Improved analytic theory of the muon anomalous magnetic moment

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(Received 14 February 1991)

We consider the recent results of Kinoshita in which he improves the accuracy of the theoretical value for the anomalous magnetic moment of the muon. This is needed now that a new, more accurate experiment has been approved at Brookhaven National Laboratory. Kinoshita's results are completely numerical. Here we perform an independent check of his results in fourth and sixth order by analytical means, using expansions in the small mass ratios which occur in the computation. Our result for the fourth-order contribution is $a_\mu^{(4)}-a_e^{(4)}=5\,904\,475.1(3)\times10^{-12}$. This is to be compared with Kinoshita's result $a_\mu^{(4)}-a_e^{(4)}=5\,904\,485\times10^{-12}$. For the sixth-order vacuum-polarization contribution we obtain $(a_\mu^{(6)}-a_e^{(6)})$(vacuum polarization)$=24\,064.8(6)\times10^{-12}$. Kinoshita's result is $(a_\mu^{(6)}-a_e^{(6)})$(vacuum polarization)$=24\,069(6)\times10^{-12}$. Our result for the total QED contribution is $a_\mu^{\rm QED}=1\,165\,846\,943(28)(27)\times10^{-12}$. This agrees with Kinoshita's result $a_\mu^{\rm QED}(K)=1\,165\,846\,961(44)(28)\times10^{-12}$. Our final result for the muon anomaly is $a_\mu^{\rm theory}=116\,591\,901(77)\times10^{-11}$. This should be compared with Kinoshita's result $a_\mu^{\rm theory}=116\,591\,919(176)\times10^{-11}$ and the experimental value $a_\mu^{\rm expt}=1\,165\,923(8.5)\times10^{-9}$. Our value makes use of a recent computation of the hadronic contribution, in which the error may be overly optimistic.

I. INTRODUCTION

One of the most important tests of quantum electrodynamics (QED) is the comparison between theory and experiment of the anomalous magnetic moment of the muon $a_\mu=(g-2)/2$. The most accurate measurements now available come from the CERN $g-2$ experiment [1] in which it was found that

$$a_{\mu^-}^{\rm expt}=1\,165\,936(12)\times10^{-9}\ (10\ {\rm ppm})\ , \tag{1}$$

$$a_{\mu^+}^{\rm expt}=1\,165\,910(11)\times10^{-9}\ (10\ {\rm ppm})\ , \tag{2}$$

and the combined result is

$$a_{\mu}^{\rm expt}=1\,165\,923(8.5)\times10^{-9}\ (7\ {\rm ppm})\ . \tag{3}$$

A new $g-2$ experiment is planned at Brookhaven National Laboratory (BNL), and an improvement in the accuracy by a factor of about 20 is expected. In order to properly compare experiment and theory one must correspondingly improve the accuracy of the theoretical prediction.

In an heroic feat, Kinoshita, Nizic, Okamoto, and Marciano have recalculated the muon anomaly [2,3] a_μ. The electron anomaly [4] a_e was also computed. Both a_μ and a_e were computed to eighth order (the tenth order was also estimated for a_μ). All of the multidimensional integrals which arise in these calculations were computed numerically. Because of the complexity and the importance of these calculations, they should be independently checked by another group.

In this paper, we check Kinoshita's results for a_μ in the fourth and sixth order, analytically, by making use of the expansions of $a_\mu^{(4)}$ and $a_\mu^{(6)}$ for large mass ratios $m_\mu/m_e\gg1$, $m_\tau/m_e\gg1$, and $m_\tau/m_\mu\gg1$. We find some small difference with Kinoshita's intermediate results in some cases; however, our final result for a_μ is consistent with his.

In order to get the complete result for a_μ, one must also include the contributions from hadronic and weak effects. However, unlike the QED contributions which can be calculated very accurately, the hadronic contribution is not known very precisely.

One of the goals of the new $g-2$ experiment at BNL is to measure the weak contribution with some precision. This can be done only if the QED and hadronic contributions are known accurately. Then one would have a very good test of the standard model and possible extensions, such as supersymmetry, composite models, etc. What we propose to do in this paper is to improve and check the QED contributions in fourth and sixth order.

II. PRELIMINARY DEFINITIONS

Generally the anomalous magnetic moment of a lepton (e, μ, or τ), which is a dimensionless quantity, can be grouped into two parts: mass-independent and mass-dependent parts. The latter can be further divided into two parts, one involving two leptons and the other involving all three leptons. That is, for a lepton with mass m_1, we can express its anomalous magnetic moment as [2,3]

$$a=A_1+A_2\left[\frac{m_1}{m_2}\right]+A_2\left[\frac{m_1}{m_3}\right]+A_3\left[\frac{m_1}{m_2},\frac{m_1}{m_3}\right]\ , \tag{4}$$

where m_2 and m_3 are the masses of the other two leptons. A_1 is the mass-independent part which is the same for all three leptons. $A_2(m_1/m_2)$ and $A_2(m_1/m_3)$ are the mass-dependent parts involving two kinds of leptons and $A_3(m_1/m_2, m_1/m_3)$ is the mass-dependent part involving all three leptons.

In Eq. (4), all of the terms can be expanded as a series in (α/π):

$$A_k = A_k^{(2)}\left[\frac{\alpha}{\pi}\right] + A_k^{(4)}\left[\frac{\alpha}{\pi}\right]^2 + A_k^{(6)}\left[\frac{\alpha}{\pi}\right]^3 + \cdots \quad (k=1,2,3)\ . \tag{5}$$

In this way, the anomalous magnetic moment of a lepton can be written as

$$a = A_1^{(2)}\left[\frac{\alpha}{\pi}\right] + \left[A_1^{(4)} + A_2^{(4)}\left[\frac{m_1}{m_2}\right] + A_2^{(4)}\left[\frac{m_1}{m_3}\right]\right]\left[\frac{\alpha}{\pi}\right]^2 + \left[A_1^{(6)} + A_2^{(6)}\left[\frac{m_1}{m_2}\right] + A_2^{(6)}\left[\frac{m_1}{m_3}\right] + A_3^{(6)}\left[\frac{m_1}{m_2}, \frac{m_1}{m_3}\right]\right]\left[\frac{\alpha}{\pi}\right]^3 + \cdots\ . \tag{6}$$

We note that, in the second order, we cannot have mass-dependent terms and, in the fourth order, we cannot have the term which involves all three leptons.

Since we already know a_e very accurately, the best way to calculate a_μ is to calculate $(a_\mu - a_e)$, from which we can easily find

$$a_\mu = a_e + (a_\mu - a_e)\ .$$

By applying Eq. (6) to the electron and muon, we have

$$\begin{aligned} a_\mu - a_e = &\left[A_2^{(4)}\left[\frac{m_\mu}{m_e}\right] + A_2^{(4)}\left[\frac{m_\mu}{m_\tau}\right] - A_2^{(4)}\left[\frac{m_e}{m_\mu}\right] - A_2^{(4)}\left[\frac{m_e}{m_\tau}\right]\right]\left[\frac{\alpha}{\pi}\right]^2 \\ &+ \left[A_2^{(6)}\left[\frac{m_\mu}{m_e}\right] + A_2^{(6)}\left[\frac{m_\mu}{m_\tau}\right] + A_3^{(6)}\left[\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}\right] - A_2^{(6)}\left[\frac{m_e}{m_\mu}\right] - A_2^{(6)}\left[\frac{m_e}{m_\tau}\right] \right. \\ &\left. - A_3^{(6)}\left[\frac{m_e}{m_\mu}, \frac{m_e}{m_\tau}\right]\right]\left[\frac{\alpha}{\pi}\right]^3 + \cdots \\ = &(a_\mu - a_e)^{(4)} + (a_\mu - a_e)^{(6)} + \cdots\ . \end{aligned} \tag{7}$$

In the following, we will use the lepton mass values $m_e = 0.510\,999\,06(15)$ MeV, $m_\mu = 105.658\,387(34)$ MeV, and $m_\tau = 1784.1(4)$ MeV. We will also use the more accurate value [5] $m_\mu/m_e = 206.768\,262(30)$ (0.15 ppm). The value of the fine-structure constant used is the latest value determined from the quantized Hall effect [6] $\alpha^{-1} = 137.0\,359\,979(32)$.

III. THE FOURTH-ORDER CONTRIBUTION TO $(a_\mu - a_e)$

From Eq. (7) we see that the lowest-order contribution to $(a_\mu - a_e)$ is the fourth-order contribution

$$(a_\mu - a_e)^{(4)} = \left[A_2^{(4)}\left[\frac{m_\mu}{m_e}\right] + A_2^{(4)}\left[\frac{m_\mu}{m_\tau}\right] - A_2^{(4)}\left[\frac{m_e}{m_\mu}\right] - A_2^{(4)}\left[\frac{m_e}{m_\tau}\right]\right]\left[\frac{\alpha}{\pi}\right]^2\ . \tag{8}$$

There are a total of nine diagrams contributing to $a_\mu^{(4)}$. Among these, there are seven mass-independent diagrams as shown in Fig. 1 and two mass-dependent diagrams as shown in Figs. 2(a) and 2(b). We have similar diagrams for $a_e^{(4)}$. The mass-independent diagrams are the same as those of $a_\mu^{(4)}$. For the mass-dependent diagrams, we just exchange μ and e, as shown in Figs. 2(c) and 2(d). So there are only four diagrams (Fig. 2) contributing to $(a_\mu - a_e)^{(4)}$: two from $a_\mu^{(4)}$ and two from $a_e^{(4)}$. The major contribution comes from the diagram in Fig. 2(a). This contribution can be expressed analytically [7] to all orders in $k = m_e/m_\mu \ll 1$ as follows:

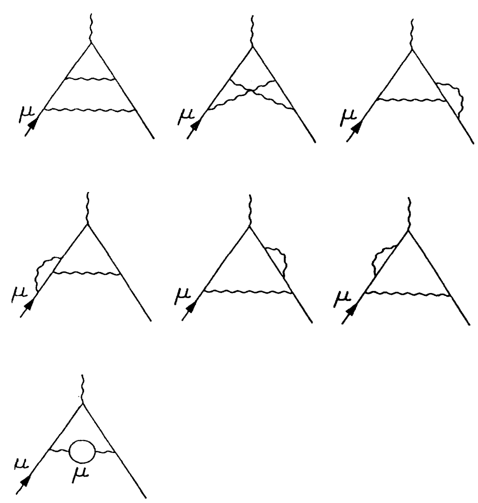

FIG. 1. Mass-independent diagrams contributing to $a_\mu^{(4)}$.

$$A_2^{(4)}\left[\frac{m_\mu}{m_e}\right]=-\frac{25}{36}+\frac{\pi^2}{4}k-\frac{1}{3}\ln k+(3+4\ln k)k^2-\frac{5}{4}\pi^2k^3+\left[\frac{\pi^2}{3}+\frac{44}{9}-\frac{14}{3}\ln k+2\ln^2 k\right]k^4+\frac{8}{15}k^6\ln k-\frac{109}{225}k^6$$
$$+\sum_{n=2}^{\infty}\left[\frac{2(n+3)}{n(2n+1)(2n+3)}\ln k-\frac{8n^3+44n^2+48n+9}{n^2(2n+1)^2(2n+3)^2}\right]k^{2n+4}=1.09\,425\,828(5)\ . \tag{9}$$

The contributions from Figs. 2(b), 2(c), and 2(d) can be obtained from the analytical expression [7] in Eq. (10):

$$A_2^{(4)}(k)=\frac{k^2}{45}+\frac{k^4\ln k}{70}+\frac{9}{19\,600}k^4-\frac{131}{99\,225}k^6+\frac{4k^6}{315}\ln k$$
$$-\sum_{n=3}^{\infty}\left[\frac{8n^3+28n^2-45}{[(n+3)(2n+3)(2n+5)]^2}\right]k^{2n+2}$$
$$+2k^2\ln k\sum_{n=3}^{\infty}\left[\frac{nk^{2n}}{(n+3)(2n+3)(2n+5)}\right], \tag{10}$$

where $k=m_2/m_1 \ll 1$ (see Fig. 3).

For Fig. 2(b) we obtain

$$A_2^{(4)}\left[\frac{m_\mu}{m_\tau}\right]=7.745(4)\times10^{-5}\ . \tag{11}$$

Our result for Fig. 2(c) is

$$A_2^{(4)}\left[\frac{m_e}{m_\mu}\right]=5.1978(5)\times10^{-7} \tag{12}$$

and the contribution due to Fig. 2(d) is negligible:

$$A_2^{(4)}\left[\frac{m_e}{m_\tau}\right]=2\times10^{-9}\ . \tag{13}$$

The accuracy in Eqs. (9), (11), (12), and (13) is limited only by the experimental accuracy of the measured masses of the charged leptons. When these values are determined more precisely, one can include more terms in the expansions in Eqs. (9) and (10), as needed. Finally we obtain the total fourth-order contribution

$$(a_\mu-a_e)^{(4)}=1.09\,433\,521(6)\left[\frac{\alpha}{\pi}\right]^2$$
$$=5\,904\,475.1(3)\times10^{-12}\ . \tag{14}$$

This differs somewhat with Kinoshita's result

$$(a_\mu-a_e)^{(4)}=1.0\,943\,370\left[\frac{\alpha}{\pi}\right]^2=5\,904\,485\times10^{-12}\ . \tag{15}$$

The difference

$$\Delta=\text{ours}-\text{Kinoshita's}$$
$$=-10(1)\times10^{-12} \tag{16}$$

is due to the fact that we have included more terms in the expansions in Eqs. (9) and (10).

IV. THE SIXTH-ORDER CONTRIBUTION TO $(a_\mu-a_e)$

In total, 122 diagrams contribute to $a_\mu^{(6)}$, including 72 mass-independent and 50 mass-dependent diagrams. Again we consider just the mass-dependent diagrams because it is only these diagrams that contribute to $a_\mu-a_e$.

From the previous section we know that the sixth-order contribution can be expressed as

$$(a_\mu-a_e)^{(6)}=\left[A_2^{(6)}\left[\frac{m_\mu}{m_e}\right]+A_2^{(6)}\left[\frac{m_\mu}{m_\tau}\right]+A_3^{(6)}\left[\frac{m_\mu}{m_e},\frac{m_\mu}{m_\tau}\right]-A_2^{(6)}\left[\frac{m_e}{m_\mu}\right]-A_2^{(6)}\left[\frac{m_e}{m_\tau}\right]-A_2^{(6)}\left[\frac{m_e}{m_\mu},\frac{m_e}{m_\tau}\right]\right]\left[\frac{\alpha}{\pi}\right]^3 . \tag{17}$$

In the following, we will present the analytical and numerical result for each term in Eq. (17) and the corresponding diagrams. $A_2^{(6)}(m_\mu/m_e)$ contains six light-by-light scattering diagrams with electron loops (as shown in Fig. 4) and 18 vacuum-polarization diagrams with second- and fourth-order electron-loop insertions into a fourth-order and a second-order muon vertex, respectively, as shown in Fig. 5.

The light-by-light scattering contribution to $A_2^{(6)}(m_\mu/m_e)$ is known numerically [8(a)]:

$$A_2^{(6)}\left[\frac{m_\mu}{m_e},\gamma\gamma\right]=20.9471(20)\ . \tag{18a}$$

There is also a new result [8(b)]:

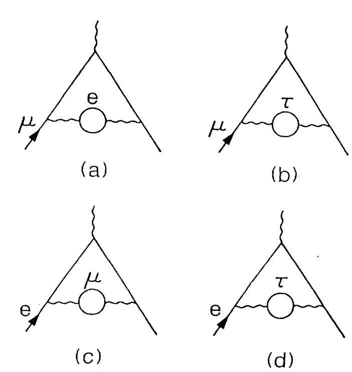

FIG. 2. Mass-dependent diagrams contributing to $a_\mu^{(4)}-a_e^{(4)}$. Diagrams (a) and (b) contribute to $a_\mu^{(4)}$ and (c) and (d) contribute to $a_e^{(4)}$.

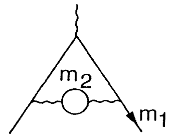

FIG. 3. General case of mass-dependent diagram in fourth order. Its contribution is $a_2^{(4)}(k), k = m_2/m_1$.

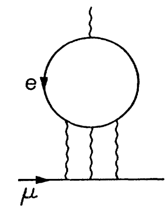

FIG. 4. Light-by-light scattering contributions to $a_\mu^{(6)} - a_e^{(6)}$. Its contribution is $A_2^{(6)}(m_\mu/m_e, \gamma\gamma)$.

$$A_2^{(6)}\left[\frac{m_\mu}{m_e}, \gamma\gamma\right] = 20.9469(18)\ . \tag{18b}$$

It can be seen that there is beautiful agreement between these two results.

The result in Eq. (18b) was obtained using VEGAS on our IBM 3090-200S. It required approximately 1500 hours of CPU time and 5×10^{10} function calls. (The integrand used is that of Aldins *et al.* [8(c)].) This result invalidates an earlier result of Samuel and Chlouber [8(d)].

The vacuum-polarization contributions are calculated separately for each subgroup corresponding to Figs. 5(a), 5(b), 5(c), and 5(d). The results are

$$\begin{aligned}A_2^{(6)}\left[\frac{m_\mu}{m_e}, \text{Fig. 5(a)}\right] &= \left[-\frac{31}{12}+\frac{5}{9}\pi^2-\frac{2}{3}\pi^2\ln2+\zeta(3)\right]\ln\frac{m_\mu}{m_e}+\frac{115}{24}-\frac{79}{54}\pi^2+\frac{5}{3}\pi^2\ln2-\frac{7}{2}\zeta(3)+3c_4\\
&\quad+\frac{\pi^2}{8}\left[\frac{m_e}{m_\mu}\right]-4\left[\frac{m_e}{m_\mu}\right]^2\ln\frac{m_\mu}{m_e}+\left[\frac{111\pi^2}{36}+7\zeta(3)-\frac{315}{54}-\frac{14\pi^2}{3}\ln2\right]\left[\frac{m_e}{m_\mu}\right]^2\\
&\quad+O\left[\left[\frac{m_e}{m_\mu}\right]^3\ln\frac{m_\mu}{m_e}\right]\\
&= -2.392\,396(6)+O\left[\left[\frac{m_e}{m_\mu}\right]^3\ln\frac{m_\mu}{m_e}\right],\end{aligned} \tag{19}$$

$$\begin{aligned}A_2^{(6)}\left[\frac{m_\mu}{m_e}, \text{Fig. 5(b)}\right] &= \frac{1}{4}\ln\left[\frac{m_\mu}{m_e}\right]+\frac{1}{2}\zeta(3)-\frac{5}{12}+\left[-\frac{13}{18}\pi^3-\frac{16}{9}\pi^2\ln2+\frac{79}{27}\pi^2\right]\left[\frac{m_e}{m_\mu}\right]+6\left[\frac{m_e}{m_\mu}\right]^2\ln^2\left[\frac{m_\mu}{m_e}\right]\\
&\quad-3\left[\frac{m_e}{m_\mu}\right]^2\ln\left[\frac{m_\mu}{m_e}\right]+\left[\pi^2+\frac{8}{3}-9\zeta(3)\right]\left[\frac{m_e}{m_\mu}\right]^2+O\left[\left[\frac{m_e}{m_\mu}\right]^3\ln^2\frac{m_\mu}{m_e}\right]\\
&= 1.49\,346(3)+O\left[\left[\frac{m_e}{m_\mu}\right]^3\ln^2\frac{m_\mu}{m_e}\right],\end{aligned} \tag{20}$$

$$\begin{aligned}A_2^{(6)}\left[\frac{m_\mu}{m_e}, \text{Fig. 5(c)}\right] &= \frac{2}{9}\left[\ln\frac{m_\mu}{m_e}\right]^2-\frac{25}{27}\ln\left[\frac{m_\mu}{m_e}\right]+\frac{317}{324}+\frac{\pi^2}{27}-\frac{4\pi^2}{45}\left[\frac{m_e}{m_\mu}\right]-\frac{8}{3}\left[\frac{m_e}{m_\mu}\right]^2\ln^2\left[\frac{m_\mu}{m_e}\right]\\
&\quad+\frac{52}{9}\left[\frac{m_e}{m_\mu}\right]^2\ln\frac{m_\mu}{m_e}+\left[\frac{56}{9}\ln^2 2-\frac{2\pi^2}{3}-\frac{400}{27}\ln2+\frac{28}{9}+\frac{128a}{27}+\frac{8b}{27}\right]\left[\frac{m_e}{m_\mu}\right]^2\\
&\quad+O\left[\left[\frac{m_e}{m_\mu}\right]^3\ln^2\frac{m_\mu}{m_e}\right]\\
&= 2.71\,857(3)+O\left[\left[\frac{m_e}{m_\mu}\right]^3\ln^2\frac{m_\mu}{m_e}\right].\end{aligned} \tag{21}$$

$$\begin{aligned}A_2^{(6)}\left[\frac{m_\mu}{m_e}, \text{Fig. 5(d)}\right] &= \left[\frac{119}{27}-\frac{4}{9}\pi^2\right]\ln\left[\frac{m_\mu}{m_e}\right]-\frac{61}{162}+\frac{\pi^2}{27}+\left[\frac{4\pi^2}{9}-\frac{115}{27}\right]\left[\frac{m_e}{m_\mu}\right]^2+O\left[\left[\frac{m_e}{m_\mu}\right]^3\right]\\
&= 0.100\,519(1)+O\left[\left[\frac{m_e}{m_\mu}\right]^3\right],\end{aligned} \tag{22}$$

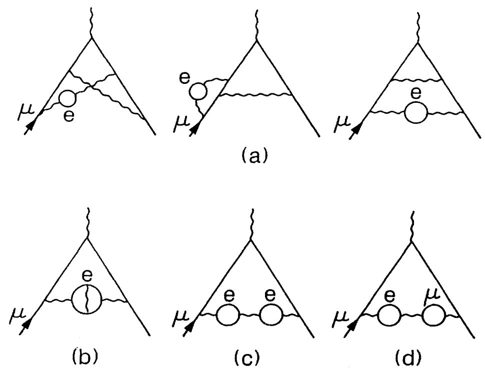

FIG. 5. Vacuum-polarization diagrams contributing to $a_\mu^{(6)}-a_e^{(6)}$. (a) Second-order electron-loop insertion into a fourth-order muon vertex. (b) Proper fourth-order electron-loop insertion into a second-order muon vertex. (c) Double-bubble second-order electron-loop insertions into a second-order muon vertex. (d) Mixed-bubble second-order loop insertions into a second-order muon vertex.

where

$$c_4=\tfrac{11}{648}\pi^4-\tfrac{2}{27}\pi^2\ln^2 2-\tfrac{1}{27}\ln^4 2-\tfrac{8}{9}a_4 \ ,$$

$$a_4=\sum_{n=1}^{\infty}\frac{1}{2^n n^4}=0.517\,479\,061 \ ,$$

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}=1.202\,056\,903 \ ,$$

$$a=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n+1)!!}{(2n+!)!!n^2}=-0.26\,758 \ ,$$

and

$$b=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!n^2}=-0.34\,201 \ . \tag{23}$$

Our errors are estimated by multiplying the next uncalculated term by ten. Our results in Eqs. (19)–(22) agree with the previously known results to $O(m_e/m_\mu)$ given in Refs. [9–12], respectively.

The presence and large contributions of the $(m_e/m_\mu)^2\ln^2(m_\mu/m_e)$ terms in Eqs. (20) and (21) are somewhat surprising.

The total vacuum-polarization contribution in sixth order is

$$A_2^{(6)}\left[\frac{m_\mu}{m_e}, \text{vacuum polarization}\right]=1.92\,015(5) \ . \tag{24}$$

After multiplying by $(\alpha/\pi)^3$, we obtain our result

$$(a_\mu^{(6)}-a_e^{(6)})(\text{vacuum polarization})=24\,064.8(6)\times 10^{-12} \ . \tag{25}$$

These results in Eqs. (19), (20), (21), and (22) should be compared with Kinoshita's new results given in Table I [13]. His total sixth-order vacuum polarization result to be compared with Eq. (25) is

$$(a_\mu^{(6)}-a_e^{(6)})(\text{vacuum polarization}, K)=24\,069(6)\times 10^{-12} \ . \tag{26}$$

The agreement is excellent.

Adding Eqs. (18b) and (24) we get the sixth-order result

$$A_2^{(6)}\left[\frac{m_\mu}{m_e}\right]=22.8671(18) \ . \tag{27}$$

Next, we consider the contribution from $A_2^{(6)}(m_\mu/m_\tau)$. The corresponding diagrams can be obtained by replacing the electron in those graphs for $A_2^{(6)}(m_\mu/m_e)$ by the τ. However, the results cannot be simply obtained in this way, due to the difference in the masses of the electron and the τ.

As in the case of $A_2^{(6)}(m_\mu/m_e)$, we again divide $A_2^{(6)}(m_\mu/m_\tau)$ into two parts: light-by-light scattering and vacuum-polarization subgraphs. They are represented by $A_2^{(6)}(m_\mu/m_\tau,\gamma\gamma)$ and $A_2^{(6)}(m_\mu/m_\tau,\text{vacuum polarization})$, respectively. We can estimate $A_2^{(6)}(m_\mu/m_\tau,\gamma\gamma)$ by using Aldins *et al.* [8(c)]. Our result is (Fig. 4 with $e\to\tau$)

$$A_2^{(6)}\left[\frac{m_\mu}{m_\tau},\gamma\gamma\right]=1.836\times 10^{-3} \ . \tag{28}$$

As for $A_2(m_\mu/m_\tau,\text{vacuum polarization})$, we have analytical expressions for all the diagrams. These formulas were derived by Barbieri and Remiddi [14] in calculating the muon contribution to the electron anomaly, and can also be applied to our cases. The results corresponding to Figs. 6 and 7 are given by, respectively,

$$A_2^{(6)}\left[\frac{m_\mu}{m_\tau}, \text{ Fig. 6}\right]=-\frac{2}{3}\left[\frac{m_\mu}{m_\tau}\right]^2\times\left[-\frac{2689}{5400}+\frac{\pi^2}{15}+\frac{23}{90}\ln\frac{m_\tau}{m_\mu}\right]=-2.063\times 10^{-3} \ , \tag{29}$$

TABLE I. Comparison of $A_2^{(6)}(m_\mu/m_e$, vacuum polarization) with Kinoshita's results. Please see Appendix, part 2.

Figure	Ours	Kinoshita	$\Delta\left[\frac{\alpha}{\pi}\right]^3(10^{-12})$	$\Delta'\left[\frac{\alpha}{\pi}\right]^3(10^{-12})$
5(a)	−2.392 396(6)	−2.39 238(43)	0(5)	−69(5)
5(b)	1.49 346(3)	1.49 373(13)	−3(2)	295(2)
5(c)	2.71 857(3)	2.71 863(5)	−1(1)	69(1)
5(d)	0.100 519(1)	0.100 519(5)	0	0
Total	1.92 015(5)	1.92 050(45)	−4(6)	295(5)

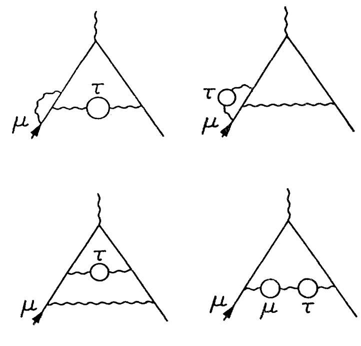

FIG. 6. Second-order τ-loop insertion into a fourth-order muon vertex.

$$A_2^{(6)}\left[\frac{m_\mu}{m_\tau}, \text{ Fig. 7}\right]=\frac{41}{486}\left[\frac{m_\mu}{m_\tau}\right]^2=0.296\times 10^{-3} . \quad (30)$$

Combining all the contributions in Eqs. (28), (29), and (30), we obtain the total value for $A_2^{(6)}(m_\mu/m_\tau)$:

$$A_2^{(6)}\left[\frac{m_\mu}{m_\tau}\right]=6.9\times 10^{-5} . \quad (31)$$

We see that a large cancellation makes the total contribution from the τ nearly negligible.

Now we come to the term $A_3^{(6)}(m_\mu/m_e, m_\mu/m_\tau)$. There are two diagrams corresponding to this contribution, as shown in Fig. 8. Generally for a two-bubble diagram shown in Fig. 9, we have the expression

$$A_3^{(6)}\left[\frac{m}{m_1}, \frac{m}{m_2}\right]=\int_0^1 dx(1-x)[-\pi(x,k_1)][-\pi(x,k_2)] , \quad (32)$$

where

$$\pi(x,k_p)=\frac{8}{9}-\frac{B_p^2}{3}+\left[\frac{1}{2}-\frac{B_p^2}{6}\right]B_p \ln\frac{B_p-1}{B_p+1}$$

and

$$B_p(x)=\left[1+\frac{4(1-x)}{x^2}k_p^2\right]^{1/2}, \quad k_p=\frac{m_p}{m} \quad (p=1,2) .$$

In principle, one can find an analytical expression for $A_3^{(6)}(m/m_1, m/m_2)$. but a numerical result is sufficient. We have

$$A_3^{(6)}\left[\frac{m_\mu}{m_e}, \frac{m_\mu}{m_\tau}\right]=5.24\times 10^{-4} . \quad (33)$$

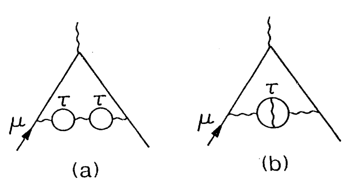

FIG. 7. Fourth-order τ-loop insertion into a second-order muon vertex.

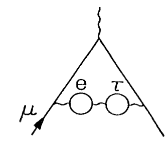

FIG. 8. Mixed-bubble contribution to $a_\mu^{(6)}$. This contribution depends on all three lepton masses.

This result agrees with Kinoshita's result to three significant figures.

Other contributions to $a_\mu^{(6)}-a_e^{(6)}$ come from the anomalous magnetic moment of the electron. That is, $A_2^{(6)}(m_e/m_\mu)$, $A_2^{(6)}(m_e/m_\tau)$ and $A_3^{(6)}(m_e/m_\mu, m_e/m_\tau)$. The corresponding diagrams can be obtained by exchanging the electron and the muon in Figs. 4, 5, 7, and 8 and by replacing the muon by the electron in Fig. 6. One can use the method used in calculating $A_2^{(6)}(m_\mu/m_e)$, $A_2^{(6)}(m_\mu/m_\tau)$ and $A_2^{(6)}(m_\mu/m_e, m_\mu/m_\tau)$ to compute these contributions. However these terms are very small and can be neglected.

Now we are in a position to get the total contribution in sixth order. Adding up Eqs. (27), (31), and (33) we have

$$(a_\mu-a_e)^{(6)}=22.8677(18)\left[\frac{\alpha}{\pi}\right]^3 . \quad (34)$$

V. THE QED CONTRIBUTION UP TO TENTH ORDER

In the previous two sections, we calculated the fourth-order and sixth-order contributions $a_\mu-a_e$. The contribution in eighth order that dominates is the contribution of the class of diagrams obtained by inserting an electron bubble in a leg of Fig. 4. Our 1977 result [15] is

$$a_\mu^{(8)}(\gamma\gamma)=117.4(5) . \quad (35)$$

This agrees with Kinoshita's recent result [2,3]

$$a_\mu^{(8)}(\gamma\gamma)=116.8(1) . \quad (36)$$

By using Kinoshita's total contributions for the eighth and tenth order,

$$(a_\mu-a_e)^{(8)}=127.00(41)\left[\frac{\alpha}{\pi}\right]^4 , \quad (37)$$

$$(a_\mu-a_e)^{(10)}=570(140)\left[\frac{\alpha}{\pi}\right]^5 , \quad (38)$$

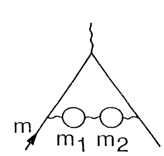

FIG. 9. General case of mass-dependent contribution $A_3^{(6)}(m/m_1, m/m_2)$.

we can obtain the total QED contribution which is given by

$$(a_\mu - a_e)^{\text{QED}} = 1.09\,433\,521(6)\left[\frac{\alpha}{\pi}\right]^2 + 22.8677(18)\left[\frac{\alpha}{\pi}\right]^3 + 127.00(41)\left[\frac{\alpha}{\pi}\right]^4 + 570(140)\left[\frac{\alpha}{\pi}\right]^5 = 6\,194\,805(27)\times 10^{-12}\ . \tag{39}$$

The anomaly of the electron, a_e has been calculated to eighth-order [4]:

$$a_e = 1\,159\,652\,140(5.3)(4.1)(27.1)\times 10^{-12}\ , \tag{40}$$

where the first and second uncertainties come from the numerical uncertainties in the sixth order and the eighth order, respectively, while the third reflects the uncertainty in α.

Adding a_e, given by Eq. (40), to Eq. (39), and subtracting

$$a_e^{\text{hadronic}} + a_e^{\text{weak}} = 1.6\times 10^{-12} \tag{41}$$

we can find the pure QED contribution to the muon anomaly which is

$$a_\mu^{\text{QED}} = 1\,165\,846\,943(28)(27)\times 10^{-12}\ . \tag{42}$$

This should be compared to Kinoshita's value

$$a_\mu^{\text{QED}}(K) = 1\,165\,846\,961(44)(27)\times 10^{-12}\ ;$$

the first error is an estimate of the theoretical uncertainty and the second reflects the uncertainty in α. Thus the difference is

$$a_\mu^{\text{QED}} - a_\mu^{\text{QED}}(K) = (-10-4-2+1-3)\times 10^{-12} = -18\times 10^{-12}\ . \tag{43}$$

This difference reflects the uncertainty coming from the uncalculated terms in our analytical calculations and the uncertainty in the numerical integrations in Kinoshita's results where the -10 comes from fourth-order, the -4 comes from sixth-order, the -2 comes from a_e^{hadronic}, the 1 comes from the τ contribution and the -3 from the light-by-light contribution.

VI. NON-QED CONTRIBUTIONS AND THE MUON ANOMALY

Unlike the anomalous magnetic moment of the electron, which is dominated by the QED effect due to the smallness of its mass, the anomaly of the muon has substantial contributions from hadronic and weak interactions. Unfortunately, those contributions have not yet been computed very accurately. So far the best estimates are [14,16,17]

$$a_\mu^{\text{hadronic}} = 7011(76)\times 10^{-11}\ , \tag{44}$$

$$a_\mu^{\text{weak}} = 195(10)\times 10^{-11}\ , \tag{45}$$

where the hadronic contribution includes fourth and sixth order and the weak contribution includes only the one-loop contribution [18].

Although we use the result of Ref. [17] for a_μ^{hadronic}, it should be noted that the error in Eq. (44) may be overly optimistic. More accurate experiments to measure $\sigma(e^+e^- \to \text{hadrons})$ are urgently needed to reduce the error in Eq. (44).

Collecting all the contributions from QED, hadronic and weak effects, we finally find the anomaly of the muon

$$a_\mu^{\text{theory}} = 116\,591\,901(77)\times 10^{-11} \tag{46}$$

which is in very good agreement with the experimental value

$$a_\mu^{\text{expt}} = 1\,165\,923(8.5)\times 10^{-9}(7\text{ ppm})\ . \tag{47}$$

Equations (46) and (47) should be compared to Kinoshita's value

$$a_\mu^{\text{theory}} = 116\,591\,919(176)\times 10^{-11}\ . \tag{48}$$

Our result given in Eq. (46) implies the following value for the gyromagnetic ratio g:

$$g = 2.00\,233\,183\,802(154)\ . \tag{49}$$

Kinoshita's result in Eq. (48) implies

$$g = 2.00\,233\,183\,838(352)\ . \tag{50}$$

These results in Eqs. (49) and (50) should be compared with the present experimental value

$$g = 2.002\,331\,846(17)\ . \tag{51}$$

In the Appendix, we will give a detailed comparison between our results and Kinoshita's results.

ACKNOWLEDGMENTS

One of us (M.A.S) would like to thank Toichiro Kinoshita for very helpful discussions. He would also like to thank John Chandler, Konrad Brandemuhl, Lorraine Goff, and J. L. Albert for their help with the computing for this project. He would also like to thank the Institute for Nuclear Research of the USSR Academy of Sciences, Moscow, where part of this work was done, for its very kind hospitality as well as the Aspen Center for Physics, where this work was completed, for its kind hospitality.

APPENDIX

So far the QED contribution to the anomaly of the muon, a_μ, has been calculated and estimated up to the tenth order. In this endeavor, Kinoshita has made a great contribution. However, we find that our results and Kinoshita's results in fourth and sixth order are slightly different, for some contributions. In this appendix, we will present this comparison in some detail.

1. Fourth order

For the fourth-order contribution, differences occur in the calculations of $A_2^{(4)}(m_\mu/m_e)$ and $A_2^{(4)}(m_\mu/m_\tau)$ terms. For the $A_2^{(4)}(m_\mu/m_e)$ term, Kinoshita just kept terms up to the second order in (m_e/m_μ) and got the result

$$A_2^{(4)}\left[\frac{m_\mu}{m_e}\right]=1.0942596 \ . \tag{A1}$$

In fact, the contributions from the third-order and even the fourth-order terms should be taken into account. For the $A_2^{(4)}(m_\mu/m_\tau)$ term, Kinoshita's result is

$$A_2^{(4)}\left[\frac{m_\mu}{m_\tau}\right]=7.794(32)\times 10^{-5} \ . \tag{A2}$$

To the accuracy required, we agree on the quantity $A_2^{(4)}(m_e/m_\mu)$ which is to be subtracted. Kinoshita's value is

$$A_2^{(4)}\left[\frac{m_e}{m_\mu}\right]=5.198\times 10^{-7} \ . \tag{A3}$$

Our values are given in Eqs. (9), (11), and (12). These should be compared with Kinoshita's results in Eqs. (A1), (A2), and (A3), respectively. Thus the total difference in the fourth-order contribution is given by

$$\Delta_1=-10\times 10^{-12} \ . \tag{A4}$$

2. Sixth order

We find that our results, corresponding to each subgroup of $A_2^{(6)}(m_\mu/m_e)$, are somewhat different from those of Kinoshita. For the vacuum-polarization part, the diagrams in Fig. 5, the results are listed in Table I.

In the table, Δ' is the difference between our results, without the m_e/m_μ and smaller terms, and Kinoshita's results. It is clear that these terms improve the agreement with Kinoshita's results and are definitely necessary at this level of precision.

TABLE II. Contributions to $A_2^{(6)}(m_\mu/m_\tau,\gamma\gamma)$ and $A_2^{(6)}(m_\mu/m_\tau,\text{vacuum polarization})$.

Figure	Contribution (10^{-3})	$\Delta\left[\frac{\alpha}{\pi}\right]^3\ [10^{-12}]$
$4(e\to\tau)$	1.836	23.0
6	−2.063	−25.9
7	0.2959	3.7
Total	0.0689	0.86

As seen, the total difference is given by

$$\Delta_2=-4(6)\times 10^{-12} \ . \tag{A5}$$

As stated above, this difference reflects the uncertainty coming from the uncalculated terms in our analytical calculations and the uncertainty in the numerical integrations in Kinoshita's results.

The other difference in the sixth order comes from Kinoshita's neglect of the contributions from τ. The contributions due to Fig. 4 with $e\to\tau$ and Figs. 6 and 7 are given in Table II. We see that the total result turns out to be relatively small due to the cancellation which occurs.

Collecting all the differences from the fourth- and the sixth-order contributions, as well as the one from a_e^{hadronic}, we get the total QED difference given by

$$\Delta a^{\text{QED}}=-18\times 10^{-12} \tag{A6}$$

which is just the one given by Eq. (43).

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