Bloch-Nordsieck cancellation of infrared divergences at finite temperature

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(Received 5 August 1991)

Virtual-photon corrections to the scattering of a heavy charged fermion, $M \gg T$, in a plasma result in infrared divergences that are worse than at $T=0$. These divergences cancel in the observable cross section, which must include a summation over the absorption and emission of real photons that are too soft to escape the plasma. The calculation requires determining the exclusive amplitudes for photon absorption and emission at $T \neq 0$. In the semiclassical approximation of Bloch and Nordsieck the problem can be solved exactly and the finite residual effects retained.

I. INTRODUCTION

At $T=0$ the Kinoshita-Lee-Nauenberg (KLN) theorem [1] guarantees that even though the perturbative expansion of S-matrix elements may be infrared singular, these singularities do not appear in the physically measurable probabilties. Experimental measurements cannot detect particles with an energy less than some threshold ε. To obtain a measurable probability one must compute the production rate for particles of energy less than ε and add them to the exclusive rate. The same theorem also guarantees that the physical probabilities are well behaved as the masses go to zero.

Infrared divergences are considerably worse at $T>0$ than at zero temperature because the usual infrared logarithms $\int dk/\omega$ are enhanced by the Bose-Einstein function:

$$\int_0^\infty \frac{dk}{\omega}\frac{1}{\exp(\omega/T)-1}\ , \tag{1.1}$$

which diverges like $\int dk/k^2$ for $\omega=k$. Strictly speaking, the actual divergence is eliminated by the bosons acquiring effective thermal masses so that $\omega \to m_{\rm eff}$ as $k \to 0$. This makes (1.1) finite and of order $T/m_{\rm eff}$. However, it is clear that this does not solve the practical problem [2] because nth-order corrections are of order $(\alpha T/m_{\rm eff})^n$ and $T/m_{\rm eff}$ is often so large that it would destroy perturbative expansions. For example, in a plasma of electrons, positrons, and photons if $T \gg m_e$ then $m_\gamma = eT/3$. Consequently $T/m_\gamma \approx 10$. In the nonrelativistic regime $T \ll m_e$, the thermal mass is $m_\gamma = e(n/m_e)^{1/2}$, where n is the electron density. This gives $T/m_\gamma \sim 10^2$ to 10^3 for main-sequence stars and even larger for less dense systems.

Since including the thermal masses is not sufficient, one must look elsewhere. The obvious extension of the KLN prescription is that an observable rate must include the possible absorption and emission of low-energy real particles in the plasma. Absorption of real bosons should be weighted by a Bose-Einstein factor N, emission by $1+N$. This extension of the KLN prescription is plausible but unproven. Explicit calculations have verified the cancellation to two-loop accuracy for dilepton production [3] and for ϕ^3 in 6 dimensions [4]. Calculations including the resummation of hard thermal loops [5] have been done for energy loss, dE/dx [6], and for real-photon production [7]. Recently Altherr [8] has shown that the KLN cancellation holds for decay rates to all orders in the coupling.

The physical context that underlies the definition of an observable cross section is quite different than at $T=0$. Measurements are performed outside the plasma, which has a finite size characterized by a length L. Any charge propagating through the plasma will radiate photons. Low-energy photons will have a sufficiently short mean free path, $\lambda(k) \ll L$, to be thermalized; high-energy ones will not. Define the energy threshold ε by

$$\lambda(\varepsilon) \approx L\ . \tag{1.2}$$

The energy ε is an essential property of the finite plasma. Photons with $k<\varepsilon$ are thermalized and the average number of photons with momentum k is given by the Bose-Einstein function $N(k)$. Photons with $k>\varepsilon$ are not thermalized; their average number is zero. This bimodal distribution has direct experimental consequences. When a charge radiates photons, those with $k>\varepsilon$ can escape and may be measured and may carry useful information. Radiation photons with $k<\varepsilon$ will have an emission probability enhanced by the stimulated-emission factor $1+N$. They will undergo too many collisions to be useful. A detector which measures their spectrum will only see a thermal distribution. At the same time, the propagating charge is also bombarded by low-energy photons ($k<\varepsilon$) from the plasma itself. The probability of absorption is proportional to $N(k)$. All the soft emissions and absorptions are unmeasurable by the detector.

The particular problem discussed in this paper is a plasma of electrons, positrons, and photons at arbitrary temperature T. A heavy fermion with mass $M \gg T$ enters a plasma with momentum p, undergoes some hard scattering, and leaves with momentum p'. The physically observable cross section to first order in α will be

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{obs}}=\frac{d\sigma}{dQ^2}+\int_0^{\varepsilon}d^3k(1+N)\frac{d\sigma^{em}}{d^3k\,dQ^2}+\int_0^{\varepsilon}d^3k\,N\frac{d\sigma^{ab}}{d^3k\,dQ^2}+O(\alpha^2)\ . \tag{1.3}$$

The first term $d\sigma/dQ^2$ is the exclusive cross section for scattering with no real photons. Since it includes virtual photons, this cross section will be infrared divergent. These divergences must be canceled by similar divergences in the rates for emission and absorption of real photons. At $T=0$, (1.3) reduces to the KLN prescription in which case ε is determined by the quality of the detector and can be made very small in a very expensive detector. At $T\neq 0$, the value of ε has nothing to do with the detector. Rather it is the maximum energy of photons that will thermalize as defined in (1.2).

This paper is concerned entirely with infrared divergences, not with collinear singularities that result from making the fermions massless. To order α one can calculate the rates in (1.3) by extending the analysis of Yennie, Frautschi, and Suura [9], to finite-temperature Feynman diagrams. However in higher orders the diagram approach fails to calculate the correct physical quantity. A correct calculation should be equivalent to computing the probabilities at $T=0$ for all processes and then thermally averaging the probabilities. The calculation may be reorganized so as to introduce exclusive probabilities that depend on T, which are later summed. But one should not indiscriminately square thermal Green's functions since they are themselves thermal averages. Squaring these thermal averages does not produce a thermal probability. This failure is equivalent to the discovery by Kobes and Semenoff [10] that the absorptive part of self-energy diagrams cannot be expressed as absolute squares of thermal Green's functions.

To go beyond one-loop order, it is instructive to defer the problem with thermal Green's functions by simplifying the problem. In the long-wavelength limit the emission and absorption of soft photons should produce a negligible recoil of the heavy fermion. This means that soft photons are emitted and absorbed independently and results in a Poisson distribution for both real and virtual photons [9]. The charged particle is treated as a classical current $j^{\mu}(x)$ coupled to the quantized radiation field $A_{\mu}(x)$. This semiclassical approximation, originated by Bloch and Nordsieck [11], makes the problem much easier than those examined in Refs. [2–8] and allows it to be solved exactly. However in order to trace the infrared cancellation of real photons against virtual photons, it is necessary to correctly define the exclusive amplitudes for absorbing and/or producing real photons at $T\neq 0$. The amplitudes are easily computed and the Bloch-Nordsieck cancellation is explicitly shown to all orders in α. The observable cross section indeed requires summing over all real photons that are absorbed or emitted by the current with energy below the threshold ε. The finite, observable cross section is

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{obs}}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}e^{B}\ , \tag{1.4}$$

$$B=-A\left[\ln\frac{\Lambda}{\varepsilon}+2\int_{\varepsilon}^{\infty}\frac{dk}{k}\frac{1}{\exp(k/T)-1}\right]\ , \tag{1.5}$$

$$A=\frac{2\alpha}{\pi}\left[\ln\frac{Q^2}{M^2}-1\right]\quad\text{for } Q\gg M\ . \tag{1.6}$$

Here Λ is an ultraviolet cutoff that must be introduced at $T=0$ since the semiclassical approximation does not treat the hard photons correctly.

The paper is organized as follows. Section II computes the one-loop corrections using finite-temperature Feynman diagrams. Section III introduces the semiclassical approximation and computes the infrared corrections to all orders. Section IV discusses the result and shows that $\varepsilon>M$ and usually $\varepsilon>T$.

II. ORDER-α INFRARED DIVERGENCES

The basic process is that of a fermion of mass $M\gg T$ entering a plasma with momentum p and leaving with momentum p'. Without radiative corrections the amplitude is $\bar{u}(p')\Gamma(p',p)u(p)$; the cross section is

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}=\frac{1}{16\pi s(s-4M^2)}\frac{1}{2!}\times\sum_{\text{spins}}|\bar{u}(p')\Gamma(p',p)u(p)|^2\ . \tag{2.1}$$

The task will be to calculate those radiative corrections that produce infrared divergences.

A. Virtual photons

The $\mathcal{O}(\alpha)$ corrections to exclusive scattering are shown in Fig. 1. The first-order vertex correction, Fig. 1(a), is

$$ie^2\int\frac{d^4k}{(2\pi)^4}D_{\mu\nu}(k)\bar{u}(p')X^{\mu\nu}u(p)\ ,$$

$$X^{\mu\nu}=\gamma^{\mu}S(p'-k)\Gamma(p'-k,p-k)S(p-k)\gamma^{\nu}\ .$$

Infrared divergences come from the fermion denominators when the virtual photon is on shell [9]:

$$D_{\mu\nu}(k)\rightarrow g_{\mu\nu}i\pi\delta(k^2)(1+2N),\quad N=\frac{1}{\exp(|\mathbf{k}|/T)-1}\ . \tag{2.2}$$

The integration is of the form

$$\int d^4k\ \delta(k^2)(1+2N)F(k^{\mu})\ . \tag{2.3}$$

The temperature-independent part will be infrared divergent if $F\sim 1/(k^0)^2$. The parts of F that behave like $1/k^0$ appear to produce temperature-dependent infrared divergences. However, since the integration (2.3) is even in k^{μ}, all $1/k^0$ parts cancel out so that only the most singular part of F, i.e, $1/(k^0)^2$, contributes. The only way F can behave like $1/(k^0)^2$ is if each fermion denominator contributes $1/k^0$. Therefore k^{μ} may be set equal to zero in the numerators and in $\Gamma(p'-k,p-k)$. This is a great simplification because it means there are only minimal changes from zero temperature. Each fermion propaga-

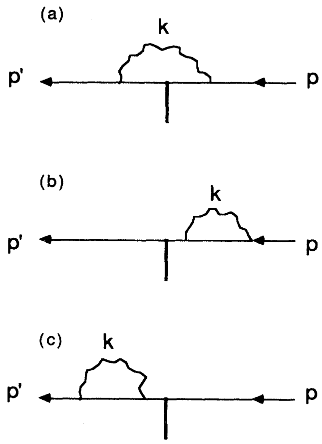

FIG. 1. Finite-temperature vertex and self-energy corrections. The vertical line represents the bare vertex $\Gamma(p_2,p_1)$ which may have complicated momentum dependence but has no virtual photons.

tor behaves like $1/k^0$ since, when $k^\mu k_\mu=0$,

$$(p-k)^2-M^2=-2(E_p k^0-p\cos\theta|k^0|)\ .$$

The fermion numerator can be rearranged using

$$(\not p-\not k+M)\gamma^\nu u(p)=u(p)(2p^\nu-k^\nu)-\tfrac{1}{2}[\not k,\gamma^\nu]u(p)\ . \tag{2.4}$$

Thus the infrared-singular term comes from

$$S(p-k)\gamma^\nu u(p)\rightarrow u(p)\frac{-p^\nu}{p\cdot k-i\eta}\ .$$

The first-order vertex correction is

$$\bar u(p')\Gamma(p',p)u(p)\frac{a}{2}\ , \tag{2.5}$$

$$\frac{a}{2}=-e^2\int\frac{d^4k}{(2\pi)^4}\pi\delta(k^2)(1+2N)\times g_{\mu\nu}\frac{p'^u}{p'\cdot k-i\eta}\frac{p^\nu}{p\cdot k-i\eta}\ . \tag{2.6}$$

Note that $a<0$.

The self-energy corrections, without mass-counterterm insertions, are shown in Figs. 1(b) and 1(c). The fermion self-energy is more complicated than at $T=0$. However, because $M\gg T$ the infrared-divergent part has the usual structure [12]

$$\Sigma_{\rm IR}(p)=\delta M+(\not p-M)f(p^2)\ . \tag{2.7}$$

The graph requires computing

$$\frac{1}{\not p-M}(\Sigma_{\rm IR}-\delta M)u(p)=f(p^2)u(p)\ .$$

Because of the structure (2.7), one can compute this by differentiating:

$$\frac{p^\alpha}{M}\frac{\partial\Sigma_{\rm IR}}{\partial p^\alpha}u(p)=f(p^2)u(p)\ .$$

Therefore Fig. 1(b) is given by

$$\frac{1}{2}\bar u(p')\Gamma(p',p)\frac{p^\alpha}{m}\frac{\partial Y}{\partial p^\alpha}u(p)\ ,$$

$$Y=ie^2\int\frac{d^4k}{(2\pi)^4}D_{\mu\nu}(k)\gamma^\mu S(p-k)\gamma^\nu\ .$$

Working out the derivative in the infrared limit gives

$$\left[\frac{p^\alpha}{M}\frac{\partial}{\partial p^\alpha}\gamma_\mu S(p-k)\gamma^\mu\right]u(p)\rightarrow-\left[\frac{p^\mu}{p\cdot k-i\eta}\right]^2u(p)\ .$$

Thus the self-energy corrections are

$$\bar u(p')\Gamma(p',p)u(p)\left[\frac{b}{2}+\frac{c}{2}\right]\ , \tag{2.8}$$

where

$$b=e^2\int\frac{d^4k}{(2\pi)^4}\pi\delta(k^2)(1+2N)\left[\frac{p^\mu}{p\cdot k-i\eta}\right]^2 \tag{2.9}$$

and c is the same but with p replaced by p'.

The total infrared-divergent part of the first-order amplitude for scattering with no photon absorption and no photon emission is of the sum of (2.5) and (2.8):

$$\mathcal{M}(0,0)=\bar u(p')\Gamma(p',p)u(p)\frac{a+b+c}{2}\ . \tag{2.10}$$

By defining

$$R^\mu(k)=e\left[\frac{p'^\mu}{p'\cdot k-i\eta}-\frac{p^\mu}{p\cdot k-i\eta}\right]$$

the combination can be written as

$$\begin{aligned}a+b+c&=\int\frac{d^4k}{(2\pi)^4}\pi\delta(k^2)(1+2N)R_\mu(k)R^\mu(k)\\&=\int_0^\Lambda\frac{d^3k}{(2\pi)^3 2k}(1+2N)\\&\quad\times\tfrac{1}{2}[R_\mu(k)R^\mu(k)+R_\mu(-k)R^\mu(-k)]\ .\end{aligned} \tag{2.11}$$

The usual ultraviolet cancellation between the vertex and self-energy corrections has been lost since only the infrared-singular parts were retained. Consequently a cutoff Λ is required for the $T=0$ part in (2.11). Since $k_\mu R^\mu=0$, R^μ is spacelike. Therefore $R^2<0$ and (2.11) is negative, infrared divergent. From (2.10) the cross section for exclusive scattering is

$$\frac{d\sigma}{dQ^2}=\left[\frac{d\sigma}{dQ^2}\right]_{\rm bare}(1+a+b+c) \tag{2.12}$$

though the infrared divergence renders it meaningless.

B. Real-photon emission and absorption

The two possible diagrams for photon emission are shown in Fig. 2. The amplitude for emission of a photon with momentum k and polarization ϵ_μ is

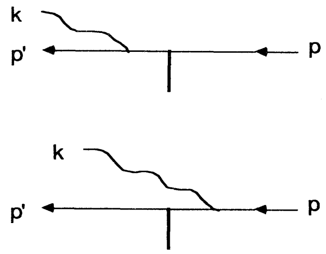

FIG. 2. Finite-temperature diagrams for the emission of a real photon.

$$\mathcal{M}(1,0)=ie\bar{u}(p')\not{\epsilon}S(p'+k)\Gamma(p'+k,p)u(p) + ie\bar{u}(p')\Gamma(p',p-k)S(p-k)\not{\epsilon}u(p) \ . \tag{2.13}$$

In the infrared limit, using (2.4), this becomes

$$\mathcal{M}(1,0)\to\bar{u}(p')\Gamma(p',p)u(p)\epsilon_\mu J^\mu(k) \tag{2.14}$$

where

$$J^\mu(k)=ie\left[\frac{p'^u}{p'\cdot k+i\eta}-\frac{p^\mu}{p\cdot k-i\eta}\right] . \tag{2.15}$$

The amplitude for scattering with the emission of one photon is

$$(2\pi)^3 2k\frac{d\sigma^{em}}{d^3k\, dQ^2}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}\sum_{\text{pol}}|\epsilon_\mu J^\mu(k)|^2 \ . \tag{2.16}$$

Similarly the amplitudes shown in Fig. 3 for photon absorption have the infrared behavior

$$\mathcal{M}(0,1)\to\bar{u}(p')\Gamma(p',p)u(p)\epsilon_\mu J^\mu(-k) \tag{2.17}$$

and the cross section for absorption

$$(2\pi)^3 2k\frac{d\sigma^{ab}}{d^3k\, dQ^2}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}\sum_{\text{pol}}|\epsilon_\mu J^\mu(-k)|^2 \ . \tag{2.18}$$

The emission and absorption cross sections are automatically equal since

$$J^\mu(-k)=J^{\mu *}(k) \ . \tag{2.19}$$

The current (2.15) will be of importance in Sec. III. For the moment, note that because of the identity

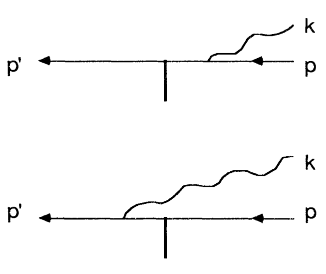

FIG. 3. Finite-temperature diagrams for the absorption of a real photon.

$$\tfrac{1}{2}[R_\mu(k)R^\mu(k)+R_\mu(-k)R^\mu(-k)]=-\sum_{\text{pol}}|\epsilon_\mu J^\mu(k)|^2 \tag{2.20}$$

the corrections to the exclusive rate can be written

$$a+b+c=-\int_0^\Lambda \frac{d^3k}{(2\pi)^3 2k}(1+2N)\sum_{\text{pol}}|\epsilon_\mu J^\mu(k)|^2 \ . \tag{2.21}$$

C. Cancellation of infrared divergences

In the plasma it is not possible to distinguish exclusive scattering (2.12) from scattering in which a very-low-energy photon is emitted or absorbed by the charge. Low-energy photons can have a mean free path that is sufficiently small to produce a thermal distribution of real photons. Let the threshold energy for photon thermalization be ε. Then the observable cross section must allow for the possibility that an undetected low-energy photon was either emitted or absorbed. The statistical weightings of these events are $1+N$ and N, respectively. Consequently,

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{obs}}=\frac{d\sigma}{dQ^2}+\int_0^\varepsilon d^3k(1+N)\frac{d\sigma^{em}}{d^3k\, dQ^2} + \int_0^\varepsilon d^3k\, N\frac{d\sigma^{ab}}{d^3k\, dQ^2} \ . \tag{2.22}$$

This combination is infrared finite when (2.12), (2.16), (2.18), and (2.21) are substituted. The negative contribution of the virtual, on-shell photons with $0<k<\varepsilon$ cancels against the positive contribution of the real emitted and absorbed photons. All that remains is the integration over the virtual photons from ε to Λ:

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{obs}}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}(1+B) \ , \tag{2.22'}$$

where B is infrared finite:

$$B=-\int_\varepsilon^\Lambda \frac{d^3k}{2k(2\pi)^3}(1+2N)\sum_{\text{pol}}|\epsilon_\mu J_\mu(k)|^2 \ . \tag{2.23}$$

III. INFRARED CANCELLATION TO ALL ORDERS

A. Semiclassical approximation

As explained in the Introduction, in order to compute to higher order it is convenient to make the semiclassical approximation of Bloch and Nordsieck [11]. The four-vector (2.15) has a simple physical interpretation. Let

$$j^\mu(x)=\int\frac{d^4k}{(2\pi)^4}\exp(-ik\cdot x)J^\mu(k) \ . \tag{3.1}$$

Explicitly calculating this gives

$$j^\mu(x)=\begin{cases} e\delta^3(\mathbf{x}-t\mathbf{p}/E)p^\mu/E & \text{if } t<0 \ ,\\ e\delta^3(\mathbf{x}-t\mathbf{p}'/E')p^{\mu\prime}/E' & \text{if } t>0 \ . \end{cases} \tag{3.2}$$

It thus is the classical current for a charge moving at constant velocity **v** before time $t=0$ and at a different constant velocity **v**′ after time $t=0$. This classical current is coupled to the quantized radiation field by

$$\mathcal{L}_1 = -j^\mu(x) A_\mu(x) \ . \tag{3.3}$$

Because only A_μ is quantized, the commutator $[\mathcal{L}_I(x), \mathcal{L}_I(x')]$ is a c number. This makes the scattering operator very simple [13,14]. Up to an overall phase it is

$$S = \exp\left[i \int d^4x \, \mathcal{L}_I(x) \right] \ , \tag{3.4}$$

$$\int d^4x \, \mathcal{L}_I(x) = \int \frac{d^3k}{(2\pi)^3 2k} [a(k)\epsilon_\mu J^{\mu *}(k) + a^\dagger(k)\epsilon_\mu J^\mu(k)] \ , \tag{3.5}$$

where a sum over the two polarization states is always implied. The photon creation operators are normalized by

$$[a(k), a^\dagger(k')] = (2\pi)^3 2k^0 \delta(\mathbf{k} - \mathbf{k}') \ . \tag{3.6}$$

At $T=0$ the virtual-photon corrections of order α are given by

$$\left\langle 0 \left| \left[i \int d^4x \, \mathcal{L}_I \right]^2 \right| 0 \right\rangle = - \int \frac{d^3k}{2k(2\pi)^3} |\epsilon_\mu J^\mu(k)|^2 \ . \tag{3.7}$$

B. Box normalization

It is very convenient to instead normalize the photons in a box of volume V so that the momenta are discrete. Then (3.6) is replaced by

$$[a_j, a_l^\dagger] = \delta_{j,l} \tag{3.8}$$

and (3.5) by

$$\int d^4x \, \mathcal{L}_I(x) = \sum_l \frac{1}{\sqrt{2k_l V}} [a_l \epsilon_\mu J^{\mu *}(k_l) + a_l^\dagger \epsilon_\mu J^\mu(k_l)] \ . \tag{3.9}$$

The scattering operator is a product over each mode. If we define

$$S_l = \exp(iX_l^* a_l + iX_l a_l^\dagger) \ , \quad X_l = \frac{\epsilon_\mu J^\mu(k_l)}{\sqrt{2k_l V}} \ , \tag{3.10}$$

then

$$S = \prod_l S_l \ . \tag{3.11}$$

Each plasma state is a direct product of free-photon modes labeled by a set of occupation numbers $\{n_l\}$:

$$|I\rangle = |n_1, n_2, n_3, \ldots\rangle = \prod_l \frac{(a_l^\dagger)^{n_l}}{\sqrt{(n_l)!}} |0, 0, \ldots\rangle \ . \tag{3.12}$$

It has energy

$$E_I = \sum_l n_l k_l \ . \tag{3.13}$$

A general final state will be of the form $|F\rangle = |m_1, m_2, m_3, \ldots\rangle$. Every matrix element therefore factors:

$$\langle F|S|I\rangle = \prod_l \langle m_l|S_l|n_l\rangle \ . \tag{3.14}$$

(Note that $E_F \neq E_I$ generally because of the external current.) Since the interaction only changes the energy levels of each mode, the partition function is the same as for free photons: $Z = \prod_l Z_l$, where

$$Z_l = \sum_{p=0}^{\infty} \exp(-p|k_l|/T) = \frac{1}{1 - \exp(-|k_l|T)} \ . \tag{3.15}$$

C. Definition of exclusive amplitudes at $T \neq 0$

The real difficulty is in deciding what to calculate. It is helpful to first review the procedure at $T=0$. Since all amplitudes factor, one can focus on the amplitude $\langle m_l|S_l|n_l\rangle$ for a single mode. At zero temperature the initial state is empty. Consequently $\langle 0|S_l|0\rangle$ is the amplitude for the purely exclusive process in which no real photons are radiated, but all virtual photons are included. Similarly $\langle 1|S_l|0\rangle$ is the amplitude for emitting one real photon with all virtual-photon corrections. A realistic photon detector will not respond to photons with an energy k_l less than some threshold energy ε. For modes below this threshold any number of photons can be emitted without triggering the detector. The appropriate probability is a sum over all the occupation numbers of that mode:

$$P_l = \sum_{m=0}^{\infty} |\langle m|S_l|0\rangle|^2 \quad (k_l < \varepsilon) \ . \tag{3.16}$$

Using completeness and unitarity, this sum is 1. That eliminates the infrared divergences. For modes with energy above threshold, $k_l > \varepsilon$, the detector vetos any photon emission process so that the single-mode amplitude is $|\langle 0|S_l|0\rangle|^2$. The observable probability is

$$P_{\text{obs}} = \prod_{k_l > \varepsilon} |\langle 0|S_l|0\rangle|^2 \tag{3.17}$$

and is free of infrared divergencies.

For $T \neq 0$, there is a threshold energy ε for thermalization defined by (1.2). There is a Bose-Einstein distribution of real photons with $k_l < \varepsilon$ that are constantly absorbed and reemitted. The appropriate probability for these modes is again an inclusive sum over all final occupation numbers. The new feature is that the initial state is not empty, but may have any occupation number n with probability $\exp(-nk_l/T)/Z_l$. Thus the probability is

$$P_l = \frac{1}{Z_l} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\langle m|S_l|n\rangle|^2 \exp(-nk_l/T) \quad (k_l < \varepsilon) \ . \tag{3.18}$$

Again using completeness and unitarity, this sum is 1. Thus all infrared divergences again cancel.

Despite the formal cancellation of infrared divergences in (3.18) the remaining nontrivial problem is what replaces (3.17) for the $k_l > \varepsilon$ modes. In particular, it is necessary to generalize the notion of exclusive amplitudes to $T \neq 0$ in order to distinguish virtual photons from real photons. The answer is as follows. The amplitude for the charge to enter the plasma with momentum p and leave with momentum p' with no emission or absorption of real photons in mode l is

$$\mathcal{M}_l(0,0) = \frac{1}{Z_l} \sum_{m=0}^{\infty} \langle m | S_l | m \rangle \exp(-mk_l/T) \ . \tag{3.19}$$

This includes all virtual photons, but no real photons. It replaces $\langle 0|S|0\rangle$. The amplitude for the emission of one real photon is

$$\mathcal{M}_l(1,0) = \frac{1}{Z_l} \sum_{m=0}^{\infty} \langle m | [a_l, S_l] | m \rangle \exp(-mk_l/T) \ . \tag{3.20}$$

The matrix element is of a commutator because $\langle m | a_l S_l | m \rangle$ without the commutator would allow the photon a_l to come directly from the heat bath as represented by $|m\rangle$. The amplitude for absorption of one real photon is

$$\mathcal{M}_l(0,1) = \frac{1}{Z_l} \sum_{m=0}^{\infty} \langle m | [S_l, a_l^\dagger] | m \rangle \exp(-mk_l/T) \ . \tag{3.21}$$

Again the commutator guarantees that the photon is absorbed by the interaction, not just lost in the heat bath. The general exclusive amplitude for absorbing r real photons and emitting s real photons is

$$\mathcal{M}_l(s,r) = \frac{1}{Z_l} \sum_{m=0}^{\infty} \langle m | [(a_l)^s S_l (a_l^\dagger)^r]_{\text{con}} | m \rangle \times \exp(-mk_l/T) \ , \tag{3.22}$$

where the subscript "con" means connected as defined by the repeated commutator. For example,

$$[(a_l)^2 S_l a_l^\dagger]_{\text{con}} = [a_l, [a_l, [S_l, a_l^\dagger]]] \ . \tag{3.23}$$

The commutators may be taken in any order because $[a, a^\dagger]$ commutes with S.

The above definitions are physically motivated but their more fundamental significance is as follows. Return to expression (3.18). Let S_l be any operator, not necessarily unitary and not necessarily related to the scattering problem. The transition probability $|\langle m | S_l | n \rangle|^2$ is thermally averaged over initial states and summed over final states in (3.18). But the initial and final states are mixtures of real and virtual photons. The exclusive amplitudes (3.22) are supposed to describe initial and final photons that are real. The temperature dependence of the exclusive amplitudes $\mathcal{M}$ may be very complicated. Real photons in the heat bath have a statistical absorption probability N_l and an emission probability $1+N_l$ where

$$N_l = \frac{1}{\exp(|k_l|/T) - 1} \ . \tag{3.24}$$

Therefore the test of the exclusive amplitudes (3.22) is that for any operator S_l they satisfy

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(N_l)^r}{r!} \frac{(1+N_l)^s}{s!} |\mathcal{M}_l(s,r)|^2 = P_l \ , \tag{3.25}$$

where P_l is defined by the inclusive sum on the right-hand side of (3.18).

D. Virtual photons

Let us first calculate the amplitude for the charge to enter the plasma with momentum p and leave with momentum p' with no emission or absorption of real photons. This is the amplitude

$$\mathcal{M}_l(0,0) = \frac{1}{Z_l} \sum_{m=0}^{\infty} \langle m | S_l | m \rangle \exp(-mk_l/T) \ . \tag{3.26}$$

To evaluate this, we first use Glauber's identity [14] to normal order S_l:

$$S_l = \exp(-|X_l|^2/2) \exp(iX_l a_l^\dagger) \exp(iX_l^* a_l) \ . \tag{3.27}$$

Consequently $\langle 0|S_l|0\rangle = \exp(-|X_l|^2/2)$. For (3.26) we need the general diagonal matrix element

$$\langle m | S_l | m \rangle = \exp(-|X_l|^2/2) \sum_{j=0}^{m} (-|X_l|^2)^j \frac{m!}{j! j! (m-j)!} \ . \tag{3.28}$$

Inserting this into (3.26) and interchanging the order of summation gives

$$\mathcal{M}_l(0,0) = \exp(-|X_l|^2/2) \sum_{j=0}^{\infty} \frac{1}{j!} (-|X_l|^2)^j \times \frac{1}{Z_l} \sum_{m=j}^{\infty} \frac{m!}{j!(m-j)!} \exp(-mk_l/T) \ . \tag{3.29}$$

The sum over m in the second line is elementary and gives powers of the Bose-Einstein function: $(N_l)^j$. The amplitude is therefore

$$\mathcal{M}_l(0,0) = \exp(-|X_l|^2/2) \sum_{j=0}^{\infty} \frac{(-|X_l|^2)^j}{j!} (N_l)^j \ . \tag{3.30}$$

Performing the summation yields

$$\mathcal{M}_l(0,0) = \exp[-|X_l|^2(N_l + \tfrac{1}{2})] \ . \tag{3.31}$$

The complete amplitude for no real photons in any mode l is the product

$$\mathcal{M}(0,0) = \prod_l \mathcal{M}_l(0,0) = \exp(B_{\text{vir}}/2) \tag{3.32}$$

where

$$B_{\text{vir}} = -\sum_l |X_l|^2 (1+2N_l) = -\int_0^{\Lambda} \frac{d^3k}{(2\pi)^3 2k} (1+2N) \sum_{\text{pol}} |\epsilon_\mu J^\mu(k)|^2 \ . \tag{3.33}$$

This is infrared divergent. The exclusive cross section for scattering with no absorption or emission is

$$\frac{d\sigma}{dQ^2}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}\exp(B_{\text{vir}}) \ . \tag{3.34}$$

The first-order result (2.12) results from the $1+B_{\text{vir}}$ expansion of the exponential. Since (3.33) is negative and infinite, the cross section is zero for the charge to scatter without absorbing or emitting radiation.

E. Real-photon emission and absorption

The amplitude for emitting one real photon (3.20) requires the commutator $[a_l,S_l]=iX_lS_l$. The amplitude for absorbing one real photon (3.21) requires $[S_l,a_l^\dagger]=iX_l^*S_l$. The general exclusive amplitude (3.22) for absorbing r photons and emitting s photons requires the repeated commutator:

$$\mathcal{M}_l(s,r)=(iX_l)^s(iX_l^*)^r\frac{1}{Z_l}\times\sum_{m=0}^{\infty}\langle m|S_l|m\rangle\exp(-mk_l/T) \ . \tag{3.35}$$

This sum was already calculated in the previous section. Thus,

$$\mathcal{M}_l(s,r)=(iX_l)^s(iX_l^*)^r\exp[-|X_l|^2(N_l+\tfrac{1}{2})] \ . \tag{3.36}$$

F. Cancellation of infrared divergences

Photons emitted by the charged particle or absorbed by the charged particle with $k_l<\varepsilon$ cannot be detected. For these modes the physical probability is

$$P_l=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(N_l)^r}{r!}\frac{(1+N_l)^s}{s!}|\mathcal{M}_s(s,r)|^2 \quad (k_l<\varepsilon) \ . \tag{3.37}$$

The virtual photons in $\mathcal{M}_l$ will now be canceled by the real photons in this sum. Inserting the explicit form (3.36) gives

$$P_l=C_IC_F\exp[-|X_l|^2(2N_l+1)] \ , \tag{3.38}$$

where

$$C_K=\sum_{r=0}^{\infty}\frac{(N_l)^r}{r!}(|X_l|^2)=\exp(|X_l|^2N_l) \ , \tag{3.39}$$

$$C_f=\sum_{s=0}^{\infty}\frac{(1+N_l)^2}{s!}(|X_l|^2)^s =\exp[|X_l|^2(1+N_l)] \ . \tag{3.40}$$

Combining gives $P_l=1$ for the below-threshold modes. This eliminates the infrared divergence, but now we are equipped to calculate the residual finite effects.

The observable probability for the charge to propagate through the plasma with no detectable photon absorption or emission is

$$P_{\text{obs}}=\prod_{k_l>\varepsilon}|\mathcal{M}_l(0,0)|^2 \ . \tag{3.41}$$

This generalizes (3.17) to $T\neq 0$. Explicitly, $P_{\text{obs}}=e^B$ where

$$B=-\sum_{k_l>\varepsilon}|X_l|^2(1+2N_l) =-\int_{\varepsilon}^{\Lambda}\frac{d^3k}{(2\pi)^3 2k}(1+2N)\sum_{\text{pol}}|\epsilon_\mu J^\mu(k)|^2 \ , \tag{3.42}$$

where Λ is the ultraviolet cutoff necessitated by the failure of the semiclassical approximation for hard photons. The physically observable cross section is

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{obs}}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}e^B \ . \tag{3.43}$$

This has no infrared divergence but does give a finite correction to the scattering.

G. Calculation of B

Using the current (2.15) the integrand of B contains

$$\sum_{\text{pol}}|\epsilon_\mu J^\mu(k)|^2=e^2\left[\frac{2p\cdot p'}{(p\cdot k)(p'\cdot k)}-\frac{m^2}{(p\cdot k)^2}-\frac{m^2}{(p'\cdot k)^2}\right] \ . \tag{3.44}$$

The integration over photon angles gives

$$\int\frac{d\Omega}{(2\pi)^3}\sum_{\text{pol}}|\epsilon_\mu J^\mu(k)|^2=\frac{e^2}{\pi}\frac{1}{k^2}(f-1) \ , \tag{3.45}$$

where f is a function of the momentum transfer $Q=|(p-p')^2|^{1/2}$:

$$f=\frac{2Q^2+4M^2}{Q(Q^2+4M^2)^{1/2}}\ln\left\{\frac{Q}{2M}+\left[1+\left[\frac{Q}{2M}\right]^2\right]^{1/2}\right\} \ . \tag{3.46}$$

In the limits of small and large momentum transfer,

$$f\to 1+\frac{Q^2}{3M^2} \quad \text{for } Q\ll M \ , \tag{3.47}$$

$$f\to \ln\left[\frac{Q^2}{M^2}\right] \quad \text{for } Q\gg M \ . \tag{3.48}$$

Consequently

$$B=-\frac{2\alpha}{\pi}(f-1)\int_{\varepsilon}^{\Lambda}\frac{dk}{k}(1+2N) =-\frac{2\alpha}{\pi}(f-1)\left[\ln\left[\frac{\Lambda}{\varepsilon}\right]+2I(\varepsilon)\right] \ , \tag{3.49}$$

where

$$I(\varepsilon)=\int_{\varepsilon}^{\infty}\frac{dk}{k}\frac{1}{\exp(k/T)-1} \ . \tag{3.50}$$

How large a contribution I is will be discussed next.

IV. DISCUSSION

The size of ε determines how large the infrared corrections will be, but the determination of ε does not depend on the semiclassical approximation. As explained in the Introduction, high-energy photons have too large a mean free path for them to thermalize. The threshold ε is the energy at which the mean free path becomes comparable to the plasma size L:

$$\lambda(\varepsilon)\approx L\ . \tag{4.1}$$

For a photon of energy k, the mean free path is $\lambda(k)=1/n\sigma(k)$ where n is the electron density and $\sigma(k)$ is the Compton cross section. The cross section is essentially constant for $k<m_e$ and decreases when $k>m_e$, which increases the mean free path. Figure 4 shows that (4.1) is always satisfied at $\varepsilon>m_e$.

(1) Minimum possible ε/T. Small values of ε/T can only occur when $T\gg m_e$ in which case the density $n\approx T^3$. The Compton cross section falls with energy like $\sigma\approx\alpha^2/m_e k$ for $\varepsilon\gg m_e$. Using (4.1) gives

$$\varepsilon\approx\frac{\alpha^2 LT^3}{m_e}\quad(\varepsilon>m_e)\ . \tag{4.2}$$

For very small L one must also check the constraint

$$\varepsilon>\frac{2\pi}{L}\ . \tag{4.3}$$

The minimum energy that satisfies (4.2) and (4.3) is $\varepsilon\approx\alpha T(2\pi T/m_e)^{1/2}$. Imposing $\varepsilon>m_e$ then gives

$$\frac{\varepsilon_{\min}}{T}\approx 1.8\alpha^{2/3}\ . \tag{4.4}$$

It does not seem possible to actually achieve such a small value. For example, in an electron-positron plasma this would occur in a plasma of dimension $L=600$ fm and temperature $T\approx 7$ MeV at an energy $\varepsilon\approx 2$ MeV. A plasma with any other L or T would have a larger ε/T.

In the unlikely case that $\varepsilon<T$ the corrections will be large. The integral I is evaluated in the Appendix with the result

$$I(\varepsilon)\approx\frac{T}{\varepsilon}-\frac{1}{2}\ln\left[\frac{2\pi T}{\varepsilon}\right]+\gamma/2\quad(\varepsilon\ll T)\ . \tag{4.5}$$

The ln ε cancels when this is substituted into (3.49):

$$B\approx-\frac{2\alpha}{\pi}(f-1)\left[\frac{2T}{\varepsilon}+\ln\frac{\Lambda}{2\pi T}+\frac{\gamma}{2}\right]\ . \tag{4.6}$$

With the worst-case value (4.4) this gives an exponent $B\approx-0.7\alpha^{1/3}(f-1)$.

(2) Typical ε. Except for the extreme case discussed above, one usually has $\varepsilon>T$. The corrections for large ε are

$$I(\varepsilon)\approx\frac{T}{\varepsilon}\exp(-\varepsilon/T)\quad(\varepsilon\gg T)\ . \tag{4.7}$$

This applies for all low-temperature ($\varepsilon>m_e>T$) and for most high-temperature plasmas.

(3) Average energy loss. The average energy lost by the charge to the heat bath in mode k_l is

$$\langle\Delta E_l\rangle=\sum_{r=0}\sum_{s=0}(sk_l-rk_l)\frac{(N_l)^r}{r!}\frac{(1+N_l)^s}{s!}|\mathcal{M}(r,s)|^2$$
$$=k_l|X_l|^2\ . \tag{4.8}$$

Summing this over all unobserved modes gives

$$\langle\Delta E\rangle=\int_0^{\varepsilon}\frac{d^3k}{2k(2\pi)^3}k\sum_{\text{pol}}|\epsilon_\mu J^\mu|^2$$
$$=\frac{2\alpha}{\pi}[f(Q)-1]\varepsilon\ . \tag{4.9}$$

This is not only finite, it is numerically small. Consequently it was consistent to ignore the recoil of the charged fermion and replace it by a classical current.

(4) Bremsstrahlung cross section. Suppose one photon in mode $k_l>\varepsilon$ is detected. Since $\mathcal{M}_l(1,0)=iX_l M_l(0,0)$ the bremsstrahlung cross section is

$$\left[\frac{d\sigma}{dQ^2}\right]_{\text{brem}}=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}|X|^2\int_{k_{\min}}^{k_{\max}}\frac{V\,d^3k}{(2\pi)^3}\prod_{k_l>\varepsilon}|\mathcal{M}_l(0,0)|^2$$
$$=\left[\frac{d\sigma}{dQ^2}\right]_{\text{bare}}\frac{2\alpha}{\pi}[f(Q)-1]\ln\left[\frac{k_{\max}}{k_{\min}}\right]e^B\ . \tag{4.10}$$

(5) Inclusive-exclusive connection. A further check on the definition (3.22) of the exclusive amplitudes is the following. Let $\bar r$ and $\bar s$ be positive integers. Consider the inclusive process in which at least $\bar r$ real photons are absorbed and at least $\bar s$ real photons are emitted. In terms of exclusive amplitudes the total probability for this process is

$$\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(N_l)^r}{r!}\frac{(1+N_l)^s}{s!}|\mathcal{M}_l(s+\bar s,r+\bar r)|^2\ . \tag{4.11}$$

One can prove that this is identically equal to

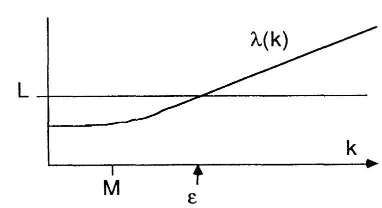

FIG. 4. Schematic plot of the energy dependence of the photon mean free path $\lambda(k)$ showing the threshold energy ε below which photons will be thermalized in a plasma of finite size L.

$$\frac{1}{Z_l}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left|\langle m|[(a_l)^{\bar{s}}S_l(a_l^\dagger)^{\bar{r}}]_{\text{con}}|n\rangle\right|^2\exp(-nk_l/T)\ . \tag{4.12}$$

Section III, in particular (3.25), only required $\bar{r}=\bar{s}=0$. The equality of (4.11) and (4.12) is true regardless of the operator S_l. For the present case the inclusive probability (calculated by either method) is $|X_l^2|^{\bar{r}+\bar{s}}$.

ACKNOWLEDGMENTS

This work was supported in part by the U.S. National Science Foundation under Grant No. PHY-9008431.

APPENDIX

In the limit $\varepsilon \ll T$ the thermal integral I defined in (3.50) can be done as follows. Let $a=\varepsilon/T$ so

$$I(a)=\int_a^\infty \frac{dx}{x}\frac{1}{(e^x-1)}\ . \tag{A1}$$

Write

$$\frac{1}{e^x-1}=\frac{1}{x}x^{-x}+\tfrac{1}{2}e^{-x}+r(x)\ . \tag{A2}$$

Then

$$I(a)=\int_0^\infty dx\left[\frac{1}{x^2}e^{-x}+\frac{1}{2x}e^{-x}\right]+H(a)\ , \tag{A3}$$

$$H(a)=\int_a^\infty \frac{dx}{x}r(x)\ . \tag{A4}$$

$H(a)$ is convergent at small x since $r(x)\to x/12+O(x^2)$ and convergent at large x since $r(x)$ falls exponentially.

The first two terms in (A.3) give exponential integral functions $E_n(a)$:

$$I(a)=\frac{1}{a}E_2(a)+\frac{1}{2}E_1(a)+H(a)\ . \tag{A5}$$

As $a\to 0$ these give

$$I(a)\to\frac{1}{a}+\frac{1}{2}\ln(a)+\frac{1}{2}\gamma-1+H(0)\ . \tag{A6}$$

The remaining integration to perform is

$$H(0)=\int_0^\infty dx\left[\frac{1}{x(e^x-1)}-\frac{1}{x}e^{-x}-\frac{1}{2}e^{-x}\right]\ . \tag{A7}$$

This is convergent both at small x and at large x. However if one splits it into three integrals then each is separately infrared divergent. For evaluation it is easier to compute

$$H_\mu(0)=\int_a^\infty dx\left[\frac{x^{\mu-1}}{e^x-1}-x^{\mu-1}e^{-x}-\frac{1}{2}x^\mu e^{-x}\right]\ . \tag{A8}$$

For $\mu>1$ none of the three integrals has an infrared divergence. One can evaluate each of these separately, add them together, and then analytically continue down to $\mu=0$. The first integral yields a Riemann zeta function and the other two are just Γ functions. Combining gives

$$H_\mu(0)=\frac{1}{2}\Gamma(\mu)\left[2\zeta(\mu)+\frac{1+\mu}{1-\mu}\right]\ . \tag{A9}$$

This is valid for $\text{Re}(\mu)>-1$. At $\mu=0$ it has the value

$$H(0)=\lim_{\mu\to 0}H_\mu(0)=1-\tfrac{1}{2}\ln(2\pi)\ . \tag{A10}$$

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