PHYSICAL REVIEW D VOLUME 44, NUMBER 2 15 JULY 1991

Gravitating topological matter in 2+1 dimensions

S. Carlip
Department of Physics, University of California, Davis, California 95616

J. Gegenberg
Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick, Canada E3B 5A3

(Received 25 February 1991)

We study topological matter minimally coupled to gravity in 2+1 dimensions. We show that the resulting system has a finite-dimensional physical phase space that can be exactly quantized. The model exhibits the mixing of "gravity" and "matter" degrees of freedom, and the impossibility of treating them independently.

I. INTRODUCTION

Faced with the difficulty of quantizing gravity in (3+1)-dimensional spacetime, a number of physicists have recently turned to 2+1 dimensions for hints.[1–7] In 2+1 dimensions, local gravitational fields do not propagate, and the physical phase space of ordinary general relativity becomes finite dimensional. This does not make the theory trivial—point particles still experience gravitational scattering, and dynamical degrees of freedom appear when the spacetime topology is anything more complicated than $\mathbb{R}^3$. But even then, quantum field theory is reduced to quantum mechanics, and the problems associated with renormalization in 3+1 dimensions disappear.[3]

As Witten[3] has shown, (2+1)-dimensional gravity can be exactly quantized. On a manifold with the topology $\mathbb{R}\times\Sigma$, where Σ is a compact genus-g surface, the gravitational Hilbert space is the space of square-integrable functions on the moduli space of Σ. The Hamiltonian is zero, and observables can be described explicitly in terms of holonomies of the *dreibein* and the spin connection. The resulting model has been used to explore several conceptual issues of quantum gravity, including the role of topology change,[8] the meaning of diffeomorphism-invariant observables,[9] and the nature of time in quantum gravity.[10]

An important step would be to couple this model to matter, allowing the exploration of such issues as the effect of quantum fluctuations of spacetime on ultraviolet divergences. Unfortunately, when realistic matter is coupled to (2+1)-dimensional gravity, many of the problems of 3+1 dimensions reappear.[11] The combined phase space of matter and gravity is no longer finite dimensional, the space of exact solutions becomes much more complex, and questions of nonrenormalizability again arise (although it is possible that some models are renormalizable in the $1/N$ expansion[12]). Following the philosophy that led us to look at (2+1)-dimensional gravity in the first place, we might therefore begin by examining gravity coupled to models of matter with finitely many degrees of freedom—"topological matter."

A first step in this direction was taken by Gegenberg, Kunstatter, and Leivo,[13] who looked at one simple form of topological matter nonminimally coupled to gravity. The goal of this paper is to describe another such system, in which the coupling of matter and gravity is more conventional.

II. ACTION, SYMMETRIES, AND FIELD EQUATIONS

In this section we outline the classical theory obtained by coupling a topological field theory of the sort considered by Horowitz and Srednicki[14,15] to Einstein gravity in 2+1 dimensions. We shall first display the symmetries of the classical action functional. Assuming that the spacetime topology is $\mathbb{R}\times\Sigma$, with Σ a compact two-dimensional space, we shall then summarize the canonical formulation, showing in particular that the constraints generate the symmetry algebra I(ISO(2,1)) and that the Hamiltonian vanishes on the constraint surface. Finally, we shall demonstrate that the diffeomorphism group is generated on shell by a particular I(ISO(2,1)) transformation.

We take as our action functional[16]

$$S=\int d^3x\,\epsilon^{\mu\nu\pi}[\tfrac{1}{2}E_{i\pi}R^{\,i}_{\mu\nu}(A)+B_{i\pi}D_{[\mu}C^{\,i}_{\nu]}]\ , \tag{1}$$

where $R^{\,i}_{\mu\nu}(A)$ is the curvature two-form corresponding to an SO(2,1) connection $A^{\,i}_{\mu}$, and the "matter" fields $B_{i\mu}$ and $C^{\,i}_{\mu}$ are SO(2,1)-valued one-forms. The first term in (1) is the standard Einstein gravitational action in first-order form;[3,7] in particular, $E_{i\pi}$ is an SO(2,1)-valued one-form that, if nonsingular, can be interpreted as the *dreibein* of the spacetime metric. In the absence of matter, the connection $A^{\,i}_{\mu}$ is compatible on shell with $E^{\,i}_{\mu}$ and so can be identified with the ordinary Lorentzian spin connection. As we shall see, this is no longer the case when the matter fields $B^{\,i}_{\mu}$ and $C^{\,i}_{\mu}$ are nontrivial. The covariant derivative D_μ in the second term of (1) is with respect to the connection $A^{\,i}_{\mu}$. On shell, the latter is flat, and this term is precisely the action for a topological field theory of the type considered in Refs. 14 and 15. Our action therefore represents topological matter minimally coupled to gravity.

The equations of motion are obtained by varying the four fields A^i_μ, E^i_μ, B^i_μ, and C^i_μ independently, and, in differential form notation, are

$$R^i := dA^i + \tfrac{1}{4}\epsilon^{ijk} A_j \wedge A_k = 0 , \tag{2a}$$

$$DE^i + \tfrac{1}{2}\epsilon^{ijk} B_j \wedge C_k = 0 , \tag{2b}$$

$$DC^i = 0 , \tag{2c}$$

$$DB^i = 0 . \tag{2d}$$

Here D denotes the exterior covariant derivative on spacetime,

$$DB^i := dB^i + \tfrac{1}{2}\epsilon^{ijk} A_j \wedge B_k , \tag{3}$$

where d is the ordinary exterior derivative on spacetime. Caretted $\hat{d}$'s and $\hat{D}$'s will appear later, where they will denote exterior derivatives on two-dimensional space.

We see from (2b) that A^i and E^i are not in general compatible. By (2c) and (2d), on the other hand, A^i *is* compatible with both B^i and C^i, and it is tempting to reinterpret one of these fields as the spacetime *dreibein*. This was the interpretation given by Horowitz[14] in a (3+1)-dimensional analogue of the model (1). However, as we shall show below, it is the spatial projection of E^i, not B^i or C^i, that is canonically conjugate to the spatial projection of A^i, and so this interpretation is not possible in the quantum theory.

We remark here that this model is an example of a "teleparallel theory,"[17,18] as is the (3+1)-dimensional version discussed by Horowitz.[14]

The action (1) is invariant under the following (infinitesimal) gauge transformation, with 12 gauge parameters ρ^i, λ^i, ξ^i, and τ^i:

$$\delta B^i = -D\rho^i - \tfrac{1}{2}\epsilon^{ijk} B_j \tau_k , \tag{4a}$$

$$\delta C^i = -D\lambda^i - \tfrac{1}{2}\epsilon^{ijk} C_j \tau_k , \tag{4b}$$

$$\delta E^i = -D\xi^i - \tfrac{1}{2}\epsilon^{ijk}(E_j\tau_k + B_j\lambda_k + C_j\rho_k) , \tag{4c}$$

$$\delta A^i = -D\tau^i . \tag{4d}$$

We now assume that spacetime is topologically $\mathbb{R}\times\Sigma$ with an induced 2+1 splitting of the fields given by $\epsilon^{0AB} = \epsilon^{AB}$ = the Levi-Civita density on Σ. The (2+1)-dimensional form of the action functional is

$$S = \int dt \int d^2x (\tilde{E}^B_i \dot{A}^i_B + \tilde{B}^A_i \dot{C}^i_A + E_{i0}F^i + A_{i0}G^i + B_{i0}J^i + C_{i0}K^i) , \tag{5}$$

where quantities with tildes are the result of multiplication by ϵ^{AB}. We have omitted possible surface terms that can arise if Σ is not compact. The 0-components of the fields are Lagrange multipliers enforcing the constraints

$$F^i := *(\hat{d}\hat{A}^i + \tfrac{1}{4}\epsilon^{ijk}\hat{A}_j \wedge \hat{A}_k) , \tag{6a}$$

$$G^i := *(\hat{D}\hat{E}^i + \tfrac{1}{2}\epsilon^{ijk}\hat{B}_j \wedge \hat{C}_k) , \tag{6b}$$

$$J^i := *(\hat{D}\hat{C}^i) , \tag{6c}$$

$$K^i := *(\hat{D}\hat{B}^i) , \tag{6d}$$

where $*$ is the spatial Hodge dual, $*(F_{AB}dx^A \wedge dx^B) = \epsilon^{AB}F_{AB}$, and caretted quantities are projections onto Σ, e.g., $\hat{A}^i = A^i_B dx^B$. The momenta canonically conjugate to the 0-components of the fields are all constrained to be zero; the remaining momenta are

$$\tilde{\Pi}^B_i := \frac{\partial L}{\partial \dot{A}^i_B} = \tilde{E}^B_i , \tag{7a}$$

$$\tilde{P}^B_i := \frac{\partial L}{\partial \dot{C}^i_B} = \tilde{B}^B_i . \tag{7b}$$

The Hamiltonian

$$H = -\int d^2x (E_{i0}F^i + A_{i0}G^i + B_{i0}J^i + C_{i0}K^i) \tag{8}$$

is weakly zero, as expected in a generally covariant theory.

The nontrivial fundamental Poisson brackets can be read off from the action (5). They are

$$\{A^i_B(y), \tilde{E}^C_j(x)\} = \delta^C_B \delta^i_j \delta(y,x) , \tag{9a}$$

$$\{C^i_A(y), \tilde{B}^C_j(x)\} = \delta^C_A \delta^i_j \delta(y,x) . \tag{9b}$$

From this, it follows that the constraint algebra is

$$\{F^i(y), F^j(x)\} = \{J^i(y), J^j(x)\} = \{K^i(y), K^j(x)\} = 0 , \tag{10a}$$

$$\{F^i(y), J^j(x)\} = \{F^i(y), K^j(x)\} = 0 , \tag{10b}$$

$$\{F^i(y), G^j(x)\} = \tfrac{1}{2}\epsilon^{ijk} F_k(x)\delta(y,x) , \tag{10c}$$

$$\{G^i(y), G^j(x)\} = \tfrac{1}{2}\epsilon^{ijk} G_k(x)\delta(y,x) , \tag{10d}$$

$$\{G^i(y), J^j(x)\} = \tfrac{1}{2}\epsilon^{ijk} J_k(x)\delta(y,x) , \tag{10e}$$

$$\{G^i(y), K^j(x)\} = \tfrac{1}{2}\epsilon^{ijk} K_k(x)\delta(y,x) , \tag{10f}$$

$$\{J^i(y), K^j(x)\} = \tfrac{1}{2}\epsilon^{ijk} F_k(x)\delta(y,x) . \tag{10g}$$

Of course, all of the constraints are preserved in time. It may also be checked that with these Poisson brackets, the constraints generate the spatial projections of the gauge transformations (4a)–(4d).

The constraint algebra (10) may be recognized as the Lie algebra of the inhomogeneized Poincaré group I(ISO(2,1)), that is, the semidirect product of the Poincaré group ISO(2,1) with its Lie algebra $\mathbb{R}^6$. I(ISO(2,1)) can be viewed as the cotangent bundle of ISO(2,1), just as ISO(2,1) is itself the cotangent bundle of the Lorentz group SO(2,1). According to Eq. (10g), the generators J^i and K^i—the constraints associated with the matter fields B_i and C_i—do not commute under the Poisson bracket algebra. Hence the "translation" subgroup $\mathbb{R}^6$ is not generated by these constraints, but rather by J^i and F^i. We see that gravitational and matter fields are mixed in a manner reminiscent of supergravity. We shall have more to say about this below when the quantum theory is discussed.

We also note that, just as in the vacuum case,[3] the action of the diffeomorphism group on the configuration space variables is generated by a linear combination of the constraints F^i, G^i, J^i, K^i with equal coefficients. Indeed, if we set $\rho^i=\lambda^i=\xi^i=\tau^i$ in (4a)–(4d), it is easily checked that the resulting transformation is precisely the diffeomorphism generated by the vector field $\rho^i E_{I\mu}$, up to terms proportional to the equations of motion.

III. QUANTIZATION

Let us now canonically quantize the system (5) on a spacetime with the topology $\mathbb{R}\times\Sigma$, where Σ is a genus-g surface. As in any constrained system, we have two options: We may first quantize and then impose the constraints as operator conditions on the states, or we may solve the constraints and quantize the physical phase space.[19] Following Witten,[3] we adopt the latter procedure. Our first task is therefore to study the space of solutions of the constraints (6a)–(6d).

The phase space of pure (2+1)-dimensional gravity has the structure of a cotangent bundle. Let us temporarily set the matter fields B^i and C^i to zero and recall the derivation of this structure. We first note that the constraint (6a) implies that $\hat{A}^i$ lies in the space $\mathcal{A}$ of flat SO(2,1) connections on Σ. Now let us consider a cotangent vector $(\delta\hat{A})^i$ at $\hat{A}^i$. If $(\hat{A}+\delta\hat{A})^i$ is to lie in $\mathcal{A}$, (6a) implies that

$$\hat{d}(\delta\hat{A})^i+\tfrac{1}{2}\epsilon^{ijk}\hat{A}_j\wedge(\delta\hat{A})_k=0\ . \tag{11}$$

This is precisely Eq. (6b) for $\hat{E}^i$ when $\hat{B}^i=\hat{C}^i=0$. We can thus identify $\hat{E}^i$ with $(\delta\hat{A})^i$, and we see that a solution $(\hat{A}^i,\hat{E}^i)$ of the constraints is a point in the cotangent bundle $T^*\mathcal{A}$.

To obtain the physical phase space, we must still factor out the gauge group. The group of ISO(2,1) gauge transformations is itself a cotangent bundle, whose base space is the space $\mathcal{G}$ of SO(2,1) gauge transformations generated by the constraint G^i. It is evident from (4d) that only this SO(2,1) subgroup acts nontrivially on $\mathcal{A}$. If we vary (4d), however, we find that a cotangent vector transforms as

$$\delta(\delta\hat{A})^i=-\hat{d}(\delta\tau)^i-\tfrac{1}{2}\epsilon^{ijk}\hat{A}_j(\delta\tau)_k-\tfrac{1}{2}\epsilon^{ijk}(\delta\hat{A})_j\tau_k\ . \tag{12}$$

This is precisely the ISO(2,1) gauge transformation (4c), with $(\delta\hat{A})^i=\hat{E}^i$ and $(\delta\tau)^i=\xi^i$. Hence, when we identify SO(2,1)-equivalent flat connections A^i, we automatically also identify ISO(2,1)-equivalent cotangent vectors. The physical phase space for pure gravity is thus the cotangent bundle $T^*(\mathcal{A}/\mathcal{G})$.

We can now allow $\hat{B}^i$ and $\hat{C}^i$ to differ from zero. $\hat{E}^i$ will no longer satisfy (11), of course, but $\hat{C}^i$ will, and so a pair $(\hat{A}^i,\hat{C}^i)$ will represent a point in $T^*\mathcal{A}$. If $((\delta\hat{A})^i,(\delta\hat{C})^i)$ is a cotangent vector at $(\hat{A}^i,\hat{C}^i)$, the constraints (6a) and (6c) then imply that

$$\begin{aligned}&\hat{d}(\delta\hat{A})^i+\tfrac{1}{2}\epsilon^{ijk}\hat{A}_j\wedge(\delta\hat{A})_k=0\ ,\\&\hat{d}(\delta\hat{C})^i+\tfrac{1}{2}\epsilon^{ijk}(\delta\hat{A})_j\wedge\hat{C}_k+\tfrac{1}{2}\epsilon^{ijk}\hat{A}_j\wedge(\delta\hat{C})_k=0\ .\end{aligned} \tag{13}$$

Comparing (6b) and (6d), we see that $(\delta\hat{A})^i$ and $(\delta\hat{C})^i$ can be identified with $\hat{B}^i$ and $\hat{E}^i$, and so fields satisfying the constraints represent points in the cotangent bundle $T^*(T^*\mathcal{A})$.

We now observe that only the ISO(2,1) gauge transformations generated by G^i and K^i act nontrivially on points $(\hat{A}^i,\hat{C}^i)$ in the base space of this bundle. As in pure gravity, factoring out this group gives us $T^*(\mathcal{A}/\mathcal{G})$; but this *phase space* for pure gravity is now the *configuration space* of our gravity-matter system. The full gauge group (4) is again a cotangent bundle, now with base space of ISO(2,1) transformations, and it is not hard to show that the physical *phase space* for our matter-gravity system is the cotangent bundle $T^*(T^*(\mathcal{A}/\mathcal{G}))$.

Now recall[20] that the space $\mathcal{A}/\mathcal{G}$ of flat SO(2,1) connections on a surface Σ is isomorphic to the Teichmüller space $\mathbf{T}(\Sigma)$.[21] As noted above, the diffeomorphisms that can be deformed to the identity are equivalent to gauge transformations and have already been divided out. But the true invariance group for quantum gravity presumably also includes the diffeomorphisms not isotopic to the identity, and we should still factor out this group. For the topologies we are considering, this means identifying configurations that differ by elements of the mapping class group of Σ. The effect is to reduce Teichmüller space to the moduli space $\mathcal{M}(\Sigma)$. Our physical phase space is thus $T^*(T^*\mathcal{M}(\Sigma))$.

With such a cotangent bundle as a phase space, quantization is straightforward. The Hilbert space is the space of square-integrable functions of the base space,

$$\mathcal{H}=L^2(T^*\mathcal{M}(\Sigma))\ , \tag{14}$$

and we have seen that the Hamiltonian (8) is weakly zero. The fundamental observables are functions on the configuration space $T^*\mathcal{M}$ and cotangent vectors to this configuration space. Equivalently, in analogy to pure gravity, we can construct observables from Wilson loops, path-ordered integrals of the fields. Indeed, if Γ is a curve in $\mathbb{R}\times\Sigma$ and $\{\mathbf{F}^i,\mathbf{G}^i,\mathbf{J}^i,\mathbf{K}^i\}$ are generators of a representation of the group (10), it is not hard to check that

$$W=\operatorname{Tr}\exp\left[\int_\Gamma(E^i\mathbf{F}_i+A^i\mathbf{G}_i+B^i\mathbf{J}_i+C^i\mathbf{K}_i)\right] \tag{15}$$

is a gauge-invariant operator, and that W depends only on the homotopy class of Γ.

It is also of interest to consider the path-integral quantization of the system (1). If M is a closed three-manifold, the partition function

$$Z_M=\int[dA][dB][dC][dE]e^{iS} \tag{16}$$

can be evaluated explicitly. To see this, we first note that the E integral imposes a δ functional $\delta[R^i]$ inside the path integral, ensuring that only flat SO(2,1) connections contribute. For simplicity, let us assume that only a finite number of such flat connections $A_{(\alpha)}$ occur (modulo gauge transformations). We can now separate out the B and C integrations,

$$\begin{aligned}&Z_M=\int[dA][dE]\exp\left[i\int_M e_i\wedge R^i(A)\right]Z^{(0)}[A]\ ,\\&Z^{(0)}[A]=\int[dB][dC]\exp\left[i\int_M B_i\wedge DC^i\right]\ ,\end{aligned} \tag{17}$$

and evaluate $Z^{(0)}$ for each $A=A_{(\alpha)}$. This is precisely the path integral studied by Schwarz,[22] who shows that it gives a topological invariant, the Ray-Singer analytic torsion $T(A_{(\alpha)})$ of the flat bundle over M with connection $A_{(\alpha)}$.[23] But Witten[8] has shown that the remaining A and E integration gives another copy of $T(A_{(\alpha)})$, and it is not hard to check that the B and C integrations do not affect this conclusion.

The partition function is therefore the topological quantity

$$Z_M=\sum_{\alpha}[T(A_{(\alpha)})]^2 . \tag{18}$$

This result may be checked by considering the supersymmetric system discussed by Witten in Ref. 8. Witten's action is essentially the same as ours, but with anticommuting fields B and C. Consequently, $Z^{(0)}$ is replaced by its inverse and has the effect of cancelling the determinants arising from the A and E integrations. Witten is therefore left with a sum over flat connections of ± 1, a quantity closely related to the Casson invariant of M.

IV. PHYSICAL INTERPRETATION

In most physical theories we think of gravity as providing an arena in which matter interacts. Our model demonstrates, however, that it is not so easy to separate "gravity" from "matter."

To see this, consider the simplest topology $\Sigma=S^2$. Note first that if $\tau^i=0$ in (4), the action (1) is invariant, up to a total divergence, under gauge transformations with *finite* values of the remaining gauge parameters. We now take the trivial configuration $A^i=E^i=B^i=C^i=0$ and act on it by the composition of the two (finite) gauge transformations

$$\begin{aligned}&\delta_1(B^i,C^i,E^i):=(-D\rho^i,0,-D\xi^i) ,\\ &\delta_2(B^i,C^i,E^i):=(0,-D\lambda^i,-\tfrac{1}{2}\epsilon^{ijk}\delta_1 B_j\lambda_k) ,\end{aligned} \tag{19}$$

to obtain

$$\begin{aligned}&E^i=-d\xi^i+\tfrac{1}{2}\epsilon^{ijk}d\rho_j\lambda_k ,\\ &B^i=-d\rho^i ,\\ &C^i=-d\lambda^i .\end{aligned} \tag{20}$$

It is easily checked that this configuration satisfies the field equations (2). But although it appears to have a nontrivial metric and nontrivial matter fields, by construction it is gauge equivalent to the configuration in which all fields vanish. In particular, let us choose $\xi^A=0$, $\rho_0=0$, and $\rho_A=x^A$. The spacetime metric constructed from E^i is then given by the line element

$$\begin{aligned}ds^2=&-[(\partial_0\xi^0)dx^0-\tfrac{1}{2}\tilde{\lambda}_2dx^1+\tfrac{1}{2}\tilde{\lambda}_1dx^2]^2\\ &+\left[\frac{\lambda_0}{2}\right]^2[(dx^1)^2+(dx^2)^2] ,\end{aligned} \tag{21}$$

where we have defined

$$\tilde{\lambda}_1:=\lambda_1+2\partial_2\xi^0 , \tag{22a}$$

$$\tilde{\lambda}_2:=\lambda_2-2\partial_1\xi^0 , \tag{22b}$$

$$\tilde{\lambda}^2:=\lambda_0^2-\tilde{\lambda}_1^2-\tilde{\lambda}_2^2 . \tag{22c}$$

This line element can be rewritten in Arnowitt-Deser-Misner (ADM) form:

$$\begin{aligned}ds^2=&-N^2(dx^0)^2\\ &+h_{AB}(dx^A+N^Adx^0)(dx^B+N^Bdx^0) ,\end{aligned} \tag{23}$$

with

$$N_1=\frac{\tilde{\lambda}_2}{2}\partial_0\xi^0,\quad N_2=-\frac{\tilde{\lambda}_1}{2}\partial_0\xi^0,\quad N=\frac{\lambda_0}{\tilde{\lambda}}\partial_0\xi^0 , \tag{24}$$

and

$$\begin{aligned}h_{AB}dx^Adx^B=&\left[\frac{\tilde{\lambda}}{2}\right]^2[(dx^1)^2+(dx^2)^2]\\ &+\tfrac{1}{4}(\tilde{\lambda}_1dx^1+\tilde{\lambda}_2dx^2)^2\\ =&e^{2\phi}|dz+\mu\, d\bar{z}|^2 ,\end{aligned} \tag{25}$$

where $z=x^1+ix^2$ and

$$e^{\phi}=\tfrac{1}{4}(\lambda_0+\tilde{\lambda}) , \tag{26a}$$

$$\mu=\left[\frac{\tilde{\lambda}_1+i\tilde{\lambda}_2}{\lambda_0+\tilde{\lambda}}\right]^2 . \tag{26b}$$

Now any metric on a two-surface can be expressed in the "Beltrami parametrization" (25), where ϕ determines the conformal factor and the Beltrami differential μ fixes a point in Teichmüller space up to diffeomorphisms.[24] For the case we are considering, μ is not arbitrary, of course, but is determined by ϕ and by the lapse and shift functions N and N_A of Eq. (24). But when space has the topology of a two-sphere, all Beltrami differentials are equivalent up to spatial diffeomorphisms.[25] Since it is evident from (24) that the lapse and shift functions may be chosen arbitrarily, we have shown that *any* metric on $\mathbb{R}\times S^2$ is diffeomorphic to a metric of the form (21).

In other words, given any metric on $\mathbb{R}\times S^2$, there exists some matter configuration (20) such that the combined system is gauge equivalent to flat empty spacetime. This provides a stark illustration of the fact that the metric cannot be treated separately from matter. Gauge transformations mix geometry and matter; it is only the combination that has physical significance. This is especially clear in the quantum theory, where states and operators depend only on gauge-invariant quantities. Indeed, for the spatial topology S^2, the moduli space $\mathcal{M}$ consists of a single point, and the Hilbert space (14) contains only one state. Although we can produce any metric by a gauge transformation, the quantum physics remains trivial.

It seems likely that the same conclusion will hold for open spacetimes $\mathbb{R}\times\mathbb{R}^2$. To show this conclusively, however, it is necessary to treat boundary terms in the action (5) more carefully and to define appropriate asymptotic conditions for the gauge transformations (4a)–(4d). Such a treatment has interesting consequences in the case of pure gravity,[4] and it would be worthwhile to examine the corresponding problem with matter present.

We have now succeeded in constructing a model for gravity interacting with topological matter in three spacetime dimensions which is solvable classically and quantum mechanically. Unlike previous models,[11,13] our gravitational and matter fields interact nontrivially. Our system illustrates an important principle—the impossibility of uniquely separating matter from gravity—which is likely to carry over to more realistic theories.

Some remaining issues include constructing the algebra of observables (in the manner of Refs. 5 and 6 for pure gravity) and better understanding the role of asymptotic conditions in an open universe. Ultimately, we hope this model will help us to better understand the coupling of nontopological matter to gravity.

ACKNOWLEDGMENTS

J. G. acknowledges the partial support of the Natural Sciences and Engineering Research Council of Canada for this research, and also thanks G. Kunstatter and G. Miller for useful conversations. Both authors thank the Canadian Association of Physicists, at whose Banff Conference on Gravitation this work was begun.

[1]G. 't Hooft, Commun. Math. Phys. **117**, 685 (1988).
[2]S. Deser and R. Jackiw, Commun. Math. Phys. **118**, 495 (1988).
[3]E. Witten, Nucl. Phys. **B311**, 46 (1988).
[4]S. Carlip, Nucl. Phys. **B324**, 106 (1989).
[5]S. P. Martin, Nucl. Phys. **B327**, 178 (1989).
[6]J. Nelson and T. Regge, Nucl. Phys. **B328**, 190 (1989).
[7]A. Ashtekar, V. Husain, C. Rovelli, J. Samuel, and L. Smolin, Class. Quantum Grav. **6**, L185 (1989).
[8]E. Witten, Nucl. Phys. **B323**, 113 (1989).
[9]S. Carlip, Phys. Rev. D **42**, 2647 (1990).
[10]S. Carlip, Class. Quantum Grav. **8**, 5 (1991).
[11]A. Ashtekar and J. D. Romano, Phys. Lett. B **229**, 56 (1989).
[12]T. Kugo, Kyoto Report No. KUNS 1014 HE(TH) 90/05, 1990 (unpublished).
[13]J. Gegenberg, G. Kunstatter, and H. P. Leivo, Phys. Lett. B **252**, 381 (1990).
[14]G. T. Horowitz, Commun. Math. Phys. **125**, 417 (1989).
[15]G. T. Horowitz and M. Srednicki, Commun. Math. Phys. **130**, 83 (1990).
[16]Greek indices take the values 0,1,2 and denote the components of tensors, while lower-case Latin indices are SO(2,1) algebra valued, and are raised and lowered by the SO(2,1) metric η_{ij}. Upper-case Latin indices, which appear below, are spatial indices with range 1,2. The Levi-Civita density $\epsilon^{\mu\nu\pi}$ satisfies $\epsilon^{012}=+1$.
[17]R. Weitzenböck, *Invariantentheorie* (Noordhoff, Gronigen, 1923).
[18]K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[19]"Solving the constraints" has a double meaning here: We must restrict ourselves to fields that obey the constraint equations, but we must also identify fields that differ by gauge transformations generated by the constraints.
[20]W. Goldman, Adv. Math. **54**, 200 (1984).
[21]Strictly speaking, $\mathcal{A}/\mathcal{G}$ has several components, only one of which is isomorphic to Teichmüller space, but only this component is physically relevant (Ref. 9).
[22]A. S. Schwarz, Lett. Math. Phys. **2**, 247 (1988); Commun. Math. Phys. **67**, 1 (1979).
[23]D. B. Ray and I. M. Singer, Adv. Math. **7**, 145 (1971).
[24]L. Baulieu and M. Bellon, Phys. Lett. B **196**, 142 (1987).
[25]It is apparent that this argument will fail for more complicated spatial topologies. The metric (25) then displays a dependence of inequivalent solutions on points in Teichmüller space.