PHYSICAL REVIEW D VOLUME 44, NUMBER 12 15 DECEMBER 1991

Coulomb field of an accelerated charge: Physical and mathematical aspects

Francis J. Alexander
Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855

Ulrich H. Gerlach
Department of Mathematics, Ohio State University, Columbus, Ohio 43210

(Received 15 January 1991)

The Coulomb field of a charge static in an accelerated frame has properties that suggest features of electromagnetism which are different from those in an inertial frame. An illustrative calculation shows that the Larmor radiation reaction equals the electrostatic attraction between the accelerated charge and the charge induced on the surface whose history is the event horizon. A spectral decomposition of the Coulomb potential in the accelerated frame suggests the possibility that the distortive effects of this charge on the Rindler vacuum are akin to those of a charge on a crystal lattice. The necessary Maxwell field equations relative to the accelerated frame, and the variational principle from which they are obtained, are formulated in terms of the technique of geometrical gauge-invariant potentials.

I. MOTIVATION AND INTRODUCTION

The classical and quantum-mechanical pictures of a charged particle together with its Coulomb field are well known in an inertial frame: The classical picture is the one where the static charge is the source of its spherical electric field which can be derived from the electrostatic potential. The quantum-mechanical picture is the one where the charge is surrounded by a cloud of virtual quanta [1,2] each one of which is emitted and reabsorbed by the charge. The (self-)interaction energy due to these processes gives rise to the "renormalized" experimentally observed rest-mass energy of the charged particle. If a second charged particle is present then there are two clouds of (virtual) quanta. In this case there is a probability that a quantum emitted by one charge can be absorbed by the other, and vice versa. The mutual interaction energy due to such exchange processes is given by

$$V(\mathbf{X}_1,\mathbf{X}_2)=e_1e_2\frac{e^{-m|\mathbf{X}_1-\mathbf{X}_2|}}{|\mathbf{X}_1-\mathbf{X}_2|}$$
$$=4\pi e_1e_2\int\int\int\frac{1}{k^2+m^2}e^{i\mathbf{k}\cdot(\mathbf{X}_1-\mathbf{X}_2)}\frac{d^3k}{(2\pi)^3}\ , \tag{1.1}$$

the familiar ("Yukawa") scalar potential, which for $m=0$ reduces to a Coulomb field. A key ingredient to this result is that the quanta responsible for this interaction are the familiar Minkowski quanta, the elementary modifications (="excitations") of the familiar translation-invariant Minkowski vacuum.

The question now is this: Does the quantum-mechanical picture of the exchange interaction between a pair of charges extend to an accelerated frame? In other words, can the classical potential energy between two uniformly accelerated charges (in the same accelerated frame) still be attributed to the exchange of virtual quanta in the accelerated frame?

To answer this question it is not enough to restrict one's attention to the quantum mechanics based on the Minkowski vacuum and its excitations. The ground state of an acceleration-partitioned field is entirely different from the Minkowski vacuum. Indeed, that ground state determines a set of quantum states which is disjoint from (i.e., unitarily unrelated to) the set of quantum states based on the Minkowski vacuum state [3]. Physically that ground state has the attributes which are reminiscent of a condensed vacuum state [4].

Very little is known about interactions between particles and an acceleration-partitioned field in its (condensed) ground state. The contrast between such interactions and those that are based on the Minkowski vacuum state of the field raises some nontrivial issues of principle which are not answered in this paper however.

Here we shall erect the framework that allows a very economic analysis of the interaction between currents and fields. The system is, of course, the classical Maxwell field with its charged sources. The utility of this framework lies in the fact that the four-dimensional problem has been reduced to two dimensions in such a way that the field and the charge current can be viewed relative to any linearly accelerated coordinate frame.

This paper accomplishes four tasks.

(1) Sections II and III formulate the full classical Maxwell electrodynamics in terms which are most natural for a linearly accelerated coordinate frame. This is done by exhibiting a reduced ("2+2") variational principle and the concomitant reduced set of decoupled inhomogeneous wave equations for the to-be-quantized transverse-electric (TE) and transverse-magnetic (TM) degrees of freedom. The reduction procedure is not new. It has already been applied to the linearized Einstein field equation of a spherically symmetric spacetime [5].

(2) Sections IV and V give what in an inertial frame corresponds to a multipole expansion. Section VI reviews the well-known quantum mechanical picture of the interaction between two charges as a spectral sum of ex-

change processes, and then gives a spectral decomposition of the Coulomb potential between a pair of linearly uniformly accelerated charges.

(3) Section VII suggests that the Coulomb attractive force between an accelerated charge and the induced charge on its event horizon be identified with Larmor's radiation reaction force.

(4) Section VIII compares a pair of charges static in the vacuum of an accelerated frame to two polarons, and then draws attention to the possibility that their interaction might be different from what one expects from quantum mechanics relative to the inertial vacuum.

II. THE REDUCED VARIATIONAL PRINCIPLE

Classical Maxwell electrodynamics is a consequence of the principle of extremizing the action integral:

$$I[A_\mu]=\int\int\int\int\left[\frac{-1}{16\pi}(A_{\nu;\mu}-A_{\mu;\nu})(A^{\nu;\mu}-A^{\mu;\nu})+J^\mu A_\mu\right]\sqrt{-g}\,d^4x\ . \tag{2.1}$$

The resulting Euler equations is the familiar system of inhomogeneous Maxwell wave equations

$$A^{\mu;\nu}{}_{;\mu}-A^{\nu;\mu}{}_{;\mu}=4\pi J^\nu\ . \tag{2.2}$$

These equations imply charge conservation:

$$J^\nu{}_{;\nu}=0\ . \tag{2.3}$$

But this fact also follows directly from the demand that I be invariant under gauge transformations $A_\mu\rightarrow A_\mu+\Lambda_{,\mu}$, i.e., from

$$I[A_\mu+\Lambda_{,\mu}]=I[A_\mu]\ . \tag{2.4}$$

A. Scalar and vector harmonics

The fact that the y-z plane accommodates the Euclidean group of symmetry operations implies that one can introduce various sets of scalar and vector harmonics which have simple transformation properties under the various group operations. We shall use the complete set of δ-function normalized scalar harmonics

$$Y^k(y,z)=\frac{1}{2\pi}e^{i(k_y y+k_z z)} \tag{2.5}$$

and the corresponding two sets of vector harmonics,

$$Y^k_{,a}\equiv\frac{\partial Y^k}{\partial x^a}=ik_a Y^k \tag{2.6a}$$

and

$$Y^k_{,c}\epsilon^c{}_a=ik_c\epsilon^c{}_a Y^k\ . \tag{2.6b}$$

Here $k\equiv(k_y,k_z)$ identifies the harmonic.

The a refers to the coordinates $x^2=y$ or $x^3=z$, which span R^2, the transverse symmetry plane on which the Euclidean group acts. The expression $\epsilon^c{}_a$ is the antisymmetric Levi-Civita symbol.

Instead of this set of translation eigenfunctions one could just as well have used the complete set of scalar Bessel harmonics (rotation eigenfunctions),

$$Y^{km}(r,\theta)=J_m(kr)e^{im\theta},\ k=\sqrt{k_y^2+k_z^2}$$

and the concomitant set of vector harmonics. Indeed, one can equally well use an orthonormalized *discrete* set of trigonometric or Bessel harmonics on a finite rectangle or a disk in the y-z plane. Which of all these possibilities one must choose depends, of course, entirely on the boundary conditions which the Maxwell field satisfies in the y-z plane. However the reduced form of the variational principle and the inhomogeneous Maxwell wave equations [see Eqs. (3.1) for the TM modes and (3.2) for the TE modes] have the same form for all these different harmonics.

For the sake of concreteness we shall use the familiar translation scalar and vector harmonics, Eqs. (2.5) and (2.6). All the normalization integrals for the vector harmonics follow directly from

$$\int\int(Y^k)^*Y^{k'}dy\,dz=\delta(k_y-k_y')\delta(k_z-k_z')\equiv\delta^2(k-k')\ . \tag{2.7}$$

Thus one has, for example,

$$\int\int(Y^k)^*_{,a}Y^{k'}_{,b}dy\,dz=k_a k_b\delta^2(k-k')\ , \tag{2.8a}$$

$$\int\int(Y^k)^*_{,c}\epsilon^c{}_b Y^{k'}_{,d}\epsilon^{db}dy\,dz=k^2\delta^2(k-k')\ , \tag{2.8b}$$

$$k^2=k_y^2+k_z^2$$

while the "longitudinal" and the "transverse" vector harmonic are always orthogonal:

$$\int\int(Y^k)^*_{,a}Y^{k'}_{,b}\epsilon^{ba}dy\,dz=0\ . \tag{2.9}$$

All integrations are over R^2, the y-z plane.

B. Transverse manifold R^2 and longitudinal manifold M^2

In order to reduce the variational principle and the Maxwell field equations, one expands the vector potential and the current density in terms of these scalar and vector harmonics. The four-vector potential is

$$A_\mu(x^\nu)=(A_B(x^\nu),A_b(x^\nu))=\left[\sum_k a^k_B(x^C)Y^k,\ \sum_k\left[a^k(x^C)\frac{\partial Y^k}{\partial x^b}+A^k(x^C)\frac{\partial Y^k}{\partial x^d}\epsilon^d{}_b\right]\right] \tag{2.10}$$

and the four-current density is

$$J_\mu(x^\nu)=(J_B(x^\nu),J_b(x^\nu))$$
$$=\left[\sum_k j_B^k(x^C)Y^k, \sum_k \left[j^k(x^C)\frac{\partial Y^k}{\partial x^b}+J^k(x^C)\frac{\partial Y^k}{\partial x^e}\epsilon^e{}_b\right]\right] . \tag{2.11}$$

Here

$$\sum_k(\cdots)\equiv\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d^2k}{2\pi}(\cdots) \tag{2.12}$$

is the mode integral over the harmonics. The scalar (on R^2) harmonic expansion coefficients are $a_B^k(x^C)$ and $j_B^k(x^C)$. They are components of vectors on M^2, the two-dimensional Lorentz spacetime spanned by $x^C=(x^0,x^1)$. Similarly the expansion coefficients for the vector (on R^2) harmonics $Y^k_{,b}$ are $a^k(x^C)$ and $j^k(x^C)$, while those for $Y_{,c}\epsilon^c{}_b$ are $A^k(x^C)$ and $J^k(x^C)$. All these coefficients are *scalars* on M^2. Evidently the four-dimensional Minkowski spacetime M^4, which is coordinatized by $x^\nu=(x^0,x^1,x^2,x^3)$, has been factored by the symmetries transverse to the acceleration into the product

$$M^4=M^2\times R^2 .$$

Here M^2 is coordinatized by $x^C=(x^0,x^1)$ and R^2 by $x^b=(x^2,x^3)=(y,z)$. The metric of M^4 relative to the accelerated coordinate frame has the form

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu$$
$$=g_{AB}(x^C)dx^A dx^B+g_{ab}(x^d)dx^a dx^b . \tag{2.13}$$

It is block diagonal

$$[g_{\mu\nu}]=\begin{bmatrix} g_{AB} & 0 \\ 0 & g_{ab}\end{bmatrix} .$$

Consequently M^2 and R^2 are mutually orthogonal submanifolds. We shall call R^2 the *transverse* submanifold because it is perpendicular to the world line of a linearly accelerated charge. The geometric objects intrinsic to it are its metric tensor field,

$$g_{ab}dx^a dx^b=dy^2+dz^2 , \tag{2.14}$$

and the scalar and vector fields given by

$$Y^k(x^a),\quad \frac{\partial Y^k}{\partial x^b},\quad \text{and}\quad \frac{\partial Y^k}{\partial x^c}\epsilon^c{}_b .$$

We shall call M^2 the *longitudinal* submanifold because it contains the world line of a linearly accelerated charge. The geometric objects intrinsic to it are not only its metric tensor field

$$g_{AB}(x^C)dx^A dx^B \tag{2.15}$$

but also the scalar and vector fields given by the coefficients of the harmonics in Eqs. (2.10) and (2.11). Relative to a linearly and uniformly accelerated coordinate frame given by

$$t=\xi\sinh\tau,\quad x=\xi\cosh\tau \tag{2.16}$$

the metric of the longitudinal submanifold M^2 has the form

$$g_{AB}dx^A dx^B=-\xi^2d\tau^2+d\xi^2 . \tag{2.17}$$

In general however, the coordinate frame is not uniformly accelerating, and the metric does not have a correspondingly simple form.

Our task of "reducing" the Maxwell field equations and its variational principle consists of formulating them strictly in terms of geometrical objects defined solely on M^2. Roughly speaking, we "factor out" the $y-z$ dependence of each harmonic degrees of freedom. Thus we introduce the harmonic expansions Eqs. (2.10) and (2.11) into the Maxwell wave equations (2.2) and equate the coefficients of the corresponding scalar and vector harmonics. The result is the reduced set of Maxwell wave equations (3.1) and (3.2) on M^2.

The reduction of the variational principle is more informative because it directly relates gauge invariance to the structure of the wave equations. Thus introduce Eqs. (2.10) and (2.11) into the action integral, Eq. (2.1). Using the fact that

$$\sqrt{-g}\; d^4x=\sqrt{-g^{(2)}}d^2x\; dy\; dz \tag{2.18}$$

with

$$g^{(2)}=\det[g_{AB}] ,$$

do the integration over R^2,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\cdots)dy\; dz .$$

Finally make use of the orthogonality and the normalization integrals Eqs. (2.7)–(2.9). After a straightforward evaluation, the action integral decomposes into two independent sums over each of the familiar transverse magnetic (TM, no magnetic field along the x direction) and the transverse electric (TE, no electric field along the x direction) modes:

$$I=\sum_k I^k_{\rm TM}+\sum_k I^k_{\rm TE} . \tag{2.19}$$

Here the mode "summation" is given by Eq. (2.12). The action for a TM mode of type $k\equiv(k_y,k_z)$,

$$I^k_{\rm TM}=\frac{1}{4\pi}\int\int[-\tfrac{1}{4}(\mathcal{A}_{B,C}-\mathcal{A}_{C,B})$$
$$\times(\mathcal{A}^*_{D,E}-\mathcal{A}^*_{E,D})g^{BD}g^{CE}$$
$$-\tfrac{1}{2}k^2\mathcal{A}^B\mathcal{A}^*_B+4\pi j^B a^*_B$$
$$+4\pi k^2 j a^*]\sqrt{-g^{(2)}}d^2x , \tag{2.20}$$

where

$$\mathcal{A}_B = a_B^k - a^k_{,B} \tag{2.21}$$

and

$$\mathcal{A}_B^* = a_B^{-k} - a_{,B}^{-k}$$

refers to the mode $-k \equiv (-k_y, -k_z)$.

Because the total Maxwell field is real, $k \to -k$ corresponds to taking the complex conjugate of an amplitude. Thus denoting Y_B^* as the complex conjugate of Y_B is consistent.

The action for a TE mode by contrast is

$$I_{\rm TE}^k = \frac{k^2}{4\pi} \int \int [-\tfrac{1}{2} A_{,B} A^*_{,C} g^{BC} - \tfrac{1}{2} k^2 A A^* + 4\pi J A^*] \sqrt{-g^{(2)}} d^2x \; . \tag{2.22}$$

Here we suppressed superscripts by letting $A = A^k$, and we set $A^* = A^{-k}$ because the total Maxwell field is real.

III. REDUCED MAXWELL WAVE EQUATION

It is now straightforward to obtain the reduced Maxwell field equation by extremizing the action. For the TM modes one has [5]

$$0 = \frac{\delta I}{\delta a_B^*}: \; (\mathcal{A}_{C|B} - \mathcal{A}_{B|C})^{|C} + k^2 \mathcal{A}_B = 4\pi j_B \; , \tag{3.1a}$$

$$0 = \frac{\delta I}{\delta a^*}: \; \mathcal{A}^B_{\;|B} = 4\pi j \; . \tag{3.1b}$$

For the TE modes one has

$$0 = \frac{\delta I}{\delta A^*}: \; -A_{,B}^{\;\;|B} + k^2 A = 4\pi J \; . \tag{3.2}$$

Here the vertical bar means covariant derivative obtained from the metric

$$g_{AB} dx^A dx^B$$

on M_2. It is clear that Eqs. (3.1) and (3.2) are geometrical vector and scalar equations on M_2. They are equivalent to the unreduced Maxwell wave equations (2.2). The reduced charge-conservation equation corresponding to Eq. (2.3) is

$$j^B_{\;|B} - k^2 j = 0 \; . \tag{3.3}$$

It is obtained from the divergence of Eq. (3.1a) combined with Eq. (3.1b).

A. Gauge invariance and charge conservation

The requirement that the action I be invariant under the gauge transformation

$$A_\mu \to \overline{A}_\mu = A_\mu + \Phi_{,\mu}$$

gives rise to charge conservation, Eq. (2.3). The gauge scalar Φ has the form

$$\Phi = \sum \phi^k(x^C) Y^k \; .$$

One sees from Eq. (2.10) that it induces the following changes on the vectors and scalars on M^2 for each mode (we are suppressing the mode index k):

$$a_B \to \overline{a}_B = a_b + \phi_{,B} \; ,$$

$$a \to \overline{a} = a + \phi \; ,$$

$$\mathcal{A}_B \to \overline{\mathcal{A}}_B = \overline{a}_B - \overline{a}_{,B} = \mathcal{A}_B \; ,$$

$$A \to \overline{A} = A \; .$$

Thus one sees that $\mathcal{A}_B$ and A are gauge-*invariant* geometrical objects on M^2, while a_B and a are gauge-*dependent* objects on M^2. If one demands that the reduced action, Eqs. (2.19), be gauge invariant, i.e.,

$$I[a_B + \phi_{,B}; a + \phi; A] = I[a_B; a; A] \; ,$$

then one has

$$j^B_{\;|B} - k^2 j = 0 \; ,$$

i.e., charge conservation for each mode. It is clear that if the action is required to be an extremum under arbitrary variations, i.e.,

$$I[a_B + \delta a_B; a + \delta a; A + \delta A] = I[a_B; a; A] \; ,$$

and thereby gives rise to the Maxwell field equations (3.1) and (3.2) then charge conservation is guaranteed. This is so because a gauge transformation is merely a special type of variation which keeps I stationary.

What is not so clear at first is why the Maxwell wave equations (3.1) and (3.2) are manifestly gauge invariant, while the action $I_{\rm TM}$ in Eq. (2.20) does not enjoy this property. The offending terms in that integral are

$$\int \int (j^B a_B^* + k^2 j a^*) \sqrt{-g^{(2)}} \, d^2x \; .$$

If one *assumes* charge conservation, i.e., Eq. (3.3), then these offending terms becomes

$$= \int \int (j^B a_B^* + j^B_{\;|B} a^*) \sqrt{-g^{(2)}} d^2x$$

$$= \int \int j^B (a_B^* - a^*_{,B}) \sqrt{-g^{(2)}} d^2x$$

$$= \int \int j^B \mathcal{A}_B^* \sqrt{-g^{(2)}} d^2x$$

which is manifestly gauge invariant. We therefore see that the *manifestly* gauge-invariant action functional

$$I_{\rm TM}^k + I_{\rm TE}^k = \frac{1}{4\pi} \int \int [-\tfrac{1}{2}(\mathcal{A}_{B|C} - \mathcal{A}_{C|B}) \mathcal{A}^{*B|C} - \tfrac{1}{2} k^2 \mathcal{A}^B \mathcal{A}_B^* + 4\pi j^B \mathcal{A}_B^*] \sqrt{-g^{(2)}} d^2x + \frac{k^2}{4\pi} \int \int [-\tfrac{1}{2} A_{,B} A^*_{,C} g^{BC} - \tfrac{1}{2} k^2 A A^* + 4\pi J A^*] \sqrt{-g^{(2)}} d^2x \tag{3.4}$$

yields all Eqs. (3.1) and (3.2) except one, namely, Eq. (3.1b). To obtain it, charge conservation has to be assumed explicitly; it cannot be obtained from the manifestly gauge-invariant action functional.

B. The electromagnetic field

The TE field modes and the TM field modes are totally decoupled from each other and thus evolve independently. It is easy to obtain the electromagnetic field. It decomposes into blocks

$$[F_{\mu\nu}]\equiv[A_{\nu,\mu}-A_{\mu,\nu}]=\begin{bmatrix} F_{BC} & F_{Bb} \\ F_{bB} & F_{bc}\end{bmatrix} . \tag{3.5}$$

With the help of Eq. (2.10) the Maxwell field of a typical TE mode has the form

$$[F^k_{\mu\nu}]_{\rm TE}=\begin{bmatrix} 0 & A_{,B}Y^k_{,d}\epsilon^d{}_b \\ -A_{,B}Y^k_{,d}\epsilon^d{}_b & -Ak^2Y^k\epsilon_{bc}\end{bmatrix} . \tag{3.6}$$

Here the gauge-invariant scalar A satisfies the inhomogeneous TE wave equation (3.2):

$$-A^{|C}{}_{,C}+k^2A=4\pi J .$$

The Maxwell field of a typical TM mode has with the help of Eq. (2.10) the form

$$[F^k_{\mu\nu}]_{\rm TM}=\begin{bmatrix} (\mathcal{A}_{C,B}-\mathcal{A}_{B,C})Y^k & -\mathcal{A}_BY^k_{,b} \\ \mathcal{A}_BY^k_{,b} & 0\end{bmatrix} . \tag{3.7}$$

The gauge-invariant vector potential $\mathcal{A}_B$ on M^2 can readily be obtained by decoupling the inhomogeneous wave equations (3.1). Observe that

$$\mathcal{A}_{C|B}-\mathcal{A}_{B|C}=\Phi\epsilon_{CB} , \tag{3.8}$$

where

$$\Phi=-\mathcal{A}_{E,F}\epsilon^{EF} . \tag{3.9}$$

This quantity is a scalar on M^2 and it is the longitudinal electric field amplitude of a TM mode. It is not to be confused with the gauge scalar in Sec III A. The quantity ϵ_{CB} are the components of the totally antisymmetric tensor on M^2. Multiply both sides of Eq. (3.1a) by ϵ^{BD}, use

$$\epsilon_{CB}\epsilon^{BD}=\delta^D_C \tag{3.10}$$

and take the divergence of both sides of Eq. (3.1a). The result is the master TM wave equation on M^2:

$$-\Phi_{,D}{}^{|D}+k^2\Phi=4\pi j_{B|D}\epsilon^{BD} . \tag{3.11}$$

From its solution one can recover all components of the TM electromagnetic field in Eq. (3.7). Indeed, the gauge-invariant vector potential $\mathcal{A}_B$ is obtained from the vector equation (3.1a). Combining it with Eq. (3.8) one obtains

$$\mathcal{A}_B=[4\pi j_B+\Phi_{,C}\epsilon^C_B]/k^2 .$$

Thus for the electromagnetic field for a TM mode, Eq. (3.7), is

$$[F^k_{\mu\nu}]_{\rm TM}=\begin{bmatrix} -\Phi\epsilon_{CB}Y^k & -\mathcal{A}_BY^k_{,b} \\ \mathcal{A}_BY^k_{,b} & 0\end{bmatrix} . \tag{3.12}$$

IV. ELECTROMAGNETIC FIELD DUE TO A PURELY LONGITUDINAL CURRENT

There is an equally good, if not slightly more direct way, of solving the vectorial TM equations (3.1). Suppose the current is purely longitudinal, i.e.,

$$J^\mu=(J^0,J^1,0,0) . \tag{4.1}$$

This happens if, for example, a charge is accelerated linearly but otherwise quite arbitrarily. This is the example in the next section where a uniformly accelerated Coulomb charge is considered. For purely longitudinal currents such as these, the reduced current on M^2 have, according to Eq. (2.11),

$$J=0,\quad j=0$$

but $j^A\neq 0$. The fact that $J=0$ implies that the TE modes satisfy the homogeneous wave equation (3.2). They do not interact with a longitudinal current. The TM modes, on the other hand, satisfy the TM wave equations (3.1) with $j=0$. With the help of Eqs. (3.8) and (3.9) they are

$$-[\mathcal{A}_{E,F}\epsilon^{EF}]_{,C}\epsilon^C_B+k^2\mathcal{A}_B=4\pi j_B , \tag{4.2a}$$

$$\mathcal{A}^B_{|B}=0 . \tag{4.2b}$$

As a consequence, the charge current j^B on M^2 also satisfies

$$j^B_{|B}=0 . \tag{4.3}$$

That $\mathcal{A}^B$ and j^B have zero divergence implies ("Helmholtz's theorem") that there exist two respective scalars ψ and η on M^2 such that

$$\mathcal{A}^B=\psi_{,C}\epsilon^{CB} , \tag{4.4a}$$

$$j^B=\eta_{,C}\epsilon^{CB} . \tag{4.4b}$$

In terms of these scalars the TM wave equation (4.2a) becomes, with the help of Eq. (3.10),

$$-[\psi_{,B}{}^{|B}]_{,C}+k^2\psi_{,C}=4\pi j^B\epsilon_{BC}=4\pi\eta_{,C} . \tag{4.5}$$

Upon integration one obtains the TM wave equation for a general longitudinal current, Eq. (4.1a),

$$-\psi_{,B}{}^{|B}+k^2\psi=4\pi\eta . \tag{4.5'}$$

This TM equation evidently has a structure identical to that of the TE wave equation (3.2). The scalar TM equation yields the electromagnetic field mode, Eq. (3.12). Its form with the help of Eqs. (4.4a), (3.9), (4.5), and

$$\mathcal{A}_{E,F}\epsilon^{EF}=\psi^{|D}_{,D}$$

is

$$[F^k{}_{\mu\nu}]_{\rm TM}=\begin{bmatrix} -\psi^{|D}_{,D}\epsilon_{CB}Y^k & -\psi_{,C}\epsilon^C_BY^k_{,b} \\ \psi_{,D}\epsilon^D_BY^k_{,b} & 0\end{bmatrix} . \tag{4.6}$$

This is a TM electromagnetic field mode due to an arbitrarily linearly moving charge distribution. We shall now consider this TM field due to a point charge in a linearly uniformly accelerating coordinate frame.

V. INTERACTION BETWEEN STATIC CHARGES

In particular, let us obtain in the accelerated frame what corresponds to the Coulomb field in an inertial frame and thereby exhibit what corresponds to a multipole expansion in the inertial frame. This expansion in terms of the appropriate special functions is a natural consequence of the 2+2 decomposition of the Maxwell field.

The current four-vector of a point charge e with four-velocity dz^μ/ds is [6]

$$J^\mu(x^\nu)=\frac{e}{\sqrt{-g}}\int_{-\infty}^{\infty}\frac{dz^\mu}{d\tau}\delta^4[x^\nu-z^\nu(\tau)]d\tau \ . \tag{5.1}$$

A charge which is static in a linearly uniformly accelerated frame has the world line

$$t=\xi_0\sinh\tau,\quad x=\xi_0\cosh\tau \ ,$$

$$y=y_0,\quad z=z_0 \ .$$

The current four-vector components relative to the coaccelerating basis are

$$\begin{aligned}&J^\tau=\frac{e}{\xi}\delta(\xi-\xi_0)\delta(y-y_0)\delta(z-z_0) \ ,\\&J^\xi=J^y=J^z=0 \ .\end{aligned} \tag{5.2}$$

It follows from Eq. (2.11) that the source for the reduced TM and TE wave equation is

$$j^0=\frac{e}{\xi}\delta(\xi-\xi_0)\frac{e^{-i(k_y y_0+k_z z_0)}}{2\pi} \ , \tag{5.3a}$$

$$j^1=0 \ , \tag{5.3b}$$

$$j=J=0 \ . \tag{5.4}$$

Applying the formalism of Sec. IV to Eqs.(5.3) and (5.4), one finds that the Coulomb potential of the charge due to (5.2) in the accelerated frame is

$$\varphi(\xi,\xi_0,y-y_0,z-z_0)=e\left[\frac{(\xi^2+R^2)^{1/2}}{R}-1\right] \ , \tag{5.5a}$$

where

$$R^2=\frac{[\xi_0^2+\xi^2+(y-y_0)^2+(z-z_0)^2]^2-4\xi^2\xi_0^2}{4\xi_0^2} \ . \tag{5.5b}$$

An alternative representation is

$$\varphi(\xi,\xi_0,y-y_0,z-z_0)=e\int_0^\infty 2\xi\xi_0 I_1(k\xi_<)K_1(k\xi_>)J_0(kr)k\,dk \ . \tag{5.6}$$

This representation is analogous to the familiar multipole expansion

$$\frac{e}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}=e\sum_{l=0}^{\infty}r_<^l r_>^{-l-1}P_l(\cos\theta)$$

of an inertial charge. Corresponding to the multipole of index l, one has in an accelerated frame multipoles of (continuous) index k. Their fields at $\xi>\xi_0$ are given by

$$[2ekI_1(k\xi_0)J_0(kr)]K_1(k\xi),\quad 0<k \ .$$

The details and the derivation of Eqs. (5.5) and (5.6) are given in the Appendix [7–9].

VI. SPECTRAL REPRESENTATION

Quantum mechanically the Coulomb energy between a pair of static charges is a spectral sum of processes in each one of which a pair of virtual quanta is exchanged.

In order to motivate the extension of this spectral decomposition from an inertial frame to a uniformly accelerated frame, let us recall its quantum mechanical basis relative to the inertial frame. The key features are already contained in the simpler Yukawa interaction, which is mediated by a scalar field instead of the vectorial Maxwell field.

Between two heavy inertial nonrelativistic nucleons situated at $\mathbf{X}_1$ and $\mathbf{X}_2$ the interaction is

$$\Delta E(\mathbf{X}_1-\mathbf{X}_2)=-\frac{g^2}{4\pi}\frac{e^{-m|\mathbf{X}_1-\mathbf{X}_2|}}{|\mathbf{X}_1-\mathbf{X}_2|} \ . \tag{6.1}$$

It is the quanta of the scalar field ϕ,

$$(\Box+m^2)\phi=0 \ , \tag{6.2}$$

which mediates this interaction. Indeed, the Hamiltonian for the interaction between this meson field and the two heavy nucleons is

$$H_{\rm int}=g\int[F(\mathbf{X}_1-\mathbf{Y})+F(\mathbf{X}_2-\mathbf{Y})]\phi(\mathbf{Y})d^3y \ . \tag{6.3}$$

Here g is the coupling constant ("charge") of a nucleon, and F expresses the finiteness of a nucleon, which in the limit of a point charge yields

$$F(\mathbf{X}-\mathbf{Y})\rightarrow\delta^3(\mathbf{X}-\mathbf{Y}) \ .$$

The interaction Hamiltonian perturbs the lowest-energy state of the unperturbed Hamiltonian

$$H_0=\int\omega_{\mathbf{k}}a_{\mathbf{k}}^*a_{\mathbf{k}}d^3k+2M \tag{6.4}$$

of the system: two nucleons each of rest mass M together with the meson field whose quanta have energy $\omega_{\mathbf{k}}$. The perturbation in the lowest-energy state is determined by second-order perturbation theory, and it is given by

$$\Delta E=\sum_n\frac{\langle 2,0|H_{\rm int}|n\rangle\langle n|H_{\rm int}|2,0\rangle}{2M-E_n} \ . \tag{6.5}$$

The meson field operator

$$\phi=\frac{1}{(2\pi)^{3/2}}\int\int\int\frac{a_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}t}}{\sqrt{2\omega_{\mathbf{k}}}}e^{i\mathbf{k}\cdot\mathbf{x}}d^3k+\text{Herm. adj.} \tag{6.6}$$

in the interaction Hamiltonian implies that the only intermediate states contributing to ΔE are those consisting of two nucleons plus one meson. Making use of

$$\sum_n\frac{(\cdots)|n\rangle\langle n|(\cdots)}{2M-E_n}\rightarrow\int\int\int\frac{(\cdots)|2,1_{\mathbf{k}}\rangle\langle 2,1_{\mathbf{k}}|(\cdots)}{-\omega_{\mathbf{k}}}d^3k\ , \tag{6.7}$$

a modest amount of algebra yields

$$\Delta E=-\frac{g^2}{(2\pi)^3}\int\int\int|\tilde{F}(\mathbf{k})|^2\frac{1}{2\omega_{\mathbf{k}}}\times(1+1+e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}+e^{i\mathbf{k}\cdot(\mathbf{x}_2-\mathbf{x}_1)})d^3k\ . \tag{6.8}$$

Thus the perturbation ΔE arises from four processes involving the emission and reabsorption of quanta. In the first, nucleon number one emits and absorbs a quantum. In the second, nucleon number two does the same. In the third and fourth, a quantum which is emitted by one nucleon gets absorbed by the other. Thus the perturbation ΔE decomposes into

$$\Delta E=\Delta E_1+\Delta E_2\ . \tag{6.9}$$

Here the exchange energy is

$$\Delta E_2=-\frac{g^2}{(2\pi)^3}\int\int\int\frac{e^{i\mathbf{k}\cdot(\mathbf{X}_1-\mathbf{X}_2)}}{k^2+m^2}d^3k \tag{6.10a}$$

$$=-\frac{g^2}{4\pi}\frac{e^{-m|\mathbf{X}_1-\mathbf{X}_2|}}{|X_1-X_2|}\ . \tag{6.10b}$$

One can probably give an analogous succinct derivation for an accelerated frame, but in that case additional qualitative features enter. See Sec. VIII. The purpose of this present section is to give in the accelerated frame a spectral decomposition of the Coulomb energy

$$V(\xi,\xi_0,y-y_0,z-z_0)=e^2\left[\frac{\xi^2+R^2}{R}-1\right]\ , \tag{6.11}$$

a decomposition which parallels the one for an inertial frame, Eq. (6.10).

A normal mode solution to the wave equation in an accelerated frame is

$$e^{-i\omega\tau}K_{i\omega}(k\xi)\frac{e^{i(k_yy+k_zz)}}{2\pi}$$

and the corresponding basis function is

$$K_{i\omega}(k\xi)\frac{e^{i(k_yy+k_zz)}}{2\pi}\ . \tag{6.12}$$

Like their inertial cousins in Eq. (6.10a), these basis functions are orthonormal and they form a complete set.

Indeed, the longitudinal (ξ-dependent) part of this function satisfies the orthogonality relation

$$\int_0^\infty\frac{2\omega}{\pi^2}\sinh\pi\omega K_{i\omega'}(k\xi)K_{i\omega'}(k\xi)\frac{d\xi}{\xi}=[\delta(\omega-\omega')+\delta(\omega+\omega')]\ , \tag{6.13}$$

the completeness relation

$$\int_0^\infty\frac{1}{\xi}K_{i\omega}(k\xi)K_{i\omega}(k\xi')\frac{\omega\sinh\pi\omega}{\pi^2}d\omega=\delta(\xi-\xi') \tag{6.14}$$

($K_{i\omega}$ is an even function of ω) and the differential equation

$$-\left[\frac{d}{d\xi}\xi\frac{d}{d\xi}-k^2\xi\right]K_{i\omega}(k\xi)=\frac{\omega^2}{\xi}K_{i\omega}(k\xi)\ . \tag{6.15}$$

Applying the completeness relation to the right-hand side of Eq. (A1), expanding the solution to Eq. (A1) in terms of the longitudinal wave function $K_{i\omega}(k\xi)$, using Eq. (6.15), and finally using the orthogonality relation Eq. (6.13), one obtains

$$\varphi_1(\xi)\equiv\xi\frac{d\psi}{d\xi}=2ee^{-i(k_yy_0+k_zz_0)}\times\int_0^\infty\frac{2\omega\sinh\pi\omega}{\pi^2}\frac{\xi K_{i\omega}(k\xi)K_{i\omega}(k\xi_0)d\omega}{\omega^2+1}\ . \tag{6.16}$$

This result can also be obtained without using the completeness of the set of wave functions $K_{i\omega}(k\xi)$. One simply accepts a well-documented integral [10] to replace the product in Eq. (A4) with its spectral representation, Eq. (6.16).

The total Coulomb potential is now given by Eq. (5.13). The spectral decomposition of the Coulomb energy $V=e\varphi$ between two static charges in a linearly accelerated frame is therefore

$$V(\xi,\xi_0,y-y_0,z-z_0)=e^2\left[\frac{(\xi^2+R^2)^{1/2}}{R}-1\right]$$

$$=2e^2\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{2\omega\sinh\pi\omega}{\pi^2}\frac{\xi K_{i\omega}(k\xi)K_{i\omega}(k\xi_0)}{\omega^2+1}\frac{e^{i[k_y(y-y_0)+k_z(z-z_0)]}}{2\pi}d\omega\,dk_ydk_z\ . \tag{6.17}$$

This is the interaction which is analogous to the Coulomb interaction, Eq. (6.10), in an inertial frame.

VII. RADIATION REACTION

With a fixed (at ξ_0, y_0, z_0) charge e giving rise to its static Coulomb potential

$$\varphi(\xi,\xi_0,y-y_0,z-z_0) = e\left[\frac{\xi_0^2+\xi^2+r^2}{\sqrt{(\xi_0^2+\xi^2+r^2)^2-4\xi^2\xi_0^2}}-1\right] \quad (7.1)$$

one might wonder: Where is the emitted Larmor radiation? We shall answer this question with a heuristic argument which is based on the idea that the future event horizon is the history of a two-dimensional resistive membrane [11–13].

Recall the Lorentz-Dirac equation of motion of a point particle of mass m and charge e under the action of an external force F^μ:

$$\frac{d}{ds}\left[m\frac{dx^\mu}{ds}-\frac{2}{3}e^2\frac{d^2x^\mu}{ds^2}\right] = -\frac{2}{3}e^2\frac{dx^\mu}{ds}\frac{d^2x^\alpha}{ds^2}\frac{d^2x^\beta}{ds^2}g_{\alpha\beta}+F^\mu . \quad (7.2)$$

This equation together with its physical interpretation [14,15–17] is a direct consequence of the conservation of total momentum-energy (="momenergy" [18]), electromagnetic together with mechanical, of the particle along its world line.

Also recall that the EM momenergy of a charge splits unambiguously into two mutually exclusive and jointly exhaustive parts: (1) that which is *radiated* and (2) that which is *bound* [14] to the charge. Each of these two parts is determined by its own respective EM stress-energy tensor, both of which are divergenceless everywhere except on the world line of the charge. The identifying feature of the radiated stress-energy tensor is that (a) it is quadratic in the acceleration and that (b) its form is that of a simple null fluid, even close to the charge [14].

There is a third part of the momenergy; it is purely mechanical and it describes the "bare" (i.e., without any EM field) particle. Its stress-energy tensor is also divergenceless everywhere except on the particle world line.

As for any total stress-energy tensor, the sum of its three individual sources vanishes. In fact, the sources and sinks of the three stress energies are balanced perfectly along the whole history of the particle. This balance is expressed by the Lorentz-Dirac equation (7.2).

Its left-hand side is the rate of change of momenergy

$$P^\mu = m\frac{dx^\mu}{ds}-\frac{2}{3}e^2\frac{d^2x^\mu}{ds^2} . \quad (7.3)$$

It is the sum of the particle's inertial momenergy ($\propto dx^\mu/ds$) and the EM momenergy ($\propto d^2x^\mu/ds^2$) which is always bound to the charge no matter what its instantaneous state of acceleration might be. Although one often talks about these two momenergies separately, physically the two go together.

The right-hand side, apart from F^μ, is the Larmor expression for the rate at which the charged particle loses momenergy in the form of EM radiation. The Lorentz-Dirac equation demands that this radiation reaction four-force change the inertial plus bound EM momenergy of the charged particle. The magnitude of the radiation four-force is given by the invariant Larmor formula

$$\text{power} = \frac{2}{3}e^2\frac{d^2x^\alpha}{ds^2}\frac{d^2x^\beta}{ds^2}g_{\alpha\beta} . \quad (7.4)$$

Let us illustrate an alternative viewpoint by deriving this power for a Coulomb charge static in a linearly uniformly accelerated coordinate frame. The line of reasoning goes roughly like this: Let the charge be fixed at (ξ_0, y_0, z_0) in the accelerated frame. The charge is surrounded by equipotential surfaces, Eq. (7.1):

$$\varphi(\xi,\xi_0,y-y_0,z-z_0) = e\left[\frac{(\xi^2+R^2)^{1/2}}{R}-1\right] = \text{const} .$$

The electric field

$$E_\xi \equiv F_{\xi\mu}u^\mu = |g^{\tau\tau}|^{1/2}F_{\xi 0} = \frac{-1}{\xi}\frac{d\varphi}{d\xi} = -e\frac{4\xi_0^2(\xi_0^2+r^2-\xi^2)}{[(\xi^2+\xi_0^2+r^2)^2-4\xi\xi_0]^{3/2}} ,$$

$$E_y \equiv F_{y\mu}u^\mu = |g^{\tau\tau}|^{1/2}F_{y0} = \frac{-1}{\xi}\frac{d\varphi}{dy} = e\frac{\xi}{\xi_0}\frac{(y-y_0)8\xi_0^3}{[(\xi^2+\xi_0^2+r^2)^2-4\xi_0^2\xi^2]^{3/2}} , \quad (7.5)$$

$$E_z \equiv F_{z\mu}u^\mu = |g^{\tau\tau}|^{1/2}F_{z0} = \frac{-1}{\xi}\frac{d\varphi}{dz} = e\frac{\xi}{\xi_0}\frac{(z-z_0)8\xi_0^3}{[(\xi^2+\xi_0^2+r^2)^4-4\xi_0^2\xi^2]^{3/2}}$$

is perpendicular to these potential surfaces and hence also to the event horizon $\xi=0$ where $\varphi=0$. There this electric field induces the surface charge density [19–21]

$$\sigma = \frac{1}{4\pi}E_\xi|_{\xi=0} = -\frac{e}{\pi}\frac{\xi_0^2}{(\xi_0^2+r^2)^2} . \quad (7.6)$$

The total charge induced at the event horizon is

$$\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\sigma\, dy\, dz = -e . \quad (7.7)$$

This, by the way, supports the fact the event horizon behaves like the history of a conductive surface [11–13].

Electrostatics implies that there is an attractive force between the point charge e and the surface charge density σ. From symmetry this force is directed along the ξ direction and has magnitude

$$|\text{force}| = \tfrac{1}{2}\int\int E_\xi\sigma|_{\xi=0}dy\, dz = \int\int \frac{E_\xi^2}{8\pi}\bigg|_{\xi=0} dy\, dz = \frac{2}{3}\frac{e^2}{\xi_0^2}$$

or

$$\text{power} = \tfrac{2}{3}e^2(\text{acceleration})^2 \tag{7.8}$$

in terms of relativistic units, which we are using. This force, or power, is a rate of change; furthermore, it is a rate of change which is tangential to the future event horizon. It therefore expresses a flow of momenergy across the two-dimensional membrane (spanned by y and z) whose history is that future event horizon. Conservation of momenergy ("for every action there is an opposite and equal reaction") implies that the momenergy gets drained from the charged particle at a rate,which yields the magnitude of the radiation reaction on the right-hand side of the Lorentz-Dirac equation (7.2).

Identifying the momenergy in the event horizon with radiation momenergy losses along the particle world line hinges on a tacit assumption: None of the particle's *bound* [14] stress-energy tensor enters into the momenergy conservation between the particle's world line and its future event horizon.

The fact that the rate given by Eq. (7.8) agrees with Larmor's formula (7.4) leads us into making two observations.

(1) The existence of charge density on the event horizon is a purely observer-dependent phenomenon. This hypothesis is illustrated by the difference in the physical interpretation of Eqs. (7.4) and (7.8). Relative to an inertial observer there is, of course, no charge on the event horizon of the accelerated frame. Equation (7.4) is interpreted as the loss of (four-) momentum from an isolated charge. By contrast, relative to an accelerated frame, Eq. (7.8) demands that the loss of momentum from this "isolated" charge is in fact due to an electrostatic interaction mediated by Eq. (7.5).

Thus the radiation field in the inertial frame becomes an electrostatic field in the accelerated coordinate frame.

This electrostatic field expresses a Coulomb interaction between the "isolated" charge and its induced twin, Eq. (7.6), on the event horizon.

It is possible to cancel (with arbitrary accuracy) the electric field due to this induced twin charge: At $\xi=0^+$ merely distribute a layer of actual charge as dictated by Eq. (7.6). This distribution will cancel the interaction of the induced charge with the "isolated" charge. Removing the distribution will restore the interaction. Thus there is no way of distinguishing the field of the charge density (7.6) from that of an actual charge on the event horizon of the accelerated frame.

(2) If we extend our considerations of a Coulomb field static in an accelerated frame to those of a moving charge, then the possibility exists of having surface currents as well as intrinsic electric and magnetic fields evolve on the event horizon. They give rise, among others, to resistive forces [11] which act back on their sources and thus could give a picturesque account of the radiation reaction force of a point charge. Such a picturesque account can often be given by referring to a black-hole analogue. An obvious example is a charged particle suspended above the equator of a rotating black hole. This can be viewed as the electrostatic analogue of a black hole rotating in an oblique magnetic field [22]. A resistive spin-down torque is exerted on the black hole [11] The back reaction on the charge can be viewed as the radiation reaction which enters the Lorentz-Dirac equation (7.2). This one can do, provided one replaces the space-time of the rotating black hole with the appropriate flat space-time analogue: The coordinate frame of a linearly accelerated observer with uniform transverse drift [23].

Although one can believe that the back reaction from the black hole corresponds to the radiation reaction in drifting Rindler spacetime, the analogue of the increase of the entropy of a black hole [24] is still rather murky. This is so because there does not yet exist for Rindler spacetime a definition of what in gravitation physics is called a black-hole entropy [24].

That such an analogue should exist is not entirely unreasonable if one recalls that radiation losses from an accelerated charge are resistive and hence irreversible in nature. It presumably is this irreversibility which would be expressed by an increase in the to-be-defined entropy of a Rindler event horizon.

VIII. ACCELERATED POLARONS?

Compare the Coulomb interaction energy between a pair of charges static in an accelerated frame as given by Eq. (6.17), with the Yukawa interaction energy between a pair of charges static in an inertial frame:

$$\begin{aligned} V(\mathbf{X}_1-X_2) &= \frac{e^2}{4\pi}\frac{e^{-m|\mathbf{X}_1-\mathbf{X}_2|}}{|\mathbf{X}_1-\mathbf{X}_2|} \\ &= e^2\int\int\int \frac{1}{k_x^2+k_y^2+k_z^2+m^2}e^{i(k_x x+k_y y+k_z z)}\frac{dk_x dk_y dk_z}{(2\pi)^3}\ . \end{aligned} \tag{8.1}$$

This interaction energy is attributed to a second-order process which expresses the exchange of quanta of momentum $\hbar(k_x,k_y,k_z)$ between two nonrelativistic charges, each of rest mass m. The emission and absorption of these virtual "Minkowski" quanta takes place in an inertial frame where the ground state of the field is the familiar Minkowski vacuum.

In an accelerated frame, however, the elementary excitations are not the Minkowski quanta. Instead one has the Fulling quanta [25]. They are elementary excitations of a different ground state, the Rindler vacuum. This ground state consists of a configuration of highly correlated photons [3].

The problem of the Coulomb interaction between a pair of charges accelerating through this correlated photon configuration is analogous to the interaction between a pair of charges in a polar crystal ("interaction between a pair of polarons"). In such a crystal a single charge is referred to as a "polaron" because it consists of the charge together with the local lattice polarization distortion which the charge produces [26]. This affects its mass ("mass renormalization") and its interaction with other charges [1] ("coupling-constant renormalization"). An accelerated charge in a correlated photon configuration may be viewed in the same way. The charge distorts this correlated configuration (the "Rindler vacuum state") and gives rise to an "accelerated polaron." Consider the interaction between two such "accelerated polarons." The description of this interaction in terms of photons is complicated when it is done in terms of photons, just as the interaction between two polarons in a crystal is complicated when done in terms of the lattice atoms. A much more natural description is in terms of elementary excitations. For "accelerated polarons" this means a description in terms of (virtual) Fulling quanta, just like for crystal polarons this means a description in terms of (virtual) sound quanta. These quanta have however an effect on the Coulomb interaction [1]. Instead of Eq. (8.1), the interaction potential (with $m=0$) is

$$V(\mathbf{X}_1-\mathbf{X}_2)=\frac{1}{\epsilon}\frac{e^2}{4\pi}\frac{1}{|\mathbf{X}_1-\mathbf{X}_2|} .$$

The potential is still of the Coulomb type, but with a change in the coupling constant

$$e^2 \rightarrow e^2/\epsilon$$

due to the dielectric constant ϵ of the crystal.

This analogy with polarons in a crystal suggests an inquiry as to whether the Coulomb interaction between two accelerating charges is also altered. In other words, does the classical Coulomb potential, Eq. (A9) or (6.17), differ from the Coulomb potential determined quantum mechanically by a factor which expresses the "polarizability" and hence the dielectric constant of the Rindler vacuum?

ACKNOWLEDGMENTS

F.J.A. was supported in part by Ohio State University and at Rutgers by NSF Grant No. DMR89-1893. F.J.A. thanks Joel Lebowitz for his support during the completion of this project.

APPENDIX: COULOMB POTENTIAL IN AN ACCELERATED FRAME

This appendix obtains (1) the total potential of an accelerated charge, (2) its resolution into what in an inertial frame corresponds to multipoles, and (3) the concomitant Bessel function integrals. The potential is obtained by solving the TM master equation (4.5) with the longitudinal source, Eq. (5.3). This source produces no TE electromagnetic field. Consequently we may equate this field, Eq. (3.6), to zero.

Furthermore, the current and the field are time (τ) independent. Consequently the $C=1$ component of the TM Eq. (4.5) becomes, with the help of Eq. (5.3a) and with $\epsilon_{01}=\xi$,

$$-\left[\frac{d^2}{d\xi^2}+\frac{1}{\xi}\frac{d}{d\xi}-\left[\frac{1}{\xi^2}+k^2\right]\right]\frac{d\psi}{d\xi} = 2e\delta(\xi-\xi_0)e^{-i(k_y y_0+k_z z_0)} . \quad \text{(A1)}$$

What are the boundary conditions that must be satisfied by the solution to this equation? They are determined by its physical significance. The solution determines all nonzero (spectral) components of the Maxwell tensor, Eq. (4.6), the electric field

$$[F^k_{10}]_{\text{TM}} = -\frac{1}{\xi}\frac{d}{d\xi}\xi\frac{d\psi}{d\xi}\epsilon_{01}Y^k(y,z) \equiv -\frac{\partial}{\partial\xi}\varphi_k(\xi)Y^k(y,z) ,$$

$$[F^k_{y0}]_{\text{TM}} = \frac{d\psi}{d\xi}\epsilon_{10}Y^k_{,y}(y,z) \equiv -\frac{\partial}{\partial y}\varphi_k(\xi)Y^k(y,z) ,$$

$$[F^k_{z0}]_{\text{TM}} = \frac{d\psi}{d\xi}\epsilon_{10}Y^k_{,z}(y,z) \equiv -\frac{\partial}{\partial z}\varphi_k(\xi)Y^k(y,z) .$$

Thus

$$\varphi_k(\xi) \equiv \xi\frac{d\psi}{d\xi}$$

is the (spectral component of the) Coulomb potential, Eq. (4.4a),

$$\mathcal{A}_0 = -\varphi_k(\xi) \equiv -\xi\frac{d\psi}{d\xi} ,$$

$$\mathcal{A}_1 = 0 .$$

Relative to the physical (=orthonormal) basis ($\xi\, d\tau, d\xi$) it satisfies the boundary conditions

$$\frac{d\psi}{d\xi} \rightarrow 0 \quad \text{as } \xi\rightarrow\infty \quad \text{(spatial infinity)} , \quad \text{(A2)}$$

$$\frac{d\psi}{d\xi} \rightarrow \text{finite as } \xi\rightarrow 0 \quad \text{(event horizon)} . \quad \text{(A3)}$$

It follows that the spectral potential, obtained from the solution to Eq. (A1), is

$$\varphi_k(\xi) = \xi\frac{d\psi}{d\xi} = 2ee^{-i(k_y y_0+k_z z_0)}\xi\xi_0 I_1(k\xi_<)K_1(K\xi_>) , \quad \text{(A4)}$$

where

$$I_1(k\xi_<)K_1(k\xi_>) \equiv \begin{cases} I_1(k\xi)K_1(k\xi_0), & \xi<\xi_0 , \\ I_1(k\xi_0)K_1(k\xi), & \xi>\xi_0 . \end{cases}$$

Here K_1 is the modified Bessel function which vanishes exponentially as $\xi\rightarrow\infty$, and I_1 is one which vanishes linearly as $\xi\rightarrow 0$. The total Coulomb potential is obtained as a sum from its spectral components:

$$\varphi(\xi,\xi_0,y-y_0,z-z_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_k(\xi)\frac{e^{i(k_y y+k_z z)}}{2\pi}dk_y dk_z$$
$$=2e\xi\xi_0\int_0^\infty I_1(k\xi_<)K_1(k\xi_>)\int_0^{2\pi}\frac{e^{ikr\cos(\theta-\alpha)}}{2\pi}d\alpha\, k\, dk\ . \quad \text{(A5)}$$

Introduce the Bessel function

$$J_0(kr)=\int_0^{2\pi}\frac{e^{ikr\cos(\theta-\alpha)}}{2\pi}d\alpha\ ,$$

where

$$r=\sqrt{(y-y_0)^2+(z-z_0)^2},\quad r\cos\theta=y-y_0\ ,$$
$$k=\sqrt{k_y^2+k_z^2},\quad k\cos\alpha=ky\ .$$

Consequently, the potential becomes an integral involving the product of the three Bessel functions

$$\varphi(\xi,\xi_0,y-y_0,z-z_0)$$
$$=2e\xi\xi_0\int_0^\infty I_1(k\xi_<)K_1(k\xi_>)J_0(kr)k\,dk\ . \quad \text{(A6)}$$

For $\xi<\xi_0$ the integral is [7]

$$\int_0^\infty I_1(k\xi)K_1(k\xi_0)J_0(kr)k\,dk$$
$$=(\xi\xi_0)^{-1}\frac{-i}{\sqrt{2\pi}}\frac{Q_{1/2}^{1/2}(u)}{(u^2-1)^{1/4}}\ ,$$

where

$$Q_{1/2}^{1/2}(\cosh\gamma)=i\left[\frac{\pi}{2\sinh\gamma}\right]^{1/2}e^{-\gamma} \quad \text{(A7a)}$$

and

$$u=\frac{\xi^2+\xi_0^2+r^2}{2\xi\xi_0}\ . \quad \text{(A7b)}$$

Then the total potential is

$$\varphi(\xi,y,z)=e\frac{u-(u^2-1)^{1/2}}{(u^2-1)^{1/2}},\quad \xi<\xi_0\ . \quad \text{(A8)}$$

For $\xi_0<\xi$ we expect the spectral integral to yield the same result. Indeed, by resorting to the identity [8]

$$\int_0^\infty I_\nu(k\xi_0)K_\nu(k\xi)J_\mu(kr)k^{\mu+1}dk$$
$$=e^{-i(\pi/2)\nu}\int_0^\infty I_\nu(ik\xi_0)J_\nu(k\xi)K_\mu(kr)k^{\mu+1}dk$$

one can use for $\xi_0<\xi$ the integral [9]

$$\int_0^\infty I_1(ik\xi_0)J_1(k\xi)K_0(kr)k\,dk$$
$$=(i\xi\xi_0)^{-1}\frac{1}{\sqrt{2\pi}}\frac{Q_{1/2}^{1/2}(u)}{(u^2-1)^{1/4}}$$

to evaluate Eq. (A6). With the help of Eq. (A7) the result is the same as Eq. (A8), except that $\xi_0<\xi$, as expected.

We conclude therefore that the Coulomb potential (A5) due to a static charge e in a (linearly uniformly) accelerated frame is

$$\varphi(\xi,\xi_0,y-y_0,z-z_0)$$
$$=e\left[\frac{(\xi^2+R^2)^{1/2}}{R}-1\right] \quad \text{(A9a)}$$
$$=e\int_0^\infty 2\xi\xi_0 I_1(k\xi_<)K_1(k\xi_>)J_0(kr)k\,dk\ . \quad \text{(A9b)}$$

where

$$R^2=\xi^2(u^2-1)$$
$$=\frac{[\xi_0^2+\xi^2+(y-y_0)^2+(z-z_0)^2]^2-4\xi^2\xi_0^2}{4\xi_0^2}$$

The representation (A9a) is the familiar expression obtained by Boulware [27]. Its representation (A9b) is analogous to the familiar multipole expansion,

$$\frac{e}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$
$$=e\sum_{l=0}^{\infty}r_<^l r_>^{-l-1}P_l(\cos\theta)\ ,$$

of an inertial charge. The "multipoles" (if one insists on introducing them) for the charge static in the accelerated frame are evidently characterized by the continuous index k instead of the discrete index l for the inertial case.

[1] See F. Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (World Scientific, Singapore, 1985), Chap. II, for an intuitive picture.

[2] S. Schweber, *Relativistic Quantum Field Theory* (Harper and Row, New York, 1961), Chap. 12.

[3] U. H. Gerlach, Phys. Rev. D **40**, 1037 (1989).

[4] U. H. Gerlach, Ann. Inst. Henri Poincaré, **49**, 397 (1988); Ann. N.Y. Acad. Sci. **571**, 331 (1989); in *Fifth Marcel Grossmann Meeting*, Proceedings, Perth, Australia, 1988, edited by D. G. Blair and M. J. Buckingham (World Scientific, Teaneck, NJ, 1989), p. 797.

[5] Except for a scale factor, these TM equations are the same as the $(-1)^{l+1}$ parity linearized Einstein field equations on an arbitrary spherically symmetric background. See U. H. Gerlach and J. F. Scott, Phys. Rev. D **34**, 3638 (1986). The scale factor r^2 is introduced by making the substitutions $k^2\to(l-1)(l+2)/r^2$, $\mathcal{A}_{C|B}{}^{|C}\to r^{-2}[r^4(r^{-2}k_C)_{|B}]^{|C}$ and similarly for $A_{B|C}{}^{|C}$. Evidently for a large radial coor-

dinate r, an $(-1)^{l+1}$ parity gravitational disturbance is the same as an electromagnetic TM mode.

[6] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Sec. 14.1.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products* (Academic, New York, 1980), p. 696, Eq. 11.

[8] W. N. Bailey, Proc. London Math. Soc. **40**, 43 (1936).

[9] Gradshteyn and Ryzhik [7], p. 695, Eq. 7.

[10] See Eq. (10) in Gradshteyn and Ryzhik [7], p. 773.

[11] T. Damour, Phys. Rev. D **18**, 3598 (1978).

[12] R. L. Znajek, Mon. Not. R. Astron. Soc. **185**, 833 (1978).

[13] T. Damour, in *Proceedings of the Second Marcel Grossmann Meeting on the Recent Developments of General Relativity,* Trieste, Italy, 1979, edited by R. Ruffini (North-Holland, New York, 1982), Part A, pp. 587–608.

[14] C Teitelboim, Phys. Rev. D **1**, 1572 (1970). This is a very useful paper. Among other things, it introduces the concept of electromagnetic four-momentum "bound" to a charge. It also delineates the key ideas of the pioneering works of P. A. M. Dirac and F. Rohrlich on the subject matter.

[15] C. Teitelboim, Phys. Rev. D **3**, 297 (1971).

[16] C. Teitelboim, Phys. Rev. D **4**, 345 (1971).

[17] C. Teitelboim, D. Villarroel, and Ch. G. Van Weert, Riv. Nuovo Cimento **3**, 1 (1980).

[18] The contraction of "momentum energy" into "momenergy" is rather natural. For a rationale, see J. A. Wheeler, *A Journey into Gravity and Spacetime* (Freeman, New York, 1990), Chap. 6.

[19] The concept of charge density induced on a black hole was first introduced by R. S. Hanni and R. Ruffini, Phys. Rev. D **8**, 3259 (1972).

[20] D. A. Macdonald and W.-Mo Suen, Phys. Rev. D **32**, 848 (1985), gives the electric and magnetic fields of a charge in various states of motion in an accelerated frame.

[21] Chapter 2, in *Black Holes: The Membrane Paradigm,* edited by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale University Press, New Haven, CT, 1986), gives an introductory treatment of the electrical properties of the surface whose history is an event horizon.

[22] W. H. Press, Astrophys. J. **175**, 243 (1972).

[23] U. H. Gerlach, Phys. Rev. D **27**, 2310 (1983).

[24] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).

[25] S. A. Fulling, Phys. Rev. D **7**, 2850 (1973).

[26] R. P. Feynman, *Statistical Mechanics; A Set of Lectures* (Benjamin/Cummings, Reading, MA, 1982), Chap. 8.

[27] D. G. Boulware, Ann. Phys. (N.Y.) **124**, 169 (1980).