}{L}\\ &=\frac{B_0}{\cosh(L/4\lambda_0)}\times\begin{cases}+\cosh[(\frac14 L-x_1)/\lambda_0] & \text{for } 0<x_1<+\frac12 L,\\ -\cosh[(\frac14 L+x_1)/\lambda_0] & \text{for } -\frac12 L<x_1<0.\end{cases} \end{aligned} \tag{7.7}$$

The magnetic field is exponentially damped from $\pm B_0$ to zero near $x_1=0$ or $\pm\frac12 L$. A complete Meissner effect operates so long as $L\gg\lambda_0$.

For a thin cylinder ($L\ll l$),

$$\begin{aligned} B_{\rm ind}(x_1)=B(x_1)-B_{\rm ext}(x_1)&=-\frac{8B_0}{L}\frac{e^2 n_e}{m}\sum_{\substack{q>0\\ \rm odd}}^{\infty}\left[\frac{L}{2\pi q}\right]^3\sin\frac{2\pi q x_1}{L}\\ &=-B_0\begin{cases} x_1(\frac12 L-x_1)/\lambda_L^2 & \text{for } 0<x_1<\frac12 L,\\ x_1(\frac12 L+x_1)/\lambda_L^2 & \text{for } -\frac12 L<x_1<0.\end{cases} \end{aligned} \tag{7.8}$$

In view of $L\ll l\ll\lambda_L$ in the anyon model of superconductivity, the induced magnetic field $B_{\rm ind}$ is negligible. In other words, since the magnetic length l is much larger than the circumference of the cylinder L, each electron mostly sees only the total external current $\int_0^L dx_1 J^2_{\rm ext}$ which vanishes.

VIII. DISCUSSIONS

On a cylinder the nonintegrable phase ϕ of the Wilson-line integral is a physical quantity. We have shown that it plays a crucial role to maintain the translation invariance of the ground state in the orthogonal direction.

The wave function of the system with given field strengths $f_{\mu\nu}$ and $F_{\mu\nu}$ is given by Eq. (3.1), $|\Psi(f,F)\rangle=\int_0^{2\pi}d\phi\, f(\phi)|\phi\rangle\otimes|\Psi_e(\phi)\rangle$. For states translationally invariant in the x_2 direction, all values of ϕ are equally probable so that $f(\phi)=$const. Based upon this observation we have determined the response functions for static, x_2 independent but otherwise arbitrary gauge field configurations.

We have confirmed the significant difference between neutral and charged anyon gases, and have observed a relationship between them. In the $L\to\infty$ limit all quantities reduce to those on a plane. For a finite L there arises a departure. In particular, at least for x_2-independent configurations the induced current $J^2_{\rm ind}$ becomes vanishingly small for a very thin cylinder ($L\ll l$).

However, this does not necessarily mean that the superconductivity in the x_2 direction disappears as L gets smaller. One of the most intriguing questions is whether or not a Josephson effect exists when a barrier is inserted in the middle of the cylinder, and how it depends on the length of the circumference of the cylinder. Here the system is not translation invariant in the x_2 direction so that nontrivial dependence of the wave function on the nonintegrable phases of Wilson-line integrals on both side is expected. We hope to come back to this problem in the near future.

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