General proof of the averaged null energy condition for a massless scalar field in two-dimensional curved spacetime

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It is by now well known that the standard local (i.e., pointwise) energy conditions always can be violated in quantum field theory in curved (and flat) spacetime, even when these energy conditions hold for the corresponding classical field. Nevertheless, some global constraints on the stress-energy tensor may exist. Indeed recent work has shown that the averaged null energy condition (ANEC), which requires the positivity of energy suitably averaged along null geodesics, holds for a wide class of states of a minimally coupled scalar field on Minkowski spacetime, and also (in the massless case) on a wide class of states in curved two-dimensional spacetimes satisfying certain asymptotic regularity properties. In this paper, we strengthen these results by proving that, for the massless scalar field in an arbitrary globally hyperbolic two-dimensional spacetime, the ANEC holds for all Hadamard states along any complete, achronal null geodesic. In our analysis, the general, algebraic notion of "state" is used, so, in particular, it is not even assumed that our state belongs to any Fock representation. Our proof shows that the ANEC is a direct consequence of the general positivity condition which must hold for the two-point function of any state. Our results also can be extended (with a restriction on states) to the massive scalar field in two-dimensional Minkowski spacetime and (with an additional restriction on states) to the (massless or massive) minimally coupled scalar field on four-dimensional Minkowski spacetime. In the case of a (curved) four-dimensional spacetime with a bifurcate Killing horizon, our proof also extends to establish the ANEC for the null geodesic generators of the horizon (provided that there exists a stationary Hadamard state of the field). This latter result implies that the ANEC must hold for the massive Klein-Gordon field in de Sitter spacetime.

I. INTRODUCTION

A variety of "energy conditions," i.e., assumptions concerning the positivity of the locally measured energy density, play a key role in the proofs of many global results in general relativity, particularly in the singularity theorems. The weakest of the local energy conditions normally considered requires that the stress-energy tensor T_{ab} satisfy $T_{ab}k^ak^b \geq 0$ for all null vectors k^a at all points in spacetime. We will refer to this condition as the *null energy condition.* This condition is sufficient to ensure the convergence of null geodesics, a key property for proving certain global results, such as the original Penrose singularity theorem (see Refs. [1,2] for details). However, it has been noted by some authors [3–5] that for a number of applications the null energy condition could be replaced by a still weaker assumption, the *averaged null energy condition* (ANEC), which requires only that the integral

$$\int_\gamma T_{ab}k^ak^b\,dv \tag{1}$$

be non-negative, where γ is a complete null geodesic, v is an affine parameter, and k^a is the tangent vector along γ. [The precise formulation of the ANEC used in this paper, which is applicable even when the integral (1) does not converge, will be given in Sec. II below.] Note that the integral (1) can also be interpreted as being the total (locally measured) energy flux through γ; so when the ANEC is satisfied, the total energy flux through any null surface (whose generators are complete) must be non-negative.

The main motivation for considering weakened versions (such as the ANEC) of the null energy condition is that, although most of the classical matter fields usually considered obey the null energy condition, all of the corresponding quantum matter fields violate it (as well as the weak, strong, and dominant energy conditions, all of which imply the null energy condition). Consider, for example, a minimally coupled (real) Klein-Gordon field ϕ with the stress-energy tensor

$$T_{ab}=\nabla_a\phi\nabla_b\phi-\tfrac{1}{2}(\nabla^c\phi\nabla_c\phi+m^2\phi^2)g_{ab}\ . \tag{2}$$

For any classical configuration ϕ, T_{ab} satisfies the null energy condition on all spacetimes. This is not true, however, for the expectation values of the (regularized) quantum operator T_{ab}. Even on Minkowski spacetime, it is easy to construct quantum states $|\psi\rangle$ in the standard Fock space for which $\langle\psi|{:}T_{ab}{:}|\psi\rangle$ violates the null energy condition at any desired point (see, e.g., Refs. [6,7]). On the other hand, whether the expected stress-energy tensor of the quantum field similarly violates the ANEC remains an open question at present, and it is this question that will be addressed by the present work.

Previous research [6,7] has shown that, for a minimally coupled scalar field on Minkowski spacetime, and also for a massless, conformally coupled field on curved two-dimensional spacetimes with certain asymptotic regularity properties, the ANEC is satisfied in all quantum states that belong to a large subset of the standard Fock space. In the present paper, we will improve these results significantly by giving an alternative analysis based on the algebraic approach to quantum field theory in curved spacetime. (The elementary aspects of the algebraic approach that will be needed in our analysis are reviewed in Sec. III below.) Our main tool is the general positivity condition on the two-point function, valid for any quantum state. Our main result is the following: For an arbitrary globally-hyperbolic two-dimensional spacetime with a massless Klein-Gordon scalar quantum field (which is conformally invariant in two dimensions), given an arbitrary Hadamard state of the field, and given a complete null geodesic γ which is achronal, we prove that ANEC holds along γ for the expected stress-energy in this state. This generalizes the results of [7] in that no topological or asymptotic conditions on the spacetime are imposed, and no restriction apart from the Hadamard condition (which is needed to define the regularized stress tensor) is placed upon the state; e.g., it is not required that the state belong to a Fock representation. Our methods also enable us to prove that the ANEC holds for the massless or massive) Klein-Gordon scalar field in four-dimensional Minkowski spacetime, provided that we impose additional restrictions on the states. (These restrictions appear to be considerably weaker than those imposed in Refs. [6,7].) By the same methods (and with the same additional restrictions on states), it also follows that for a minimally coupled massive scalar field on de Sitter spacetime, ANEC must hold, and more generally, for a massive or massless Klein-Gordon field on a stationary, globally hyperbolic spacetime with a Killing horizon $\mathcal{H}$, the ANEC must hold along the null generators of $\mathcal{H}$, provided that there exists a stationary Hadamard state of the field on that spacetime.

It should be noted that ANEC alone does not suffice to replace the energy condition hypotheses of any of the standard singularity theorems. (The theorems of Ref. [4] require a version of ANEC to hold along all half-infinite null geodesics.) Furthermore, our main results apply only to two-dimensional spacetimes (where Einstein's equation is trivial and, in any case, null convergence cannot occur) and to flat spacetimes. Nevertheless, our results do show that nontrivial global restrictions on the quantum stress-energy tensor, of the general type needed for the singularity theorems, hold at least in the cases amenable to our analysis. Thus our results can be interpreted as lending support to the view that the conclusions of the singularity and other global theorems of classical general relativity may continue to hold in quantum theory despite the fact that violations of the standard local energy conditions can occur. In addition, as noted above, our results establish (in the cases described above) the non-negativity of the total energy flux through a null surface. Finally, our results show two new relationships which may be of some interest in their own right. (1) There is a direct relationship between averaged energy conditions and the positivity condition on states (in the algebraic sense) of the quantum field. (2) There is a close connection between the validity of the ANEC along a null geodesic and the achronality of that geodesic.

In Sec. II we give the precise formulation of the ANEC that will be used in this paper. In Sec. III we briefly review the elements of the algebraic approach to quantum field theory in curved spacetime which will be needed in our analysis. Our main results are presented in Secs. IV and V where the ANEC is proved to hold for the massless Klein-Gordon field in two-dimensional flat and curved spacetimes. The extension of these results to the massive field and to four-dimensional flat spacetime is given in Sec. VI.

Our notation and conventions follow [2].

II. PRECISE FORMULATION OF THE ANEC AND ITS SIGNIFICANCE

As already discussed in Sec. I, the basic idea of the ANEC is simply that the integral (1) be non-negative. However, we do not wish to restrict consideration of the validity of the ANEC to spacetimes and states which satisfy sufficiently stringent asymptotic conditions to ensure convergence of the integral (1). Thus we seek a formulation of the ANEC which does not require convergence of this integral.

There are many possible generalizations of the ANEC to the case where (1) does not converge. The choice we shall make corresponds precisely to the condition that emerges in our general proof given later in this paper. We now state our formulation of the ANEC.

Let T_{ab} be any smooth stress-energy tensor on spacetime and γ be a complete null geodesic, with affine parameter v and corresponding tangent vector k^a. Let $c(x)$ be a bounded real-valued function of compact support on $\mathbb{R}$ whose Fourier transform $\hat{c}(k)$ is such that, for some $\delta > 0$,

$$(1+k^2)^{1+\delta}|\hat{c}(k)|$$

is bounded [i.e., $|\hat{c}(k)|$ decays at least as fast as $|k|^{-2-2\delta}$ as $|k| \to \infty$]. [This implies that $c(x)$ is C^1.] Then we say that T_{ab} satisfies the ANEC along γ if and only if, for all such $c(x)$ (and all choices of origin of affine parameter v), we have

$$\liminf_{\lambda \to \infty} \int_{-\infty}^{\infty} T_{ab}k^a k^b [c(v/\lambda)]^2 dv \geq 0 \ . \tag{3}$$

For each fixed $\lambda > 0$, the integrand in Eq. (3) is a continuous function of compact support. Hence the integral always exists, and $\liminf_{\lambda \to \infty}$ of the integral is always well defined (but is possibly equal to $\pm\infty$). It is not difficult to show that if the integral (1) exists, then our formulation of the ANEC is equivalent to the non-negativity of (1). Thus our formulation generalizes the usual definition without being restricted to the convergence of (1).

Note that our definition of the ANEC does not require that

$$\liminf_{\lambda \to \infty} \int_{-\lambda}^{\lambda} T_{ab}k^a k^b dv \geq 0 \ ,$$

since that would correspond to the choice of $c(x)$ being a step function, which is not continuous and thus does not satisfy our condition on its Fourier transform. Nevertheless, our formulation of the ANEC is of a sufficient strength to enable the proof of null convergence in appropriate circumstances. To illustrate this point, we prove the following result.

Proposition. Let p be a point on a complete null geodesic $\gamma(v)$ satisfying the property that, for all $c(x)$ in our formulation of the ANEC,

$$\liminf_{\lambda\to\infty}\int_0^\infty R_{ab}k^a k^b[c(v/\lambda)]^2 dv \geq 0 ,$$

where R_{ab} denotes the Ricci tensor (and where $v=0$ at p). (If the ANEC together with Einstein's equation holds, then this condition must hold for at least one direction along γ from p.) Consider a null geodesic congruence containing γ whose expansion $\theta(v)$ along γ is nonpositive at p. Then either θ vanishes identically along γ or there exists a finite $v_0>0$ at which $\lim_{v\to v_0}\theta(v)=-\infty$.

To prove this, we absorb the shear term $-2\sigma^2$ into the source term $-R_{ab}k^a k^b$ in the Raychaudhuri equation [1]

$$\frac{d\theta}{dv}=-\frac{1}{2}\theta^2-2\sigma^2-R_{ab}k^a k^b ,$$

and put $u\equiv v/2$. The proposition is then an immediate consequence of the following lemma.

Lemma. For all $u\geq 0$, let $\theta(u)$ satisfy the ordinary differential equation

$$\frac{d\theta}{du}=-\theta^2-f(u) , \tag{4}$$

with initial value $\theta(u=0)=-\alpha$, $\alpha\geq 0$, where $f(u)$ is continuous on $[0,\infty)$ and has the property

$$\liminf_{\lambda\to\infty}\int_0^\infty f(u)[c_\lambda(u)]^2 du \geq 0 , \tag{5}$$

for every function $c(x)$ in the class defined above Eq. (3). [Here $c_\lambda(u)$ is shorthand notation for the function $c_\lambda(u)\equiv c(u/\lambda)$]. Then either $\theta(u)$ vanishes identically on $[0,\infty)$ [in which case α and $f(u)$ must also be identically zero], or there exists a finite $u_0>0$ at which $\lim_{u\to u_0}\theta(u)=-\infty$.

Proof. It is sufficient to prove that if $\theta(u)$ exists as a C^1 function $\forall u\in[0,\infty)$, then $\theta(u)\equiv 0$. Let

$$\begin{aligned} &b(x)=(1-x^2)^4, \quad |x|<1 ,\\ &b(x)=0, \quad |x|\geq 1 . \end{aligned} \tag{6}$$

Multiplying both sides of Eq. (4) by $b_\lambda(u)\equiv b(u/\lambda)$ and integrating by parts, we find

$$\int_0^\infty f b_\lambda du=-\alpha-\int_0^\infty \theta^2 b_\lambda du+\int_0^\infty \theta b'_\lambda du . \tag{7}$$

After substituting Eq. (6) in the last term of Eq. (7), we arrive at the identity

$$\int_0^\infty f b_\lambda du=-\alpha-\left[\int_0^\infty \theta^2 b_\lambda du+\frac{8}{\lambda}\int_0^\lambda \theta(u)(u/\lambda)[1-(u/\lambda)^2]^3 du\right] . \tag{8}$$

The second term inside the large parentheses in Eq. (8) can be bounded by

$$\left|\frac{8}{\lambda}\int_0^\lambda \theta(u)(u/\lambda)[1-(u/\lambda)^2]^3 du\right| \leq \frac{8}{\lambda}\int_0^\lambda |\theta(u)|[1-(u/\lambda)^2]^2 du \leq \frac{8}{\sqrt{\lambda}}\left[\int_0^\infty \theta^2 b_\lambda du\right]^{1/2} , \tag{9}$$

where we used $|u/\lambda|\leq 1$ and $|1-(u/\lambda)^2|\leq 1$ in the first line and the Schwarz inequality in the second. Equation (9) shows that as $\lambda\to\infty$ the positive-definite first term in the large parentheses on the right side of Eq. (8) always dominates the indefinite second term. Therefore, unless $\theta(u)$ is identically zero, the right-hand side of Eq. (8) has to be negative and bounded away from zero for all sufficiently large λ. But this is impossible since the function $c(x)=\sqrt{b(x)}$ satisfies the conditions formulated above Eq. (3); so Eq. (5) implies that the lim inf of the left-hand side of Eq. (8) as $\lambda\to\infty$ is always non-negative. Hence $\theta(u)$ must vanish identically if it exists globally over the entire interval $[0,\infty)$. □

To illustrate an application of the above result, we point out that in the presence of the ANEC, there cannot exist a nonsingular, static, spherically symmetric spacetime (with static Killing field timelike everywhere) consisting of two asymptotically flat regions joined by a "wormhole" [8]. Namely, if such a spacetime existed, the null geodesic congruences emanating from the minimal area surface of the wormhole will have a vanishing initial convergence in both directions, but could not have an identically vanishing convergence. Consequently, at least one of these congruences will satisfy the hypotheses of the above lemma, and the type of argument used in the Penrose singularity theorem (see, e.g., Refs. [1,2]) could then be employed to obtain a contradiction. [The assumption of spherical symmetry is used in this argument mainly to assure that all the geodesics in one of the congruences satisfy the hypotheses of the lemma; thus this assumption (as well as the static assumption) could be replaced by much weaker assumptions.]

By means of similar arguments, it appears likely that one could prove that if our formulation of the ANEC holds and the null generic condition is satisfied, then every complete null geodesic contains a pair of conjugate points. (See [1] for the standard proof when the null energy condition holds pointwise and [5] for a proof using an averaged energy condition different from our formulation of the ANEC.)

III. OVERVIEW OF THE ALGEBRAIC APPROACH TO QUANTUM FIELD THEORY IN CURVED SPACETIME

In this section we briefly review the elements of the algebraic approach to linear quantum field theory in curved spacetime which will be needed for our analysis given in the following sections. We refer the reader to Ref. [9] for a more extensive discussion of these basic ideas.

The main idea of the algebraic approach is to view quantum states as objects which act upon the (smeared) quantum field operators in a manner which corresponds in the usual Hilbert-space approach, to the operation of taking expectation values. The approach is implemented by defining an abstract $*$-algebra $\mathcal{A}$ with identity I (corresponding to the operator algebra that would be generated by the smeared quantum field operators on a Hilbert space), and then defining a state ω to be a linear map into complex numbers ω: $\mathcal{A}\to\mathbb{C}$, satisfying the positivity condition $\omega(A^*A)\geq 0$ for all $A\in\mathcal{A}$, as well as the normalization condition $\omega(I)=1$. The GNS construction then shows that every state defined in this manner actually can be realized as a state in the ordinary sense, i.e., as a vector in some Hilbert space $\mathcal{H}$, which carries a representation ρ of the algebra $\mathcal{A}$.

The main advantage of the algebraic approach is that one does not need to specify a particular choice of $\mathcal{H}$ and representation ρ in order to define the theory; i.e., one can simultaneously consider all of the states which arise in all possible (unitarily inequivalent) Hilbert-space constructions of the quantum field theory. In Minkowski spacetime this advantage is not of critical significance and for most purposes, since one has available the criterion of Poincaré invariance to define a preferred vacuum state and, thereby, a preferred Fock Hilbert space and preferred representation of the field algebra. Usually, only the states in this Fock space are considered to be physically relevant. However, in curved spacetime it does not appear that there is available any such natural prescription for choosing a preferred Hilbert-space representation. The algebraic approach allows one to formulate quantum field theory in curved spacetime without requiring one to introduce such a choice.

For definiteness, we now focus attention on the case of a real Klein-Gordon scalar field ϕ satisfying

$$(\Box-m^2)\phi=0\ , \tag{10}$$

although an exactly similar discussion (with appropriate changes of commutators to anticommutators for fermion fields) would hold for all other linear fields. Let (M,g) be a globally hyperbolic spacetime, so that the initial value formulation for Eq. (10) on any smooth, spacelike Cauchy surface $\mathcal{C}$ is well posed. In the usual Hilbert-space formulation, the field is described as an operator-valued distribution on spacetime; i.e., for each (C_0^∞) test function f on spacetime, we specify an operator $\phi(f)$ on some Hilbert space $\mathcal{H}$. The satisfaction of the wave equation by ϕ is expressed by the condition that $\phi(f)=0$ for any f of the form $f=(\Box-m^2)g$, where g is a test function. The canonical commutation relations imposed upon ϕ take the form

$$[\phi(f),\phi(g)]=i\Delta(f,g)I\ , \tag{11}$$

where $\Delta(f,g)\equiv\sigma[Ef,Eg]$, where E denotes the advanced minus the retarded Green's function, and for any two solutions F and G with initial data of compact support on $\mathcal{C}$, we have

$$\sigma[F,G]\equiv\int_{\mathcal{C}}(F\nabla^aG-G\nabla^aF)\,d\Sigma_a\ . \tag{12}$$

(It is easily verified that $\sigma[F,G]$ is independent of the choice of $\mathcal{C}$.)

It is useful to note two properties of the map $f\to Ef$ taking the space T of C_0^∞ functions on spacetime to the space S of smooth solutions of Eq. (10) with initial data of compact support on any Cauchy surface $\mathcal{C}$: (i) Every smooth solution with initial data of compact support on $\mathcal{C}$ can be obtained in this manner, i.e., the range of this map is all of S. (See, e.g., Appendix B of Ref. [9] for a proof of this result.) (ii) The kernel of this map consists precisely of the test functions f of the form $f=(\Box-m^2)g$, where g is a test function. These properties allow one to view the field operator ϕ as a distribution defined on S rather than a distribution defined on T, which vanishes on elements of T of the form $(\Box-m^2)g$. We shall adopt this viewpoint here. In order to distinguish notationally between the action of ϕ on T and its corresponding action on S, we will use parentheses [e.g., $\phi(f)$] when denoting the action of ϕ on T and square brackets (e.g., $\phi[F]$) to denote its action on S. The canonical commutation relations for ϕ viewed as a distribution on S are simply

$$[\phi[F],\phi[G]]=i\sigma[F,G]I\ . \tag{13}$$

As already indicated above, in the algebraic approach one defines a $*$-algebra corresponding to the algebra of field operators in the Hilbert-space approach. There are a number of ways in which this can be done. For our purposes (since we will be interested only in the two-point function), the most convenient choice of algebra $\mathcal{A}$ is the one denoted $\mathcal{A}'$ in Sec. 3.2 of Ref. [9]. It is constructed by starting with the free algebra over $\mathbb{C}$ generated by the formal objects $\phi[F]$ (for all $F\in S$) together with the identity element I; i.e., one takes all formal finite sums (with complex coefficients) of finite products of these objects. One then imposes the commutation relations Eq. (13) as well as the linearity of ϕ as a distribution on S by equating any two such expressions if they can formally be reduced to each other using Eq. (13) and the linearity of ϕ. A $*$-operation is then determined on the resulting algebra $\mathcal{A}$ by setting $\phi[F]^*\equiv\phi[F]$ for all $F\in S$ and by extending it antilinearly to all of $\mathcal{A}$ with the rule $(AB)^*=B^*A^*$, for all $A,B\in\mathcal{A}$.

Thus, by definition of the algebra $\mathcal{A}$, a state ω acts on any finite linear combination (with complex coefficients) of finite products $\phi[F_1]\cdots\phi[F_n]$. We define the two-point distribution λ by

$$\lambda[F,G]\equiv\omega(\phi[F]\phi[G])\ . \tag{14}$$

Since for any $A\in\mathcal{A}$ we have $\omega(A^*)=\overline{[\omega(A)]}$ [as follows from the positivity of ω on the elements $(A+I)^*(A+I)$ and $(A+iI)^*(A+iI)$], Eq. (14) implies that

$$\mathrm{Im}\lambda[F,G]=\tfrac{1}{2}\sigma[F,G] . \tag{15}$$

We define

$$\mu[F,G]\equiv\mathrm{Re}\lambda[F,G] . \tag{16}$$

Using the correspondence described above, we may view λ and μ either as bidistributions on S or as bidistributions on T which satisfy the wave equation (10) in each variable.

When the positivity condition $\omega(A^*A)\geq 0$ for an arbitrary state ω is imposed for algebra elements A of the form

$$A=\alpha\phi[F]+i\beta\phi[G] ,$$

where α,β are arbitrary real numbers, it follows that the inequality

$$\mu[F,F]\mu[G,G]\geq\tfrac{1}{4}|\sigma[F,G]|^2 \tag{17}$$

must hold for all F in S. This is the fundamental inequality used in this paper. The positivity restrictions we shall obtain on the stress energy tensor will be derived directly from this inequality. Note that the positivity condition also implies $\mu[F,F]>0$ for all $F\neq 0$ in S.

We close this section with a comment about a special feature of the two-dimensional case. It is well known that for the massless Klein-Gordon field in two-dimensional Minkowski spacetime, infrared divergences prevent one from defining the smeared field operator in the standard Fock representation for all test functions. Rather, the smeared field operator is defined only for the subspace T' of test functions f which satisfy $\int f=0$. Under the map $f\rightarrow Ef$, the subspace T' corresponds precisely to the subspace S' of solutions in S which can be expressed in the form $h(u)+k(v)$, where u and v are the standard Minkowski null coordinates and $h:\mathbb{R}\rightarrow\mathbb{R}$ and $k:\mathbb{R}\rightarrow\mathbb{R}$ are arbitrary C_0^∞ functions. Thus, in two dimensions, the usual requirement that states be defined on the algebra generated in the manner described above by the objects $\phi[F]$ for all F in S must be weakened to require that it be defined only on the subalgebra generated by $\phi[F]$ for F in S'. Similar remarks apply for curved two-dimensional spacetimes. Thus, for the massless field in two dimensions treated in the next section, we shall require that the two-point function Eq. (14) be well defined only in the case where F and G are in S'.

IV. PROOF OF ANEC IN MINKOWSKI SPACETIME: TWO-DIMENSIONAL CASE

In this section, we shall prove that the ANEC holds for the massless scalar field in two-dimensional Minkowski spacetime. The generalization of the result to two-dimensional curved spacetime will be given in the next section, and the generalization to the four-dimensional Minkowski case for both massless and massive fields will be given in Sec. VI.

Without any loss of generality, we focus attention to a null geodesic of the form $\gamma\equiv\{u=0\}$ in two-dimensional Minkowski spacetime $(\mathbb{R}^2,\eta)$, where $\eta=-du\,dv$. We consider an arbitrary state of the field (in the algebraic sense described in the previous section), restricted only by the requirement that the two-point distribution $\lambda(f,g)$ be of the Hadamard form, so that the expected stress-energy tensor be well defined and smooth everywhere in spacetime. Although the definition of the "Hadamard form" for a distribution in a general curved spacetime is rather intricate (see Sect. 3.3 of Ref. [9] for a precise, general definition in four dimensions), for a massless scalar field in two-dimensional Minkowski spacetime, the Hadamard condition requires simply that for any test functions f, g in T' (see the end of the previous section) the symmetrized two-point distribution μ be of the form

$$\mu(f,g)=\int[\mu_0(x,x')+w(x,x')]f(x)g(x)dx\,dx' , \tag{18}$$

where μ_0 is the symmetrized two-point function of the Minkowski vacuum, given explicitly by

$$\mu_0(x,x')=-\frac{1}{4\pi}\ln|\Gamma|=\frac{1}{4\pi}\ln|(u-u')(v-v')| \tag{19}$$

(here Γ denotes squared geodesic distance between x and x'), and where $w(x,x')$ is a smooth bisolution. The expected stress-energy tensor $\langle T_{ab}\rangle$ is obtained by subtracting from μ the "Minkowski contribution" μ_0, then performing appropriate differentiations, and finally taking the coincidence limit $x'\rightarrow x$. Thus, the components of $\langle T_{ab}\rangle$ are all given by simple formulas involving coincidence limits of partial derivatives of $w(x,x')$. In particular, along the geodesic γ, we have

$$\langle T_{ab}(v)\rangle k^a k^b=\langle T_{vv}(v)\rangle=\left.\frac{\partial^2 w(0,v',0,v'')}{\partial v'\partial v''}\right|_{v'=v''=v} . \tag{20}$$

We shall now show that the positivity condition (17) on states implies the version of the ANEC formulated at the beginning of Sec. II. For this purpose, it is useful to view μ as a bilinear map on solutions in S' rather than as a bilinear map on test functions in T'. Clearly, for any $F\in S'$ we have

$$\mu[F,F]=\mu_0[F,F]+w[F,F] . \tag{21}$$

The relevant solutions for our purpose are the ones of the form $F=F(v)$.

The "Minkowski vacuum contribution" $\mu_0[F,F]$ for a solution in S' of the form $F=F(v)$, can be evaluated using the methods of Appendix B of Ref. [9] (with γ playing the role of a component of a Killing horizon). We obtain

$$\mu_0[F,F]=-\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ln|v-v'|\frac{\partial}{\partial v}F(v)\times\frac{\partial}{\partial v'}F(v')\,dv\,dv' . \tag{22}$$

To put this in a more useful form, we write

$$\ln|v-v'|=\tfrac{1}{2}\ln[(v-v')^2]$$
$$=\tfrac{1}{2}[\ln(v-v'-i\epsilon)+\ln(v-v'+i\epsilon)] ,$$

where ϵ is a positive real number which we will let vanish at the end of the calculation. We decompose F into its positive- and negative-frequency parts:

$$F(v)=F^+(v)+F^-(v)\ , \tag{23}$$

where

$$F^+(v)\equiv\frac{1}{\sqrt{2\pi}}\int_0^\infty \hat{F}(k)e^{ikv}dk,$$

$$F^-(v)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 \hat{F}(k)e^{ikv}dk\ , \tag{24}$$

and $\hat{F}(k)$ is the Fourier transform

$$\hat{F}(k)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikv}F(v)dv \tag{25}$$

which decays (as $|k|\to\infty$) faster than any inverse power of $|k|$ since $F(v)$ is C_0^∞. By analytically continuing Eqs. (24), in v, it is clear that $F^+(v)$ is the boundary value of a function holomorphic in the upper half v plane, and $F^-(v)$ is the boundary value of a function holomorphic in the lower half v plane. By applying a partial integration, we can now write Eq. (22) in the form

$$\mu_0[F,F]=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{F^+(v)+F^-(v)}{v-v'-i\epsilon}+\frac{F^-(v)+F^+(v)}{v-v'+i\epsilon}\right]\left[\frac{\partial}{\partial v'}F^+(v')+\frac{\partial}{\partial v'}F^-(v')\right]dv\,dv'\ .$$

The v integral can be evaluated by closing the contour in the upper half plane for $F^+(v)$, and in the lower half plane for $F^-(v)$. The result is, after letting $\epsilon\to 0$,

$$\mu_0[F,F]=i\int_{-\infty}^{\infty}\left[F^+(v')\frac{\partial}{\partial v'}F^-(v')-F^-(v')\frac{\partial}{\partial v'}F^+(v')\right]dv'=2\int_0^\infty k|\hat{F}(k)|^2dk\ , \tag{26}$$

which is the usual formula for the Klein-Gordon norm.

On the other hand, the contribution to $\mu[F,F]$ from $w(x,x')$ is given by the "double symplectic smearing" of f with w along γ:

$$w[F,F]=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(v)\frac{\overleftrightarrow{\partial}}{\partial v}w(0,v,0,v')\frac{\overleftrightarrow{\partial}}{\partial v'}F(v')\,dv\,dv'=4\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F(v)F(v')dv\,dv'\ , \tag{27}$$

where

$$Y(v,v')\equiv\frac{\partial^2}{\partial v\partial v'}w(0,v,0,v')\ . \tag{28}$$

Note that by Eq. (20), the coincidence limit of Y yields the component of $\langle T_{ab}\rangle$ of interest:

$$\langle T_{ab}(v)\rangle k^ak^b=Y(v,v)\ . \tag{29}$$

Thus, the positivity condition (17) on states yields the following condition on Y: For all functions $F_1,F_2\in C_0^\infty(\mathbb{R})$, we have

$$\left[2\mu_0[F_1,F_1]+8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F_1(v)F_1(v')\,dv\,dv'\right]\times\left[2\mu_0[F_2,F_2]+8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F_2(v)F_2(v')\,dv\,dv'\right]\geq|\sigma[F_1,F_2]|^2 \tag{30}$$

where μ_0 is given by Eq. (26), and

$$\sigma[F_1,F_2]=\int_{-\infty}^{\infty}\left[F_1(v)\frac{\partial}{\partial v}F_2(v)-F_2(v)\frac{\partial}{\partial v}F_1(v)\right]dv$$
$$=-4\,\mathrm{Im}\int_0^\infty k\overline{\hat{F}_1(k)}\hat{F}_2(k)dk\ . \tag{31}$$

We shall show shortly that the inequality (30) implies the ANEC, without any assumptions on w (and hence on Y) apart from smoothness. In particular, no restrictions on the asymptotic behavior of $w(x,x')$ for large x and x' shall be imposed. However, before giving this general proof, we will illustrate the main idea behind it by first considering the case where $Y(v,v')$ is in Schwartz space; i.e., Y is C^∞ and it and all its derivatives decay at infinity faster than any inverse polynomial. In that case, the Fourier transform $\hat{Y}(k,k')$ of Y [Eq. (34)] also is in Schwartz space, and the interchanges of orders of integration in the calculation below are easily justified. (Note that the Schwartz space restriction is much stronger than needed for the purposes of this simplified argument; it would suffice to require that the function $Y(v,v')$ and its appropriate restrictions on subsets of $\mathbb{R}^2$ belong to suitable L^1 and L^2 spaces. Note also that similar restrictions on the quantum state (in addition to the restriction that the state belongs to the standard Fock space and contains finitely many particles, which we are not imposing here) are implicitly assumed in the arguments used in Refs. [6] and [7] in order to justify interchanges of integration similar to those used below.) Using the Plancherel theorem to convert the integrals over v and v' appearing in Eq. (30) to corresponding integrals involving the Fourier-transformed quantities, we obtain, after some algebra, the inequality

$$\left[\int_0^\infty k|\hat{F}_1(k)|^2dk+\xi(\hat{F}_1,\hat{F}_1)-\eta(\hat{F}_1,\hat{F}_1)\right]$$
$$\times\left[\int_0^\infty k|\hat{F}_2(k)|^2dk+\xi(\hat{F}_2,\hat{F}_2)-\eta(\hat{F}_2,\hat{F}_2)\right]$$
$$\geq\left[\mathrm{Im}\int_0^\infty k\overline{\hat{F}_1(k)}\hat{F}_2(k)dk\right]^2, \quad (32)$$

where

$$\xi(\hat{F},\hat{F})\equiv 4\int_0^\infty\int_0^\infty \hat{Y}(k,-k')\overline{\hat{F}(k)}\hat{F}(k')\,dk\,dk',$$
$$\eta(\hat{F},\hat{F})\equiv 4\,\mathrm{Re}\int_0^\infty\int_0^\infty \hat{Y}(k,k')\overline{\hat{F}(k)\hat{F}(k')}\,dk\,dk', \quad (33)$$

and $\hat{Y}(k,k')=\hat{Y}(k',k)$ denotes the Fourier transform

$$\hat{Y}(k,k')\equiv\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty Y(v,v')e^{-i(kv+k'v')}dv\,dv'. \quad (34)$$

[Note that the reality and symmetry property of $Y(v,v')$ imply that

$$\overline{\hat{Y}(k,-k')}=\hat{Y}(k',-k),$$

and thus the quantity $\xi(\hat{F},\hat{F})$ is real.] The inequality (32) must hold for all $\hat{F}_i(k)$ $(i=1,2)$ that are Fourier transforms of C_0^∞ functions $F_i(v)$, but by continuity (since $\hat{Y}$ is in Schwartz space and, in particular, in L^2) it extends to all complex-valued functions $\hat{F}_i(k)$ on $[0,\infty)$ that have a finite Klein-Gordon norm [Eq. (26)] and are in $L^2[0,\infty)$. Given any such function $\hat{F}(k)$, we put $\hat{F}_1(k)\equiv\hat{F}(k)$, and $\hat{F}_2(k)\equiv i\hat{F}(k)$, $k\in[0,\infty)$. Denoting

$$n\equiv\int_0^\infty k|\hat{F}(k)|^2dk,\quad \xi\equiv\xi(\hat{F},\hat{F}),\quad \eta\equiv\eta(\hat{F},\hat{F}), \quad (35)$$

the inequality (32) then takes the form

$$(n+\xi-\eta)(n+\xi+\eta)\geq n^2. \quad (36)$$

In addition, each of the factors on the left side of Eq. (36) must be non-negative individually [see the comment below Eq. (17)]. We view Eq. (36) as a quadratic inequality on ξ. Satisfaction of such an equality requires that either $\xi\leq r_1$ or $\xi\geq r_2$, where $r_1\leq r_2$ are the two roots of the quadratic equation which in this case takes the explicit form

$$\xi^2+2\xi n-\eta^2=0.$$

However, if $\xi\leq r_1$, then the individual factors on the left side of Eq. (36) would be negative. Hence we obtain

$$\xi\geq r_2=-n+\sqrt{n^2+\eta^2}\geq 0. \quad (37)$$

Thus, recalling the definition (33), we obtain the inequality

$$\int_0^\infty\int_0^\infty \hat{Y}(k,-k')\overline{\hat{F}(k)}\hat{F}(k')\,dk\,dk'\geq 0 \quad (38)$$

for all $\hat{F}(k)$ of finite Klein-Gordon norm and in L^2. Given any $\kappa\in[0,\infty)$, we choose an $\hat{F}(k)$ supported in a sufficiently small neighborhood of $\kappa\in[0,\infty)$, and (since $\hat{Y}$ is in Schwartz space, hence in particular continuous) deduce from Eq. (38) that

$$\hat{Y}(\kappa,-\kappa)\geq 0\quad\forall\kappa\in[0,\infty). \quad (39)$$

But a simple calculation shows that

$$\int_{-\infty}^\infty Y(v,v)dv=2\int_0^\infty\hat{Y}(k,-k)dk; \quad (40)$$

therefore, Eq. (39) proves the ANEC statement $\int Y(v,v)dv\geq 0$ [see Eq. (29)], which is equivalent to the generalized ANEC (3) in this case since the ANEC integral (1) properly exists when Y is in Schwartz space.

We now dispense with any assumptions about $Y(v,v')$ apart from smoothness, and prove that the ANEC as formulated in Eq. (3) holds along the null geodesic γ for an arbitrary Hadamard state. Let $c(x)$ be any function satisfying the conditions appearing in our definition of the ANEC; i.e., $c(x)$ is any compact-supported (C^1) real-valued function whose Fourier transform $\hat{c}(k)$ is such that for some $\delta>0$ the function $(1+k^2)^{1+\delta}|\hat{c}(k)|$ is bounded. For convenience, we normalize $c(x)$ so that

$$\int_{-\infty}^\infty c^2(x)dx=1. \quad (41)$$

The previous argument [see Eqs. (38) and (39)] suggests that the most efficient use of the fundamental inequality (30) will arise by choosing F_1 and F_2 so that their Fourier transforms are sharply peaked around $k=\kappa$. We set, for all $\lambda>0$ and $\kappa\geq 0$,

$$F_1(v)=F_{\lambda,\kappa}(v)\equiv c(v/\lambda)\cos(\kappa v),$$
$$F_2(v)=G_{\lambda,\kappa}(v)\equiv c(v/\lambda)\sin(\kappa v). \quad (42)$$

Let us first compute the Klein-Gordon norm $\mu_0[F_{\lambda,\kappa},F_{\lambda,\kappa}]$ appearing in the fundamental inequality (17). A straightforward calculation gives

$$\hat{F}_{\lambda,\kappa}(k)=\tfrac{1}{2}\lambda\{\hat{c}[(k-\kappa)\lambda]+\hat{c}[(k+\kappa)\lambda]\}, \quad (43)$$

and, thus, changing the variable of integration to $t=k\lambda$, we obtain

$$\mu_0[F_{\lambda,\kappa},F_{\lambda,\kappa}]=2\int_0^\infty k|\hat{F}_{\lambda,\kappa}(k)|^2dk$$
$$=\tfrac{1}{2}\int_0^\infty t|\hat{c}(t-\kappa\lambda)|^2dt$$
$$+\mathrm{Re}\int_0^\infty t\,\overline{\hat{c}(t-\kappa\lambda)}\hat{c}(t+\kappa\lambda)dt$$
$$+\tfrac{1}{2}\int_0^\infty t|\hat{c}(t+\kappa\lambda)|^2dt. \quad (44)$$

To evaluate the first term on the right side of Eq. (44), we change the integration variable to $s=t-\kappa\lambda$ and write

$$\int_0^\infty t|\hat{c}(t-\kappa\lambda)|^2dt$$
$$=\int_{-\kappa\lambda}^\infty(s+\kappa\lambda)|\hat{c}(s)|^2ds$$
$$=\int_{-\infty}^\infty s|\hat{c}(s)|^2ds-\int_{-\infty}^{-\kappa\lambda}s|\hat{c}(s)|^2ds$$
$$+\kappa\lambda\int_{-\infty}^\infty|\hat{c}(s)|^2ds-\kappa\lambda\int_{-\infty}^{-\kappa\lambda}|\hat{c}(s)|^2ds. \quad (45)$$

The first term vanishes since $\overline{\hat{c}(s)}=\hat{c}(-s)$ because $c(x)$ is real. In the third term, we have $\int_{-\infty}^\infty|\hat{c}(s)|^2ds=1$ on account of our normalization condition (41). The remaining terms are small for large $\kappa\lambda$ since

$|\hat{c}(s)|=O(|s|^{-2-2\delta})$ as $|s|\to\infty$. Thus, the first term on the right-hand side of Eq. (44) takes the form

$$\tfrac{1}{2}\int_0^\infty t|\hat{c}(t-\kappa\lambda)|^2dt=\tfrac{1}{2}\kappa\lambda+\epsilon_0(\kappa\lambda) \ . \tag{46}$$

Here and throughout the remainder of this section, any quantity $\epsilon_l(y)$ $(l=0,1,2,3,\ldots)$ will always denote a continuous function on $[0,\infty)$ which for some $\delta>0$ decays at least as fast as $y^{-1-\delta}$ as $y\to\infty$ [i.e., $|\epsilon_l(y)|y^{1+\delta}$ is bounded]. Thus, in particular, the integral $\int_0^\infty\epsilon_l(y)dy$ exists and is finite. No further details about any $\epsilon_l(y)$ will be used in our arguments below.

Now, by a similar calculation, the remaining two integrals in Eq. (44) can be seen to make only a contribution of the form $\epsilon_l(\kappa\lambda)$, and by exactly the same reasoning as above, the remaining quantities $\mu_0[G_{\lambda,\kappa},G_{\lambda,\kappa}]$ and $\sigma[F_{\lambda,\kappa},G_{\lambda,\kappa}]$ can be calculated; the final result is

$$\begin{aligned}\mu_0[F_{\lambda,\kappa},F_{\lambda,\kappa}]&=\tfrac{1}{2}\lambda\kappa+\epsilon_1(\lambda\kappa) \ ,\\ \mu_0[G_{\lambda,\kappa},G_{\lambda,\kappa}]&=\tfrac{1}{2}\lambda\kappa+\epsilon_2(\lambda\kappa) \ ,\\ \sigma[F_{\lambda,\kappa},G_{\lambda,\kappa}]&=\lambda\kappa+\epsilon_3(\lambda\kappa) \ .\end{aligned} \tag{47}$$

We now compute the $Y(v,v')$ contribution to the inequality (30). We introduce

$$Y_\lambda(v,v')\equiv Y(v,v')c(v/\lambda)c(v'/\lambda) \ , \tag{48}$$

which is a compact-supported smooth function. We have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F_{\lambda,\kappa}(v)F_{\lambda,\kappa}(v')\,dv\,dv'&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y_\lambda(v,v')\cos(\kappa v)\cos(\kappa v')\,dv\,dv'\\ &=\tfrac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y_\lambda(v,v')(e^{i\kappa v}+e^{-i\kappa v})(e^{i\kappa v'}+e^{-i\kappa v'})\,dv\,dv'\\ &=\tfrac{1}{8}(\alpha+\beta) \ ,\end{aligned} \tag{49}$$

where

$$\begin{aligned}\alpha(\lambda,\kappa)&\equiv 4\pi[\hat{Y}_\lambda(-\kappa,\kappa)+\hat{Y}_\lambda(\kappa,-\kappa)]=8\pi\hat{Y}_\lambda(\kappa,-\kappa) \ ,\\ \beta(\lambda,\kappa)&\equiv 4\pi[\hat{Y}_\lambda(-\kappa,-\kappa)+\hat{Y}_\lambda(\kappa,\kappa)] \ ,\end{aligned} \tag{50}$$

and

$$\hat{Y}_\lambda(k,k')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y_\lambda(v,v')e^{-ikv}e^{-ik'v'}dv\,dv' \ .$$

Note that, as in Eq. (40), we have

$$\begin{aligned}\int_0^\infty\alpha\,d\kappa&=4\pi\int_{-\infty}^{\infty}Y_\lambda(v,v)\,dv\\ &=4\pi\int_{-\infty}^{\infty}\langle T_{vv}\rangle[c(v/\lambda)]^2dv \ .\end{aligned} \tag{51}$$

It is easy to verify by a similar calculation that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')G_{\lambda,\kappa}(v)G_{\lambda,\kappa}(v')\,dv\,dv'=\tfrac{1}{8}(\alpha-\beta) \ . \tag{52}$$

We now combine Eqs. (42), (47), (49), and (52) with the inequality (30); the result is the inequality

$$(\lambda\kappa+\epsilon_4+\alpha+\beta)(\lambda\kappa+\epsilon_5+\alpha-\beta)\geq(\lambda\kappa+\epsilon_3)^2 \ . \tag{53}$$

Again, each factor appearing on the left-hand side of Eq. (53) must be non-negative. The non-negativity of the sum of these factors yields

$$\alpha\geq-\kappa\lambda-\tfrac{1}{2}(\epsilon_4+\epsilon_5) \ ,$$

which shows that α is bounded from below for small $\lambda\kappa$. On the other hand, expression (53), viewed as a quadratic inequality on α, is of the same basic nature as Eq. (36). By the same type of argument used to obtain Eq. (37), α must be greater than the larger root of the quadratic equation obtained by using the equality sign in the inequality Eq. (53), which takes the form

$$\alpha^2+(2\lambda\kappa+\epsilon_6)\alpha+\epsilon_7\lambda\kappa+\epsilon_8-(\beta+\epsilon_9)^2=0 \ . \tag{54}$$

Thus, when $\lambda\kappa$ is large enough to ensure that Eq. (54) has real roots, we obtain the inequality

$$\begin{aligned}\alpha&\geq\tfrac{1}{2}\{-(2\lambda\kappa+\epsilon_6)+[(2\lambda\kappa+\epsilon_6)^2+4(\beta+\epsilon_9)^2\\ &\qquad-4\epsilon_7\lambda\kappa-4\epsilon_8]^{1/2}\}\\ &\geq\tfrac{1}{2}\{(2\lambda\kappa+\epsilon_6)+[(2\lambda\kappa+\epsilon_6)^2-4\epsilon_7\lambda\kappa-4\epsilon_8]^{1/2}\} \ .\end{aligned} \tag{55}$$

It may be verified that the right-hand side of inequality (55) vanishes as $\lambda\kappa\to\infty$ faster than $|\lambda\kappa|^{-1-\delta}$ for some $\delta>0$. Combining this with the lower bound on α for $\lambda\kappa$, we obtain

$$\alpha(\lambda,\kappa)\geq\epsilon_{10}(\lambda\kappa) \ . \tag{56}$$

Integrating both sides of Eq. (56) with respect to κ, we find

$$\int_0^\infty\alpha\,d\kappa\geq\int_0^\infty\epsilon_{10}(\lambda\kappa)\,d\kappa=\frac{1}{\lambda}\int_0^\infty\epsilon_{10}(y)\,dy \ . \tag{57}$$

Since $\int_0^\infty\epsilon_{10}(y)dy$ is finite, the inequality (57) implies [cf. Eq. (51)]

$$\frac{1}{4\pi}\liminf_{\lambda\to\infty}\int_0^\infty\alpha\,d\kappa=\liminf_{\lambda\to\infty}\int_{-\infty}^{\infty}\langle T_{vv}\rangle[c(v/\lambda)]^2dv\geq 0 \ , \tag{58}$$

which is the statement of the ANEC [Eq. (3)] that we set out to prove. Indeed, we obtain the stronger result that there exists a constant $C>0$ such that, for all $\lambda>0$,

$$\int_{-\infty}^{\infty}\langle T_{vv}\rangle[c(v/\lambda)]^2dv\geq-\frac{C}{\lambda} \ . \tag{59}$$

Equation (59) is very similar in form and content to a recent result of Ford [10].

V. PROOF OF ANEC IN TWO-DIMENSIONAL CURVED SPACETIME

Let (M,g) be an arbitrary globally hyperbolic (curved) two-dimensional spacetime, ϕ a massless Klein-Gordon field on M, and $\gamma \subset M$ any achronal, complete null geodesic. In this section, we will prove that the ANEC [as formulated in Eq. (3)] is satisfied along γ in any Hadamard state of the quantum field ϕ, by carrying the analysis to a point at which the general proof given in Sec. IV for flat spacetime applies word for word to (M,g) and the null geodesic γ.

Any two-dimensional spacetime is locally conformally flat: If we let u and v denote any (locally defined) null coordinates, then the metric (locally) takes the form

$$g = -C(u,v)\,du\,dv \ . \tag{60}$$

Furthermore, the massless scalar field [Eq. (10) with $m=0$] is conformally invariant in two dimensions, with conformal weight zero. Thus, any solution of the massless scalar wave equation in the flat spacetime $\eta = -du\,dv$ defines a solution (at least locally) in the spacetime of Eq. (60). Consequently, the Hadamard condition on a state can be formulated exactly as in flat spacetime: Locally, in any null coordinate system, the symmetrized two-point distribution is required to be of the form $\mu = \mu_c + w_c$ [see Eq. (18)], where

$$\mu_c(x,x') = -\frac{1}{4\pi}\ln|(u-u')(v-v')| \ , \tag{61}$$

and $w_c(x,x')$ is a smooth bisolution. Note that the splitting, Eq. (61), of μ into the pieces μ_c and w_c depends upon the choice of null coordinates (u,v), but the requirement that μ can be expressed in this form is independent of the choice of coordinates. However, in the point-splitting prescription for computing the renormalized $\langle T_{ab}\rangle$ in the curved spacetime (60), one does not simply subtract μ_c from μ as in flat spacetime. Indeed, as already mentioned, in the curved spacetime (60), μ_c [Eq. (61)] depends upon the choice of the coordinates u and v and has no local geometrical significance [in contrast with the flat case where locally Cartesian coordinates are used and $u_0 = -(1/4\pi)\ln|\Gamma|$, where $\Gamma(x,x')$ is the squared geodesic distance between x and x']. In order that the point-splitting procedure produce a renormalized $\langle T_{ab}\rangle$ with proper causal behavior, it is necessary that one subtract from μ a Hadamard distribution which is constructed entirely from the local spacetime geometry [11]. [Note that, although the quantity $-(1/4\pi)\ln|\Gamma|$ is constructed from the local spacetime geometry, it fails to be a bisolution in curved spacetime, and thus it also cannot be used for the point-splitting prescription.]

In order to obtain the desired locally constructed Hadamard distribution, one may proceed as follows. Given $p \in M$, let η and γ denote, respectively, the unique "left-" and "right-" moving null geodesics through p. [Global hyperbolicity of (M,g) implies that the distinction between "left-" and "right-" moving null geodesics is globally well defined.] Let U and V denote affine parameters along η and γ respectively, with $U=V=0$ at p and with U scaled so that the tangents l^a and k^a to η and γ satisfy $k^a l^a = -\frac{1}{2}$ at p. Choose achronal segments of both η and γ which contain p. (Such achronal segments must exist since global hyperbolicity implies that strong causality holds at p.) In a sufficiently small neighborhood $\mathcal{O}$ of p each $q \in \mathcal{O}$ will be connected to this segment of η by a unique "right-" moving null geodesic and be connected to this segment of γ by a unique "left-" moving null geodesic. We label q by the affine parameter values U and V of these intersection points on η and γ, respectively, thereby defining a null coordinate system in $\mathcal{O}$.

The expected stress-energy tensor $\langle T_{ab}\rangle$ at p is now defined by the same prescription as in flat spacetime, where we subtract from μ the Hadamard distribution

$$\mu_0 \equiv -\frac{1}{4\pi}\ln|(U-U')(V-V')| \ , \tag{62}$$

where (U,V) are the null coordinates locally constructed according to the prescription of the preceding paragraph applied at p. In particular, the component $\langle T_{ab}\rangle k^a k^b$ at p is again given by

$$\langle T_{ab}(p)\rangle k^a k^b = \left.\frac{\partial^2 w(0,V,0,V')}{\partial V \partial V'}\right|_{V=V'=0} , \tag{63}$$

where

$$w(x,x') \equiv \mu(x,x') - \mu_0(x,x') \ . \tag{64}$$

This prescription for $\langle T_{ab}\rangle$ satisfies all the axioms of [11], which uniquely determine $\langle T_{ab}\rangle$ up to the addition of conserved local curvature terms. It is not difficult to verify that this prescription is in fact equivalent to the usual point-splitting procedure as described, e.g., in Chapter 6 of Ref. [12]. In particular, for a spacetime $(\mathbb{R}^2,g)$ with metric (60) globally conformal to Minkowski spacetime, this prescription for computing $\langle T_{ab}\rangle$ for the conformal vacuum state [with two-point function given by the right-hand side of Eq. (61)] agrees with the standard expressions given in Eqs. (6.136) and (6.137) of Ref. [12].

We proceed now to the proof of the ANEC. Let γ be a complete, achronal, null geodesic, which, for definiteness, we assume is "right" moving. Choose an affine parametrization of γ and let k^a denote its tangent. Let $p \in \gamma$. Since γ is achronal, the open neighborhood $\mathcal{O}$ of p for which the above construction of the null coordinates (U,V) holds can be chosen to include all of γ. (Indeed, even if γ were not achronal, we still could construct in a similar manner null coordinates covering an open neighborhood of γ.) Then, by the above discussion, Eq. (63) holds at p. On the other hand, at another point $\tilde{p} \in \gamma$, Eq. (63) will hold with w replaced by $\tilde{w} = \mu - \tilde{\mu}_0$, where $\tilde{\mu}_0$ is given by Eq. (62) with the null coordinates (U,V) constructed at p replaced by the corresponding null coordinates $(\tilde{U},\tilde{V})$ for $\tilde{p}$. However, it is clear from the construction of these null coordinates that throughout $\mathcal{O} \cap \tilde{\mathcal{O}}$ we have $\tilde{U} = \tilde{U}(U)$ (i.e., $\tilde{U}$ is a function of U only), and $\tilde{V} = V - c$, where $c = V(\tilde{p})$ is a constant. Furthermore, we have $U = \tilde{U} = 0$ on γ. Thus, for all $(x,x') \in (\mathcal{O} \cap \tilde{\mathcal{O}}) \times (\mathcal{O} \cap \tilde{\mathcal{O}})$ we have

$$\tilde{w}(x,x')=w(x,x')+\frac{1}{4\pi}\ln\left|\frac{\tilde{U}-\tilde{U}'}{U-U'}\right| , \tag{65}$$

and, hence, in particular,

$$\left.\frac{\partial^2 \tilde{w}^2}{\partial\tilde{V}\partial\tilde{V}'}\right|_{\tilde{V}=\tilde{V}'=0,\,\tilde{U}=\tilde{U}'=0}=\left.\frac{\partial^2 w}{\partial V\partial V'}\right|_{V=V'=c,\,U=U'=0} . \tag{66}$$

Consequently, if we choose a fixed $p\in\gamma$ to define μ_0 and w in a neighborhood of γ by Eqs. (62) and (64), then everywhere along γ we have

$$\langle T_{ab}(V)\rangle k^a k^b=\left.\frac{\partial^2 w(0,V,0,V')}{\partial V\partial V'}\right|_{V'=V} , \tag{67}$$

which expresses this component of $\langle T_{ab}\rangle$ along γ in terms of w by exactly the same formula as in flat spacetime [see Eq. (20)].

We consider, now, the positivity condition (17) on our quantum state. Any smooth function F of compact support on γ locally gives rise to a smooth solution $F(V)$ in the neighborhood $\mathcal{O}$ on which our null coordinates (U,V) are defined. However, since (M,g) is globally hyperbolic and γ is achronal, each F actually gives rise globally to a solution in S. To prove this, we note that by the same argument as given in Ref. [9] (cf. the discussion near Fig. 4 of that reference), we can deform γ outside of the support of F to obtain a Cauchy surface $\mathcal{C}$, which is spacelike apart from the finite segment of γ included in it (see Fig. 1). Then, given any $q\in M$, the unique "left" moving null geodesic through q must intersect $\mathcal{C}$ at one and only one point. We define $F(q)=0$ if this intersection occurs off of γ, and $F(q)=F(V)$ if this intersection occurs on γ at the point labeled by the affine parameter V. This construction globally defines the desired solution in S. Note that the assumption that γ is achronal is crucially used here; if this property fails, then only a restricted class of functions F on γ would give rise to global solutions in S. Indeed, if γ fails to be achronal, the ANEC need not hold: a simple counterexample to ANEC in that case is the flat cylindrical $(S^1\times\mathbb{R})$ spacetime, with the field in the static vacuum state, where the negative Casimir energy and pressure imply a violation of the ANEC on all null geodesics.

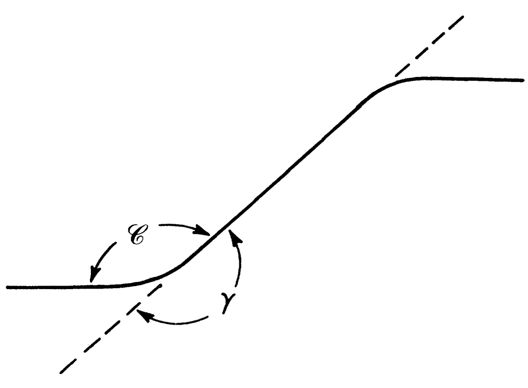

FIG. 1. In a two-dimensional globally hyperbolic spacetime, any null geodesic γ can be deformed outside a finite segment to produce a Cauchy surface $\mathcal{C}$ which is spacelike apart from the null segment of γ that it contains.

The explicit form of the positivity condition (17), with solutions F_1, F_2, of the type discussed in the previous paragraph, for a state with two-point function μ given by Eq. (64), now becomes identical to Eq. (30) in the case of Minkowski spacetime with the variables v,v' replaced by V,V'. (Furthermore, all integrals over the affine parameter V of γ continue to range from $-\infty$ to ∞ since γ is complete). Thus, all equations and arguments in Sec. IV, from Eq. (26) onwards up to Eq. (59), apply without modification. This establishes the validity of the ANEC for two-dimensional (globally hyperbolic) curved spacetimes as stated in the beginning of this section.

VI. A PROOF OF ANEC FOR MASSLESS AND MASSIVE FIELDS IN FOUR-DIMENSIONAL MINKOWSKI SPACETIME

In this section, we shall extend the results of Sec. IV to the massive Klein-Gordon field in two-dimensional Minkowski spacetime, and to the massless and massive Klein-Gordon field in four-dimensional Minkowski spacetime. To do so, however, we will need to impose some conditions on the quantum state which restrict the asymptotic behavior of the regularized two-point function $w(x,x')$ at large x and x'.

We treat, first, the case of a massive Klein-Gordon field in two-dimensional Minkowski spacetime. The Hadamard condition on the state of the field takes the form

$$\mu_m(x,x')=\mu_{0m}(x,x')+w(x,x') , \tag{68}$$

where μ_{0m} is the symmetrized two-point function of the Minkowski vacuum state for a Klein-Gordon field of mass m, and w is a smooth bi-solution of the massive Klein-Gordon equation; the explicit form of μ_{0m} will not be needed in the following discussion. Again, the relevant component $\langle T_{ab}\rangle k^a k^b$ along γ is given in terms of w by Eq. (20). In the massless case, we obtained a condition on w which led directly to the ANEC by applying the general positivity condition (17) to a particular class of solutions. The following fact played a crucial role in this argument: Given a null geodesic γ and given a C_0^∞ function F_0 on γ, then there exists a solution F in S which induces this "data" F_0 on γ. [This fact followed trivially from the fact that the function $F(u,v)\equiv F_0(v)$ on two-dimensional Minkowski spacetime satisfies the massless Klein-Gordon equation.] In order to extend our results to the massive case, a similar property is needed.

To analyze this issue, given a C_0^∞ function E_0 on γ let $\mathcal{C}$ be a Cauchy surface obtained by deforming γ outside the support of F_0 to a spacelike surface, as illustrated in Fig. 1 above. Consider the initial-value problem for a solution of the massive Klein-Gordon equation

$$\frac{\partial^2 F}{\partial u\,\partial v}=-m^2 F , \tag{69}$$

with data on $\mathcal{C}$ specified as follows: we choose $F=F_0$ on the portion of $\mathscr{C}$ coinciding with γ, and set $F=\dot{F}=0$ on the spacelike portion of $\mathscr{C}$. Then the standard theorems of the spacelike initial-value formalism imply that F must vanish in the domain of dependence of the spacelike portion of $\mathscr{C}$, and, in particular, on their null boundaries. The standard theorems of the null initial-value formalism then yield the following results: There exists a unique smooth solution F_+ on $J^+(\gamma)$ and a unique smooth solution F_- on $J^-(\gamma)$ each of which induces the desired initial data on $\mathscr{C}$. Hence, by "merging" these solutions, we obtain a continuous solution (in the distributional sense) whose data on $\mathscr{C}$ (and hence on all Cauchy surfaces) is of compact support, and whose restriction to γ is F_0. However, as pointed out in the "note in proof" of Ref. [9], in general, F will be merely continuous across γ and thus will not define a solution in S. Indeed, integrating Eq. (69) along γ, and using the fact that $\partial F_+/\partial u=0$ in the domain of dependence of the spacelike section of $\mathscr{C}$, we find that along the portion of γ lying in $I^+(\mathscr{C})$ we have

$$\frac{\partial F_+}{\partial u}(v)=-\int_{-\infty}^{v} m^2 F_0\,dv=-m^2\int_{-\infty}^{\infty} F_0\,dv\ , \quad (70)$$

whereas $\partial F_-/\partial u=0$ on that portion of γ. Thus, if $m\neq 0$, in order for F to define a C^1 solution, we must have $\int_{-\infty}^{\infty}F_0 dv=0$, which is equivalent to the condition that on γ we have $F_0=dG_0/dv$, where $G_0\in C_0^\infty(\mathbb{R})$. More generally, the necessary and sufficient condition that the C_0^∞ initial data F_0 on γ yields a C^n solution F to Eq. (69) with data of compact support on Cauchy surfaces is that F_0 be of the form

$$F_0=\frac{d^nG_0}{dv^n}\ , \quad (71)$$

with $G_0\in C_0^\infty(\mathbb{R})$. It is not difficult to show that no $F_0\in C_0^\infty$ can yield a C^∞ solution in S. This contrasts sharply with the massless case, where every $F_0\in C_0^\infty(\mathbb{R})$ gives rise to a C^∞ solution $F\in S$.

The fact that the solutions F obtained by the above procedure fail to be in S means that the field algebra $\mathcal{A}$ does not contain representatives $\phi[F]$ of these solutions F, and, hence *a priori* the positivity condition (17) need not apply to such F. Nevertheless, by the argument sketched in the note in proof to [9], in two dimensions any Hadamard state on $\mathcal{A}$ can be extended, by continuity, to a Hadamard state acting on an enlarged field algebra $\hat{\mathcal{A}}$ containing representatives of all solutions whose data on a spacelike Cauchy surface is merely C^4 and of compact support. Hence, the positivity condition analogous to Eq. (30),

$$\left[2\mu_{0m}[F_{01},F_{01}]+8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F_{01}(v)F_{01}(v')\,dv\,dv'\right]\times\left[2\mu_{0m}[F_{02},F_{02}]+8\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F_{02}(v)F_{02}(v')\,dv\,dv'\right]\geq|\sigma[F_{01},F_{02}]|^2\ , \quad (72)$$

holds for all F_{01},F_{02} of the form (71) with $n=4$. Here Y is again defined by Eq. (28) and $\sigma[F_{01},F_{02}]$ is again given by Eq. (31). Furthermore, $\mu_{0m}[F_0,F_0]$ is again given by precisely the same formula, Eq. (26), as held in the massless case; indeed, the methods of Appendix B of [9] establish that the two-point function of any stationary Hadamard state of the massive or massless scalar field takes the universal form (26) (in two dimensions) on solutions of the type we are considering on any component of a bifurcate Killing horizon.

Thus, in the massive case, the inequality (72) is identical in form to the inequality (30) of the massless case, except that now F_{01} and F_{02} are restricted by Eq. (71) with $n=4$. Unfortunately, this restriction on F_{01} and F_{02} prevents one from paralleling the proof given in Sec. IV for the massless case. Nevertheless, if $w(x,x')$ is suitably restricted, then Eq. (72) will automatically hold by continuity for all F_{01}, $F_{02}\in C_0^\infty$. In particular, suppose w is such that for all F_{01}, $F_{02}\in C_0^\infty$ we have

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Y(v,v')F_{01}(v)F_{02}(v')\,dv\,dv'\right|^2\leq C\mu_{0m}[F_{01},F_{01}]\mu_{0m}[F_{02},F_{02}]\ , \quad (73)$$

where $C>0$ is a constant. Then Eq. (72) will hold for all F_{01}, $F_{02}\in C_0^\infty$. Namely, it is not difficult to show (see the note in proof of [9]) that any $F_0\in C_0^\infty$ can be approximated in the norm μ_{0m} by a sequence of C_0^∞ functions satisfying Eq. (71) (this is true for any fixed n). When Eq. (73) holds, the vacuum norm μ_{0m} bounds the full norm μ_m, so convergence in the norm μ_{0m} implies convergence in the norm μ_m. Note that a sufficient (but not necessary) condition for Eq. (73) to hold is that both $w(0,v,0,v')$ and $Y(v,v')$ be square integrable on $\gamma\times\gamma$.

It is not difficult to show that Eq. (73) holds for all Hadamard states in the standard Minkowski Fock space for which the expected number of particles is finite. Thus, Eq. (73) holds for a wide class of physically relevant states. Nevertheless, Eq. (73) represents a nontrivial restriction on the quantum state, and indeed, there exist Hadamard states in the standard Fock space which violate it.

Given a Hadamard state of the massive Klein-Gordon field which satisfies Eq. (73) along γ, then all the ingredients used in our proof of ANEC for the massless case hold. Namely, Eq. (72) holds for all F_{01}, $F_{02}\in C_0^\infty$, $\mu_{0m}[F_0,F_0]$ is given by Eq. (26), and $\langle T_{ab}\rangle k^ak^b$ is given by Eq. (20). Thus, if a Hadamard state of the massive Klein-Gordon field in two-dimensional Minkowski spacetime satisfies Eq. (73) on a null geodesic γ, then the

ANEC as formulated in Eq. (3) holds for γ.

Note that Eq. (73) is not necessary to ensure that Eq. (72) holds for all F_{01}, $F_{02}\in C_0^\infty$, i.e., other (inequivalent) conditions would suffice. For example, if $Y(v,v')$ [but not necessarily $w(x,x')$] is square integrable on $\gamma\times\gamma$, then an argument similar to that given above establishes Eq. (72) for all F_{01},F_{02} which are in L^2 and have finite Klein-Gordon norm. Thus Eq. (73) could be replaced by the condition that $Y\in L^2(\gamma\times\gamma)$.

We turn, now, to the case of a massive or massless Klein-Gordon field in four-dimensional Minkowski spacetime. Once again, in both the massless and massive cases the Hadamard condition is expressed by the analog of Eq. (68):

$$\mu(x,x')=\mu_0(x,x')+w(x,x') , \tag{74}$$

where μ_0 denotes the two-point function of the Minkowski vacuum state, and w is a smooth bisolution. The relevant component of the stress-energy tensor is again given in terms of w by Eq. (20).

In the four-dimensional case, the analog of the class of solutions needed for our positivity argument may be defined as follows. Given a complete null geodesic γ, there exists a unique null plane Σ containing γ. Let v denote a null coordinate on Σ coinciding with an affine parameter along each generator, and let s denote the remaining pair of spacelike coordinates on Σ, with $s=s_0$ along γ. We seek solutions F, which have data of compact support on Cauchy surfaces and whose restriction to Σ yields a function $F_0\in C_0^\infty(\Sigma)$. As in the two-dimensional massive case, such solutions always exist, but, in general will only be C^0; integral constraints (for both the massive and massless case) on F_0 along each generator of Σ must be satisfied in order to obtain a C^n solution. Once again, however, if F_0 is of the form

$$F_0=\frac{\partial^n G_0}{\partial v^n} , \tag{75}$$

where $G_0\in C_0^\infty(\Sigma)$, then the solution F will be in C^n. Again, by the argument sketched in the note in proof to [9], any Hadamard state on $\mathcal{A}$ can be extended by continuity to a Hadamard state acting on an enlarged field algebra $\hat{\mathcal{A}}$ containing representatives of all solutions whose data on a spacelike Cauchy surface is C^5 and of compact support. Thus, we obtain the analog of Eq. (72), namely,

$$\left[2\mu_0[F_{01},F_{01}]+8\int_{\Sigma\times\Sigma}Y(v,s,v',s')F_{01}(v,s)F_{01}(v',s')\,dv\,d^2s\,dv'd^2s'\right]$$
$$\times\left[2\mu_0[F_{02},F_{02}]+8\int_{\Sigma\times\Sigma}Y(v,s,v',s')F_{02}(v,s)F_{02}(v',s')\,dv\,d^2s\,dv'\,d^2s'\right]\ge|\sigma[F_{01},F_{02}]|^2 \tag{76}$$

for all F_{01},F_{02} of the form (75) with $n=5$. Here d^2s denotes the volume element on the spacelike surface $\{v=\text{const}\}$, σ is given by the analog of Eq. (31),

$$\sigma[F_{01},F_{02}]=-4\,\mathrm{Im}\int\int_0^\infty k\overline{\hat{F}_{01}(k,s)}\hat{F}_{02}(k,s)\,dk\,d^2s , \tag{77}$$

Y is defined by the analog of Eq. (28),

$$Y(v,s,v',s')=\frac{\partial^2}{\partial v\partial v'}w(v,s,v',s') , \tag{78}$$

and again by the results of Appendix B of [9] (which hold for both the massive and massless cases), the analog of Eq. (26) holds:

$$\mu_0[F_0,F_0]=2\int\int_0^\infty k|\hat{F}_0(k,s)|^2dk\,d^2s . \tag{79}$$

Here $\hat{F}_0(k,s)$ denotes the Fourier transform of $F_0(v,s)$ with respect to the variable v only.

As in the massive case in two dimensions, Eq. (76) with F_0 restricted by Eq. (75) with $n=5$ does not suffice to enable us to prove ANEC. However, if w is restricted by the analog of Eq. (73), namely

$$\left|\int_{\Sigma\times\Sigma}Y(v,s,v',s')F_{01}(v,s)F_{02}(v',s')\,dv\,d^2s\,dv'd^2s'\right|^2$$
$$\le C\mu_0[F_{01},F_{01}]\,\mu_0[F_{02},F_{02}] , \tag{80}$$

then, by continuity, Eq. (76) will hold for all F_{01}, $F_{02}\in C_0^\infty(\Sigma)$. [As in the two-dimensional case, other (inequivalent) restrictions on w would suffice to ensure this.] We now choose F_{01} and F_{02} to be analogs of Eq (42):

$$\begin{aligned}F_{01}(v,s)&=c(v/\lambda)\cos(\kappa v)f(s) ,\\ F_{02}(v,s)&=c(v/\lambda)\sin(\kappa v)f(s) ,\end{aligned} \tag{81}$$

where $f(s)$ denotes an arbitrary smooth function of compact support normalized by

$$\int|f(s)|^2d^2s=1 .$$

Carrying through the steps of Sec. IV which led to Eq. (58), we now obtain

$$\liminf_{\lambda\to\infty}\int\int_{-\infty}^{\infty}Y(v,s,v,s')[c(v/\lambda)]^2$$
$$\times f(s)f(s')\,dv\,d^2s\,d^2s'\ge 0 . \tag{82}$$

However, Eq. (82), by itself, does not imply the desired result

$$\liminf_{\lambda\to\infty}\int_{-\infty}^{\infty}Y(v,s_0,v,s_0)[c(v/\lambda)]^2dv$$
$$=\liminf_{\lambda\to\infty}\int_\gamma\langle T_{ab}\rangle k^ak^b[c(v/\lambda)]^2dv\ge 0 .$$

In order to obtain this result, we need to impose further restrictions on Y (and hence on w) which, in effect, permit us to interchange the limit in λ with the integration over s and s' in a neighborhood of $s=s'=s_0$. A sufficient condition to ensure this is that, for any fixed s, s' in an open neighborhood $\mathcal{U}$ of s_0, the function $y_{s,s'}(v)\equiv Y(v,s,v,s')$ is in $L^1(\mathbb{R})$ and, as a vector in $L^1(\mathbb{R})$, varies continuously with s and s':

$$\forall(s,s')\in\mathcal{U}\times\mathcal{U},\quad y_{s,s'}\in L^1(\mathbb{R})\ ,$$

and (83)

$$y:\mathcal{U}\times\mathcal{U}\to L^1(\mathbb{R})\ \text{is continuous}\ .$$

[Note that this condition implies that the ANEC integral (1) exists.] In that case, for the support of f chosen within the neighborhood $\mathcal{U}$, the inequality (82) takes the form

$$\int d^2s\,d^2s' f(s)f(s')\int_{-\infty}^{\infty} Y(v,s,v,s')\,dv \ge 0 \tag{84}$$

[where the v integral converges and depends continuously on s, s' by virtue of assumption (83)]. However, since $f(s)$ is arbitrary, Eq. (84) immediately implies that

$$\int_{-\infty}^{\infty} Y(v,s_0,v,s_0)\,dv=\int_\gamma \langle T_{ab}\rangle k^a k^b dv \ge 0\ . \tag{85}$$

Thus, we have proved that the ANEC holds along γ provided that Y satisfies both conditions (80) and (83).

Note that conditions (80) and (83) (or some weakened versions of them) are needed for our proof. However, our proof makes use of the general positivity condition (17) only on a restricted class of solutions. It is possible that the satisfaction of the positivity condition on *all* solutions could imply the ANEC without the need to impose any additional restrictions on the state other than the Hadamard condition.

The above argument can be extended directly to prove the ANEC for a null geodesic generator γ of a bifurcate Killing horizon in a stationary, globally hyperbolic curved spacetime (possessing a Cauchy surface which contains the bifurcation surface), provided that an isometry-invariant Hadamard state of the field exists on that spacetime (see Ref. [9]). Namely, in such a spacetime, given an arbitrary compact subset K of a component of the horizon, we can find a Cauchy surface which contains K [9]. If a (necessarily unique [9]) stationary Hadamard state ω_0 exists, we can write the two-point function of an arbitrary Hadamard state as

$$\mu=\mu_0+w\ , \tag{86}$$

where now μ_0 denotes the two-point function of ω_0. The expected stress-energy tensor can be written as $\langle T_{ab}\rangle_0+\langle T_{ab}\rangle_w$, where $\langle T_{ab}\rangle_0$ is the expected stress-energy tensor in the state ω_0, and $\langle T_{ab}\rangle_w$ is given by the curved-spacetime analog of the coincidence limit formula of flat spacetime so that $\langle T_{ab}\rangle_w k^a k^b$ is again given by Eq. (20). However, along γ, the component $\langle T_{ab}\rangle_0 k^a k^b$ of the stress tensor in the state ω_0 vanishes. (Proof: Let ξ^a denote the Killing field which generates the Killing horizon. Then we have $\langle T_{ab}\rangle_0\xi^a\xi^b=e^{2\kappa t}\langle T_{ab}\rangle_0 k^a k^b$, where t denotes the Killing parameter and κ is the surface gravity of the horizon. We also have $\langle T_{ab}\rangle_0\xi^a\xi^b=0$ at the intersection of γ with the bifurcation surface, since $\xi^a=0$ there. However, $\langle T_{ab}\rangle_0\xi^a\xi^b$ is constant along γ by virtue of the isometry invariance of ω_0. Thus, $\langle T_{ab}\rangle_0 k^a k^b=0$ along γ.) Therefore, with Y defined as in Eq (78), the relevant component $\langle T_{ab}\rangle k^a k^b$ of the expected stress tensor along the generator γ is again given by the coincidence limit $Y(v,s_0,v,s_0)$. Furthermore, as shown in [9] the symmetrized two-point function μ_0 of ω_0 satisfies Eq. (79) along each component Σ of the bifurcate Killing horizon. In addition, it is shown in the "note in proof" of Ref. [9] that if F_0 satisfies a strengthened version of Eq. (75), then it gives rise to a C^5 solution. Hence, for F_{01} and F_{02} of this form, Y again satisfies the fundamental inequality (76). It then follows that if $w(x,x')$ of the Hadamard state (86) obeys the restrictions (80) and (83), then the ANEC holds along the generator γ.

Remarkably, the result of the above paragraph can be applied to establish the ANEC for a massive minimally coupled scalar field in de Sitter spacetime. Namely, since de Sitter spacetime admits a maximum number of Killing fields, any complete null geodesic γ is contained as a generator in some bifurcate Killing horizon. Moreover, the de Sitter vacuum [13] for the massive Klein-Gordon field is a Hadamard state ω_0, which is isometry invariant with respect to all Killing fields. Therefore, for a massive scalar field, the ANEC holds along every complete null geodesic γ in de Sitter space-time in any Hadamard state which obeys the conditions (80) and (83) with respect to the decomposition (86).

Note added. The following argument establishes that for a massless field the ANEC cannot hold in a general curved four-dimensional spacetime. Namely, if ANEC is to hold generally on all curved four-dimensional spacetimes, then for a metric of the form $\eta_{ab}+\gamma_{ab}$, the contribution to the ANEC integral which is linear in γ_{ab} must vanish (since otherwise by taking γ_{ab} small and reversing its sign if necessary, we would obtain a counterexample to the ANEC). This linear order contribution to $\langle T_{ab}\rangle$ for massless fields in the "in" vacuum state has been calculated by Horowitz [14]. A scaling argument (similar to the argument leading to Eq. (18) of Ref. [14]) shows that if the ANEC integral vanishes for all γ_{ab}, then the contribution arising from the linearized local curvature term $a\dot{A}_{ab}+b\dot{B}_{ab}$ (in the notation of Ref. [14]) also must vanish for all γ_{ab}. However, it is easily verified that (for $a\neq 0$) this is not the case in general for null geodesics that intersect the support of γ_{ab}. Nevertheless, it remains possible that the ANEC holds in all four-dimensional curved spacetimes along (achronal) null geodesics for which the contribution to the ANEC from the corresponding exact local curvature term vanishes. (Note that this is the case for the four-dimensional examples discussed in Sec. VI above.)

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