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Renormalization of the δ expansion in curved space-time

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Renormalization of a recently proposed δ expansion for a self-interacting scalar field theory in curved space-time is examined. The explicit calculation is carried out up to order δ^2, which indicates that the expansion is renormalizable, but reduces to essentially the $\lambda\phi^4$ theory when the cutoff is removed. A similar conclusion has been reached in a previous paper where the case of flat space-time is considered.

I. INTRODUCTION

A new perturbative expansion [1, 2] in quantum field theory has recently been proposed. In the case of self-interacting scalar theory in four dimensions, this expansion, called the δ expansion, amounts to replacing the interacting term $\lambda\phi^4$ in the Lagrangian by $\lambda\phi^{2(1+\delta)}$. The relevant quantities, such as the Green's functions, of this theory are then evaluated as power series in δ. This expansion has been proven to be renormalizable using a momentum-cutoff regularization scheme [3]. In fact, when the cutoff is taken to go to infinity, the renormalized Green's functions reduce to the ones in the $\lambda\phi^4$ theory. Thus it has been argued that all theories with the potential $\lambda\phi^{2(1+\delta)}$, for $\delta \leq 1$, are equivalent. The problem of renormalization of the δ expansion from other points of view has also been discussed in Refs. [4] and [5].

In this paper we would like to address the question of renormalization of the δ expansion in curved space-time following the treatment in Ref. [3]. The interest in studying quantum fields in general background space-time is twofold. It may be considered as a first step towards constructing a consistent theory of quantum gravity. Hopefully it will provide some hint on how to deal with the uncontrollable ultraviolet divergences present in quantizing Einstein gravity. On the other hand, there are physical situations, for example, the evolution of the early Universe or phenomena near a black hole, where the effects of curved space-time are crucial and must be taken into account [6]. To apply the δ expansion to these more general conditions, one has to first resolve the question of renormalization.

To begin with we have to adopt an appopriate regularization scheme. In Ref. [3] a momentum-cutoff regularization was used. In curved space-time the formalism is much simpler if we stay in coordinate space. Thus we use instead a coordinate-space cutoff, or point-splitting regularization [7–9]. Since we stay in coordinate space, we will work with the connected Green's functions instead of the one-particle-irreducible ones.

The paper is organized as follows. In the next section we give the Feynman rules for the δ expansion [10]. In particular, the divergent parts of the propagator and the vertex functions are discussed in some detail. In Sec. III, we calculate the connected Green's functions of up to δ^2 and show that the theory is renormalizable at least up to this order. In the last section we summarize our results and give some discussion on the behavior of the Green's functions in higher powers of δ. In the two appendices, we evaluate a divergent integral and an infinite sum which are frequently used in Sec. III.

II. FEYNMAN RULES

In this section we give the Feynman rules for the δ expansion [10] generalized to curved space-time. The asymptotic behavior of the Feynman propagator is discussed using the Schwinger-DeWitt expansion [11, 12]. The regularization scheme we adopt here is the covariant coordinate, or point-splitting, regularization.

To construct the δ expansion for a self-interacting scalar field, we begin by replacing the $\lambda\phi^4$ Lagrangian (in Euclidean signature)

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{1}{2}\xi_0 R\phi^2 + \lambda\phi^4, \tag{2.1}$$

with

$$\mathcal{L} = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{1}{2}\xi_0 R\phi^2 + \lambda M^2\phi^2\left(\frac{\phi^2}{M^2}\right)^\delta, \tag{2.2}$$

where ∇_μ is the covariant derivative and M is some arbitrary mass parameter. To renormalize the theory order by order in δ, we add counterterms to Eq. (2.2) to form the renormalized Lagrangian

$$\mathcal{L}_R = \frac{1}{2}(1+A)(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\xi R\phi^2 + \lambda Z M^2\phi^2\sum_{n=1}^{\infty}\frac{\delta^n}{n!}\left(\ln\frac{\phi^2}{M^2}\right)^n, \tag{2.3}$$

where the renormalization constants are defined by

$$A = \delta A_1 + \delta^2 A_2 + \cdots,$$
$$m^2 = m_0^2 + \delta m_1^2 + \delta^2 m_2^2 + \cdots, \tag{2.4}$$
$$\xi = \xi_0 + \delta\xi_1 + \delta^2\xi_2 + \cdots,$$
$$Z = Z_0 + \delta Z_1 + \delta^2 Z_2 \cdots,$$

with $m_0^2 = \mu^2 + 2\lambda M^2$.

The Feynman propagator is given by

$$G_0(x,y) = \left\langle x \left| \frac{1}{-\nabla_\mu\nabla^\mu + m_0^2 + \xi_0 R} \right| y \right\rangle = \int_0^\infty dt\, e^{-m_0^2 t} \langle x | e^{t(\nabla_\mu\nabla^\mu - \xi_0 R)} | y \rangle. \tag{2.5}$$

For the asymptotic region $y \to x$, we can use the Schwinger-DeWitt expansion to express the matrix element:

$$\langle x | e^{t(\nabla_\mu\nabla^\mu - \xi_0 R)} | y \rangle = \frac{1}{(4\pi t)^2}\, e^{-\sigma(x,y)/2t}\, \Delta^{1/2}(x,y) \sum_{n=0}^{\infty} t^n a_n(x,y), \tag{2.6}$$

where $\sigma(x,y)$ and $\Delta(x,y)$ are symmetric biscalar functions defined by

$$\sigma = \frac{1}{2}\sigma_\mu \sigma^\mu, \tag{2.7}$$

$$\Delta = \frac{1}{4}\nabla_\mu(\Delta\sigma^\mu), \tag{2.8}$$

with the coincident limits $\sigma(x,x) = 0$ and $\Delta(x,x) = 1$. $\sigma_\mu \equiv \nabla_\mu \sigma$ denotes the covariant derivative of σ. Note that in our notation covariant derivatives act on the first argument of all the bifunctions. $\sigma(x,y)$ is related to the geodesic distance between x and y. More precisely, if we take s as the geodesic distance, then $\sigma(x,y) = s^2/2$. The so-called Schwinger-DeWitt coefficients $a_n(x,y)$ are defined by

$$\sigma^\mu \nabla_\mu a_0 = 0, \tag{2.9}$$

$$(n+1)a_{n+1} + \sigma^\mu \nabla_\mu a_{n+1} = \Delta^{-1/2}\nabla_\mu\nabla^\mu(\Delta^{1/2} a_n) - \xi_0 R a_n, \quad n \geq 0, \tag{2.10}$$

with the coincident limit $a_0(x,x) = 1$. Thus after doing the integration over t, Eq. (2.5) can now be written in terms of the modified Bessel functions $K_n(x)$:

$$G_0(x,y) = \frac{\Delta^{1/2}(x,y)}{8\pi^2} \sum_{n=0}^{\infty} a_n(x,y) \left(\frac{\sigma(x,y)}{2m_0^2}\right)^{(n-1)/2} \times K_{n-1}(\sqrt{2\sigma m_0^2}). \tag{2.11}$$

Using the asymptotic behavior of the modified Bessel functions and the coincident limits [12]

$$\nabla_\mu \Delta^{1/2}|_{y\to x} = 0, \tag{2.12}$$
$$\nabla_\mu\nabla_\nu \Delta^{1/2}|_{y\to x} = \frac{1}{6}R_{\mu\nu}, \tag{2.13}$$
$$\nabla_\mu a_0|_{y\to x} = 0, \tag{2.14}$$
$$\nabla_\mu\nabla_\nu a_0|_{y\to x} = 0, \tag{2.15}$$
$$a_1|_{y\to x} = \left(\frac{1}{6} - \xi_0\right) R, \tag{2.16}$$

one can expand $G_0(x,y)$ in a power series of $\sigma(x,y)$:

$$G_0(x,y) = \frac{1}{8\pi^2\sigma}\left(1 + \left[m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right]\sigma \ln\sqrt{\frac{1}{2}\sigma m_0^2} + \sigma\left\{\frac{1}{12}R_{\mu\nu}\frac{\sigma^\mu\sigma^\nu}{\sigma^2} - m_0^2\left[\psi(1) + \frac{1}{2}\right] - \left(\xi_0 - \frac{1}{6}\right)R\psi(1)\right\} + \cdots\right), \tag{2.17}$$

where $\psi(x) = d\ln\Gamma(x)/dx$. With this form of G_0, the ultraviolet divergences of various Feynman diagrams can be examined. For example, the one-loop diagram in Fig. 1 gives

$$I \equiv G_0(x,y)|_{y\to x}, \tag{2.18}$$

which is ultraviolet divergent. To deal with that, we adopt a symmetric point-splitting regularization, which amounts to make the replacements [9],

$$\sigma \to \frac{s^2}{2}, \tag{2.19}$$
$$\sigma_\mu \to 0, \tag{2.20}$$
$$\sigma_\mu\sigma_\nu \to \frac{1}{4}g_{\mu\nu}s^2. \tag{2.21}$$

Then, from Eq. (2.17),

$$I = \frac{1}{4\pi s^2}\left(1 + \frac{1}{2}s^2\left\{\left[m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right]\left(\ln\frac{1}{2}m_0^2 s - \psi(1)\right) - \frac{1}{2}m_0^2 + \frac{1}{24}R\right\} + \cdots\right), \tag{2.22}$$

where the remaining terms are of higher powers of s^2 together with some powers of $\ln s$.

Now we can give the Feynman rules for the vertices, which are summarized in Table I. In the table,

$$v_{2n}^{(m)} \equiv \frac{\partial^m}{\partial \delta^m} v_{2n}(\delta)\bigg|_{\delta=0} , \tag{2.23}$$

where

$$v_{2n}(\delta) = \frac{\lambda M^2 (2\delta+2)!(I/M^2)^\delta}{(\delta+1-n)! 2^{\delta+1-n} I^{n-1}} . \tag{2.24}$$

Using the expansion of I in Eq. (2.22), we can express the vertex functions $v_{2n}^{(m)}$ in a power series of s^2,

$$v_2^{(1)} = 2\lambda M^2 \left[\ln\frac{2I}{M^2} + \psi(3/2) + 1 \right] = 2\lambda M^2 L + \lambda M^2 s^2 P + \cdots , \tag{2.25}$$

$$v_2^{(2)} = 2\lambda M^2 \left\{ \left[\ln\frac{2I}{M^2} + \psi(3/2) + 1 \right]^2 + \psi'(3/2) - 1 \right\} = 2\lambda M^2 \left[L^2 + \psi'(3/2) - 1 \right] + 2\lambda M^2 s^2 L P + \cdots , \tag{2.26}$$

and, for $n \geq 2$,

$$v_{2n}^{(1)} = 2\lambda M^2 (-1)^n (n-2)! \left(\frac{2}{I}\right)^{n-1} = 2\lambda M^2 (-1)^n (n-2)! (8\pi^2 s^2)^{n-1} \left[1 - \left(\frac{n-1}{2}\right) s^2 P + \cdots \right] , \tag{2.27}$$

$$\begin{aligned} v_{2n}^{(2)} &= 4\lambda M^2 (-1)^n (n-2)! \left(\frac{2}{I}\right)^{n-1} \left\{ \left[\ln\frac{2I}{M^2} + \psi(3/2) + 1 \right] + \psi(1) - \psi(n-1) \right\} \\ &= 4\lambda M^2 (-1)^n (n-2)! (8\pi^2 s^2)^{n-1} ([L + \psi(1) - \psi(n-1)] \\ &\qquad + \frac{1}{2} s^2 \{ -(n-1)[L + \psi(1) - \psi(n-1)] + 1 \} P + \cdots) , \end{aligned} \tag{2.28}$$

where

$$L = -\ln 2\pi^2 M^2 s^2 + \psi(3/2) + 1 , \tag{2.29}$$

$$P = \left[m_0^2 + \left(\xi_0 - \frac{1}{6} \right) R \right] \left[\ln\frac{1}{2} m_0 s - \psi(1) \right] - \frac{1}{2} m_0^2 + \frac{1}{24} R . \tag{2.30}$$

Now we have the Feynman rules for the vertices and the propagator. Together with the usual symmetry counting, various Feynman diagrams can thus be evaluated. We shall do that in the next section to calculate the connected Green's functions in the δ expansion for the self-interacting scalar theory in Eq. (2.2).

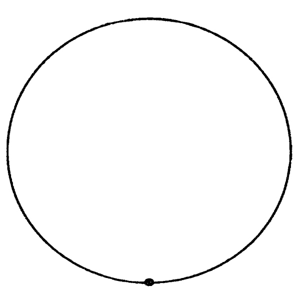
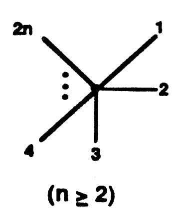


FIG. 1. One-loop diagrams.

III. RENORMALIZATION

Here we consider the renormalization of the δ expansion for the theory given by Eq. (2.2) up to δ^2. Since we are in general curved space-time, it is much simpler to stay in coordinate space. We choose the covariant coordinate, or point-splitting, regularization. In addition, we have to consider the connected Green's functions instead of the one-particle-irreducible ones. Higher-order behaviors of the Green's functions will be discussed in the next section.

TABLE I. Feynman rules for the vertices.

Vertex	Feynman rules
	$\delta A_1 \nabla_\mu \nabla^\mu$, $\delta^2 A_2 \nabla_\mu \nabla^\mu$, ... $-\delta m_1^2$, $-\delta^2 m_2^2$, ... $-\delta \xi_1 R$, $-\delta^2 \xi_2 R$, ... $-\delta v_2^{(1)} Z_0$, $-\delta^2 v_2^{(2)} Z_0/2$, ... $-\delta^2 v_2^{(1)} Z_1$, ...
 $(n \geq 2)$	$-\delta v_{2n}^{(1)} Z_0$, $-\delta^2 v_{2n}^{(1)} Z_0/2$, ... $-\delta^2 v_{2n}^{(1)} Z_1$, ...

A. Order δ

First we look at the 4-point connected Green's function $G_c^{(4)}$. At order δ, the only contribution comes from the Feynman diagram as shown in Fig. 2(a). Using the Feynman rules given in the last section and Table I, we have

$$G_c^{(4)}(x_1,x_2,x_3,x_4)$$
$$= \int d^4y\sqrt{g(y)}\prod_{i=1}^{4} G_0(x_i,y)\ (-\delta v_4^{(1)}Z_0). \quad (3.1)$$

As we can see from Eq. (2.27),

$$v_4^{(1)} \sim s^2, \quad (3.2)$$

where s is the geodesic distance, our regularization parameter. To render $G_c^{(4)}$ nonzero to order δ, we have to choose at least

$$Z_0 \sim \frac{1}{s^2}. \quad (3.3)$$

Here we specify our renormalization prescription by choosing

$$Z_0 = \frac{1}{16\pi^2M^2s^2}. \quad (3.4)$$

Then,

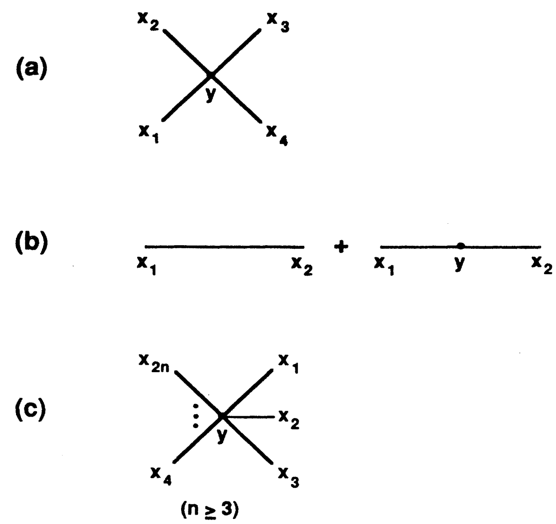


FIG. 2. Order-δ diagrams for (a) $G_c^{(4)}$, (b) $G_c^{(2)}$, and (c) $G_c^{(2n)}$, $n \geq 3$.

$$G_c^{(4)}(x_1,x_2,x_3,x_4) = -\delta\lambda\int d^4y\sqrt{g(y)}\prod_{i=1}^{4}G_0(x_i,y), \quad (3.5)$$

which is now finite.

Next we consider the 2-point Green's function. The relevant Feynman diagrams are shown in Fig. 2(b), which gives

$$G_c^{(2)}(x_1,x_2) = G_0(x_1,x_2) + \int d^4y\sqrt{g(y)}G_0(x_1,y)\left[\delta A_1\nabla_\mu\nabla^\mu - \delta m_1^2 - \delta\xi_1R(y) - \delta v_2^{(1)}Z_0\right]G_0(y,x_2). \quad (3.6)$$

From Eqs. (2.25) and (3.4), we see that

$$v_2^{(1)}Z_0 = \frac{\lambda}{8\pi^2s^2}L + \frac{\lambda}{16\pi^2}P, \quad (3.7)$$

where L and P are defined in Eqs. (2.29) and (2.30). Here we adopt an oversubtraction scheme and set

$$A_1 = 0, \quad (3.8)$$

$$m_1^2 = -\frac{\lambda}{8\pi^2s^2}L - \frac{\lambda m_0^2}{16\pi^2}\left(\ln\frac{1}{2}m_0s - \psi(1) - \frac{1}{2}\right), \quad (3.9)$$

$$\xi_1 = -\frac{\lambda}{16\pi^2}\left[\left(\xi_0 - \frac{1}{6}\right)\left(\ln\frac{1}{2}m_0s - \psi(1)\right) + \frac{1}{24}\right]. \quad (3.10)$$

The second term in Eq. (3.6) vanishes, and

$$G_c^{(2)}(x_1,x_2) = G_0(x_1,x_2). \quad (3.11)$$

For $n \geq 3$, Fig. 2(c) is the only Feynman diagram that contributes to the n-point Green's function. Since

$$v_{2n}^{(1)}Z_0 \sim (s^2)^{n-1}/s^2 \to 0 \quad (3.12)$$

as $s \to 0$, we have

$$G_c^{(2n)}(x_1,...,x_{2n}) = 0, \quad \text{for } n \geq 3. \quad (3.13)$$

B. Order δ^2

Now we consider the Green's functions to order δ^2. For the 4-point function, we have the Feynman diagrams shown in Fig. 3. Because of the oversubtraction scheme we have adopted, the contributions from Figs. 3(b) and 3(c) vanish. Using the Feynman rules in Sec. II and the usual symmetry counting, Fig. 3(d) gives

$$C_{3d} = \sum_{l=2}^{\infty} \frac{1}{(2l)!} \int d^4y \sqrt{g(y)} \int d^4z \sqrt{g(z)} \prod_{i=1}^{4} G_0(x_i, y) \{\delta^2 v_{2l+4}^{(1)} v_{2l}^{(1)} Z_0^2 [G_0(y,z)]^{2l}\}$$

$$= \sum_{l=2}^{\infty} \frac{1}{(2l)!} \int d^4y \sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y)\ \delta^2 v_{2l+4}^{(1)} v_{2l}^{(1)} Z_0^2 \mathcal{O}_{2l,0}(y), \quad (3.14)$$

where

$$\mathcal{O}_{n,p}(y) \equiv \int d^4x \sqrt{g(x)} \left[2\sigma(x,y)\right]^{p/2} \left[G_0(x,y)\right]^n, \quad (3.15)$$

and is evaluated in Appendix A. From Eqs. (2.27) and (3.4),

$$v_{2l+4}^{(1)} v_{2l}^{(1)} Z_0^2 \sim (s^2)^{2l-2}. \quad (3.16)$$

With the asymptotic behavior of $\mathcal{O}_{2l,0}$ as shown in Appendix A [Eq. (A7)], it is easy to see that, in the limit $s \to 0$, the surviving term in C_{3d} is finite and is equal to

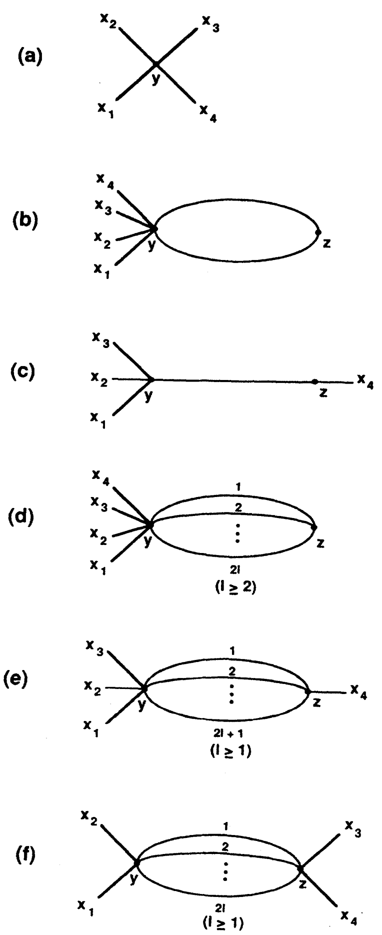


FIG. 3. Order-δ^2 diagrams for $G_c^{(4)}$. There are three other similar diagrams in (c), 3 in (e) and 2 in (f) with permutations of the x's.

$$C_{3d} = \int d^4y \sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y) \times \sum_{l=2}^{\infty} \frac{\delta^2 \lambda^2 \Gamma(l-1)}{128\pi^{3/2}(l-1)\Gamma(l+1/2)}. \quad (3.17)$$

The infinite sum above and its generalized form

$$f(a) \equiv \sum_{l=0}^{\infty} \frac{\Gamma(l+1)}{(l+a)\Gamma(l+3/2)} \quad (3.18)$$

are discussed in Appendix B. The values of different similar sums that we need in this section are tabulated in Table II. In the above equation

$$\sum_{l=2}^{\infty} \frac{\Gamma(l-1)}{(l-1)\Gamma(l+1/2)} = 2\left[f(1) - f(3/2)\right]$$

$$= \frac{2}{\sqrt{\pi}} \psi'(3/2). \quad (3.19)$$

Therefore,

$$C_{3d} = \int d^4y \sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y) \left[\frac{\delta^2 \lambda^2}{64\pi^2} \psi'(3/2)\right]. \quad (3.20)$$

Now the contributions from Figs. 3(e) and 3(f) (with $l \geq 2$) can be obtained similarly, giving

$$C_{3e} = \int d^4y \sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y) \left[-\frac{\delta^2 \lambda^2}{8\pi^2}(2G - 1)\right], \quad (3.21)$$

where $G \cong 0.9$ is the Catalan's constant, and

TABLE II. Specific values of the function defined in Appendix B and the corresponding derivatives. G is the Catalan's constant and $\psi'(x) \equiv d^2 \ln\Gamma/dx^2$.

x	$f(x)$	$f'(x)$
$\frac{1}{2}$	$8G/\sqrt{\pi}$	−5.07
1	$[\psi'(\frac{3}{2}) + 4]/\sqrt{\pi}$	−1.49
$\frac{3}{2}$	$4/\sqrt{\pi}$	−0.76
2	$[\psi'(\frac{3}{2}) + 6]/2\sqrt{\pi}$	−0.48
$\frac{5}{2}$	$28/9\sqrt{\pi}$	−0.34

$$C_{3f}(l \geq 2) = \int d^4y\sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y) \left[\frac{3\delta^2\lambda^2}{128\pi^2}\left(2 - \psi'(3/2)\right)\right]. \tag{3.22}$$

For $l = 1$, Fig. 3(f) gives

$$C_{3f}(l = 1) = \frac{1}{2}\int d^4y\sqrt{g(z)} G_0(x_1, y)G_0(x_2, y)G_0(x_3, z)G_0(x_4, z)\delta^2 v_4^{(1)} v_4^{(1)} Z_0^2 \mathcal{O}_{2,0}(y) + \cdots, \tag{3.23}$$

where the remaining terms come from two other similar diagrams with permutations of the x's. The vertex functions

$$v_4^{(1)} v_4^{(4)} Z_0^2 \sim s^0, \tag{3.24}$$

and this combination does not vanish as $s \to 0$. Thus to evaluate $C_{3f}(l = 1)$, we need to find both the divergent and the finite parts of $\mathcal{O}_{2,0}$. From Appendix A [Eq. (A9)],

$$\mathcal{O}_{2,0}(y) = \frac{1}{8\pi^2}[-\ln\tfrac{1}{2}m_0 s + B(y)], \tag{3.25}$$

where $B(y)$ is the finite part of $\mathcal{O}_{2,0}(y)$ and is space-time dependent in general. Note that with $l = 1$, Fig. 3(f) is exactly the same Feynman diagram that one will encounter in the $\lambda\phi^4$ theory.

To summarize, we have

$$C_{3d} + C_{3e} + C_{3f} = \int d^4y\sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y)\left(-\frac{\delta^2\lambda^2}{64\pi^2}\right)\left[\ln\frac{1}{2}m_0 s + \frac{1}{2}\psi'(3/2) + 16G - 11 - B(y)\right]. \tag{3.26}$$

This is of course divergent as $s \to 0$. The 4-point function can be rendered finite to order δ^2 by choosing an appropiate Z_1 in the remaining Feynman diagram in Fig. 3(a), where the vertex functions give

$$-\frac{\delta^2}{2}v_4^{(2)} Z_0 - \delta^2 v_4^{(1)} Z_1$$
$$= -\delta^2\lambda L - \delta^2\lambda(16\pi^2 M^2 s^2) Z_1\left[1 + O(s^2)\right]. \tag{3.27}$$

Hence, choosing

$$Z_1 = -\frac{1}{16\pi^2 M^2 s^2}\left[L + \frac{\lambda}{64\pi^2}\left(\ln\frac{1}{2}m_0 s + \frac{1}{2}\psi'(3/2) + 16G - 11\right)\right], \tag{3.28}$$

and with Eq. (3.26), we see that the renormalized 4-point function to order δ^2 is

$$G_c^{(4)}(x_1, x_2, x_3, x_4) = \int d^4y\sqrt{g(y)} \prod_{i=1}^{4} G_0(x_i, y) \times\left[-\delta\lambda + \frac{\delta^2\lambda^2}{64\pi^2}B(y)\right]. \tag{3.29}$$

The choice of Z_1 here is consistent with the oversubtraction scheme that we have adopted so far. Note again that this renormalized 4-point function is basically the same one in the $\lambda\phi^4$ theory.

The Feynman diagrams relevant to the connected 2-point function are shown in Fig. 4. The contributions from Figs. 4(b) and 4(c) vanish because of our oversubtraction scheme. Consider first the Feynman diagrams in Fig. 4(d),

$$C_{4d} = \sum_{l=2}^{\infty}\frac{1}{(2l)!}\int d^4y\sqrt{g(y)} G_0(x_1, y)G_0(x_2, y) \times\delta^2 v_{2l+2}^{(1)} v_{2l}^{(1)} Z_0^2 \mathcal{O}_{2l,0}(y). \tag{3.30}$$

Using Eqs. (2.27), (3.4), and (A7), and the value of $f(x)$ in Table II, we can evaluate the above equation following the procedure similar to that in calculating the 4-point function, and we have

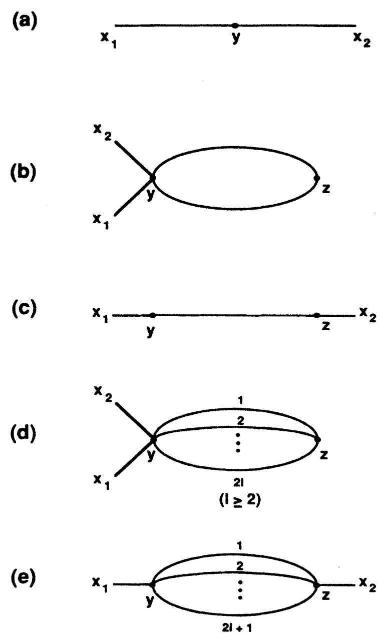


FIG. 4. Order-δ^2 diagrams for $G_c^{(2)}$.

$$C_{4d} = \int d^4y\sqrt{g(y)}G_0(x_1,y)G_0(x_2,y)\left(-\frac{\delta^2\lambda^2}{1024\pi^4}\right)$$
$$\times\left\{\frac{1}{s^2}\left[3\psi'(3/2)-2\right]+P\left[-\frac{1}{2}\psi'(3/2)+8G-5\right]\right.$$
$$\left.+\left[-\frac{\sqrt{\pi}}{4}f'(1/2)-2G+1\right]\left[m_0^2+\left(\xi_0-\frac{1}{6}\right)R\right]-\frac{1}{6}R\left[\frac{1}{6}\psi'(3/2)+\frac{4}{3}G-1\right]\right\}. \quad (3.31)$$

Next let us consider the Feynman diagram in Fig. 4(e). The contribution is

$$C_{4e} = \sum_{l=1}^{\infty}\frac{1}{(2l+1)!}\int d^4y\; G_0(x_1,y)G_0(x_2,z)\delta^2 v_{2l+2}^{(1)}v_{2l+2}^{(1)}Z_0^2\left[G_0(y,z)\right]^{2l+1}. \quad (3.32)$$

First, we covariantly Taylor expand $G_0(x_2,z)$,

$$G_0(z,x_2) = G_0(y,x_2) - \left[\nabla_\mu G_0(y,x_2)\right]\sigma^\mu(y,z) + \frac{1}{2}\left[\nabla_\mu\nabla_\nu G_0(y,x_2)\right]\sigma^\mu(y,z)\sigma^\nu(y,z)+\cdots, \quad (3.33)$$

in order to separate the divergent part of Eq. (3.32). Then,

$$C_{4e} = \sum_{l=1}^{\infty}\frac{1}{(2l+1)!}\int d^4y\; G_0(x_1,y)G_0(x_2,y)\delta^2 v_{2l+2}^{(1)}v_{2l+2}^{(1)}Z_0^2\mathcal{O}_{2l+1,0}$$
$$\times\sum_{l=1}^{\infty}\frac{1}{(2l+1)!}\int d^4y\; G_0(x_1,y)\frac{1}{8}\left[\nabla_\mu\nabla^\mu G_0(y,x_2)\right]\delta^2 v_{2l+2}^{(1)}v_{2l+2}^{(1)}Z_0^2\mathcal{O}_{2l+1,2}+\cdots. \quad (3.34)$$

Again for $l=1$, the combination of the vertex functions $v_{2l+2}^{(1)}v_{2l+2}^{(1)}Z_0^2$ does not vanish as $s\to 0$, and we have to treat this case separately. For $l\geq 2$, the surviving terms are

$$C_{4e}(l\geq 2) = \int d^4y\sqrt{g(y)}\; G_0(x_1,y)G_0(x_2,y)\left(-\frac{\delta^2\lambda^2}{256\pi^4}\right)$$
$$\times\left\{\frac{1}{s^2}\left[\psi'(3/2)-4G+\frac{8}{3}\right]+\left[\frac{\sqrt{\pi}}{8}f'(3/2)+\frac{1}{4}\psi'(3/2)-\frac{1}{6}\right]P\right.$$
$$\left.-\frac{3}{16}\left[m_0^2+\left(\xi_0-\frac{1}{6}\right)R\right]\left[\psi'(3/2)-\frac{2}{3}\right]-\frac{1}{48}R\left[\psi'(3/2)-\frac{10}{9}\right]\right\}$$
$$+\int d^4y\sqrt{g(y)}\; G_0(x_1,y)\left[\nabla_\mu\nabla^\mu G_0(y,x_2)\right]\left(-\frac{\delta^2\lambda^2}{4096\pi^4}\right)\left[\psi'(3/2)-\frac{10}{9}\right]. \quad (3.35)$$

For $l=1$,

$$C_{4e}(l=1) = \frac{1}{6}\int d^4y\sqrt{g(y)}G_0(x_1,y)G_0(x_2,y)\delta^2 v_4^{(1)}v_4^{(1)}Z_0^2\mathcal{O}_{3,0}$$
$$+\frac{1}{48}\int d^4y\sqrt{g(y)}G_0(x_1,y)\left[\nabla_\mu\nabla^\mu G_0(y,x_2)\right]\delta^2 v_4^{(1)}v_4^{(1)}Z_0^2\mathcal{O}_{3,2}+\cdots. \quad (3.36)$$

Only the first two terms above are divergent. From Eqs. (A8) and (A9),

$$\mathcal{O}_{3,0}(y) = \frac{1}{32\pi^4}\left\{\frac{1}{2s^2}-\frac{3}{4}\left[m_0^2+\left(\xi_0-\frac{1}{6}\right)R\right]\left(\ln\frac{1}{2}m_0 s\right)^2\right.$$
$$\left.+\left[\frac{3}{2}\psi(1)\left[m_0^2+\left(\xi_0-\frac{1}{6}\right)R\right]+\frac{3}{4}m_0^2-\frac{1}{48}R\right]\ln\frac{1}{2}m_0 s+E_1(y)\right\}, \quad (3.37)$$

and

$$\mathcal{O}_{3,2}(y) = \frac{1}{32\pi^4}\left[-\ln\tfrac{1}{2}m_0 s+E_2(y)\right], \quad (3.38)$$

where $E_1(y)$ and $E_2(y)$ are the finite parts of $\mathcal{O}_{3,0}(y)$ and $\mathcal{O}_{3,2}(y)$, respectively. Thus we have

$$C_{4e}(l=1) = \int d^4y\sqrt{g(y)}G_0(x_1,y)G_0(x_2,y)\left(\frac{\delta^2\lambda^2}{384\pi^4}\right)$$
$$\times\left\{\frac{1}{s^2} - \frac{3}{2}\left[m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right]\left(\ln\frac{1}{2}m_0 s\right)^2\right.$$
$$+\left[\left(m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right)\left(3\psi(1) - 1\right) + \frac{3}{2}m_0^2 - \frac{1}{24}R\right]\left(\ln\frac{1}{2}m_0 s\right)$$
$$\left.+\left[m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right]\psi(1) + \frac{1}{2}m_0^2 - \frac{1}{24}R + \frac{1}{2}E_1\right\}$$
$$+\int d^4y\sqrt{g(y)}G_0(x_1,y)\left[\nabla_\mu\nabla^\mu G_0(y,x_2)\right]\left(\frac{\delta^2\lambda^2}{1536\pi^4}\right)\left\{-\ln\frac{1}{2}m_0 s + E_2\right\} + E_3(x_1,x_2), \tag{3.39}$$

where E_3 is the finite contribution from the remaining terms in Eq. (3.36). Now we come to the last diagram for $G_c^{(2)}$ in Fig. 4(a):

$$C_{4a} = \int d^4y\sqrt{g(y)}G_0(x_1,y)\left(\delta^2 A_2\nabla_\mu\nabla^\mu - \delta^2 m_2^2 - \delta^2\xi_2 R - \frac{\delta^2}{2}v_2^{(2)}Z_0 - \delta^2 v_2^{(1)}Z_1\right)G_0(y,x_2)$$
$$= \int d^4y\sqrt{g(y)}G_0(x_1,y)\left(\delta^2 A_2\nabla_\mu\nabla^\mu - \delta^2 m_2^2 - \delta^2\xi_2 R\right)G_0(y,x_2)$$
$$+\int d^4y\sqrt{g(y)}G_0(x_1,y)G_0(x_2,y)\left(\frac{\delta^2}{16\pi^2}\right)\left\{\frac{\lambda}{s^2}\left[L^2 - \psi'(3/2) + 1\right]\right.$$
$$\left.+\frac{\lambda^2}{32\pi^2}\left(\frac{L}{s^2} + \frac{P}{2}\right)\left[\ln\frac{1}{2}m_0 s + \frac{1}{2}\psi'(3/2) + 16G - 11\right]\right\}. \tag{3.40}$$

Finally, $G_c^{(2)}(x_1,x_2)$ is given by the sum of the contributions in Eq. (3.40), (3.31), (3.39), and (3.35). To render it finite we choose counterterms

$$A_2 = \frac{\lambda^2}{1536\pi^4}\left[\ln\frac{1}{2}m_0 s + \frac{3}{8}\psi'(3/2) - \frac{5}{12}\right], \tag{3.41}$$

$$m_2^2 = \frac{1}{16\pi^2 s^2}\left\{\lambda\left[L^2 - \psi'(3/2) + 1\right] + \frac{\lambda^2 L}{32\pi^2}\left[\ln\frac{1}{2}m_0 s + \frac{1}{2}\psi'(3/2) + 16G - 11\right] + \frac{\lambda^2}{64\pi^2}\left[-7\psi'(3/2) + 16G - 6\right]\right\}$$
$$+\frac{\lambda^2 m_0^2}{1024\pi^4}\left\{-3\left(\ln\frac{1}{2}m_0 s\right)^2 + \left(\ln\frac{1}{2}m_0 s\right)\left[-\frac{\sqrt{\pi}}{2}f'(3/2) + 7\psi(1) + 8G - \frac{9}{2}\right]\right.$$
$$\left.+\psi(1)\left[\frac{\sqrt{\pi}}{2}f'(3/2) - 8G + 8\right] + \frac{\sqrt{\pi}}{4}f'(3/2) + \frac{\sqrt{\pi}}{4}f'(1/2) + \frac{3}{4}\psi'(3/2) - 2G + \frac{5}{2}\right\}, \tag{3.42}$$

$$\xi_2 = \frac{\lambda^2}{1024\pi^4}\left\{\left(\xi_0 - \frac{1}{6}\right)\left[-3\left(\ln\frac{1}{2}m_0 s\right)^2 + \left(\ln\frac{1}{2}m_0 s\right)\left(-\frac{\sqrt{\pi}}{2}\psi'(3/2) + 7\psi(1) + 8G - 8\right)\right.\right.$$
$$\left.+\psi(1)\left(\frac{\sqrt{\pi}}{2}f'(3/2) - 8G + 8\right) + \frac{\sqrt{\pi}}{4}f'(1/2) + \frac{3}{4}\psi'(3/2) + 2G - \frac{3}{2}\right]$$
$$\left.-\frac{5}{72}\left(\ln\frac{1}{2}m_0 s\right) - \frac{\sqrt{\pi}}{48}f'(3/2) + \frac{1}{9}\psi'(3/2) + \frac{5}{9}G - \frac{16}{27}\right\}. \tag{3.43}$$

Then,

$$G_c^{(2)}(x_1,x_2) = G_0(x_1,x_2) + \int d^4y\sqrt{g(y)}G_0(x_1,y)G_0(x_2,y)\left(\frac{\delta^2\lambda^2}{384\pi^4}E_1(y)\right)$$
$$+\int d^4y\sqrt{g(y)}G_0(x_1,y)\left[\nabla_\mu\nabla^\mu G_0(y,x_2)\right]\left(\frac{\delta^2\lambda^2}{1536\pi^4}E_2(y)\right) + E_3(x_1,x_2). \tag{3.44}$$

Note that E_1, E_2, and E_3 all come from Fig. 4(e) with $l=1$, and this is the same diagram one would encounter in the $\lambda\phi^4$ theory.

For $n \geq 3$, we need the Feynman diagrams in Fig. 5 to calculate the n-point Green's function. For Fig. 5(a),

$$C_{5a} = \int d^4y\sqrt{g(y)} \prod_{i=1}^{2n} G_0(x_i,y)\left(-\frac{\delta^2}{2}v_{2n}^{(2)}Z_0 - \delta^2 v_{2n}^{(1)}Z_1\right). \tag{3.45}$$

From Eqs. (2.28), (3.4), (2.27) and (3.28),

$$v_{2n}^{(2)}Z_0 \sim (s^2)^{n-2} \to 0, \tag{3.46}$$

$$v_{2n}^{(1)}Z_1 \sim (s^2)^{n-2}\ln s \to 0, \tag{3.47}$$

as $s \to 0$ for $n \geq 3$. Thus $C_{5a}=0$. For Fig. 5(b), with $l \geq 2$, a typical diagram gives

$$C_{5c} = \frac{1}{l!}\int d^4y\sqrt{g(y)}\int d^4z\sqrt{g(z)} \prod_{i=1}^{r} G_0(x_i,y) \prod_{j=r+1}^{2n} G_0(x_j,z)\left(\delta^2 v_{r+l}^{(1)}v_{2n-r+l}^{(1)}Z_0^2\left[G_0(y,z)\right]^l\right). \tag{3.48}$$

From the asymptotic behavior of $\mathcal{O}_{n,p}(x)$ in Appendix A, we see that, for $l=2$,

$$v_{r+l}^{(1)}v_{2n-r+l}^{(1)}Z_0^2\mathcal{O}_{l,0} \sim (s^2)^{n-2}\ln s \to 0, \tag{3.49}$$

and, for $l \geq 3$,

$$v_{r+l}^{(1)}v_{2n-r+l}^{(1)}Z_0^2\mathcal{O}_{l,0} \sim (s^2)^{n-2} \to 0, \tag{3.50}$$

as $s \to 0$. Thus $C_{5b}(l \geq 2) = 0$ also. Finally, for $l=1$ in Fig. 5(b), we have

$$C_{5b}(l=1) = \int d^4y\sqrt{g(y)}\int d^4z\sqrt{g(z)} \prod_{i=1}^{r} G_0(x_i,y) \prod_{j=r+1}^{2n} G_0(x_j,z)\left(\delta^2 v_{r+1}^{(1)}v_{2n-r+1}^{(1)}Z_0^2 G_0(y,z)\right), \tag{3.51}$$

and

$$v_{r+1}^{(1)}v_{2n-r+1}^{(1)}Z_0^2 \sim (s^2)^{n-3}, \tag{3.52}$$

which vanishes as $s \to 0$ except for $n=3$. That is, up to order δ^2,

$$G_c^{(2n)}(x_1,\ldots,x_{2n}) = 0, \tag{3.53}$$

for $n \geq 4$. For $G_c^{(6)}(x_1,\ldots,x_6)$, we can have $r=1$ or $r=3$ in Fig. 5(b) with $l=1$. However, in our oversubtraction scheme, the diagrams with $r=1$ vanish. Thus the only surviving diagrams are the ones with three external lines on each side, and

$$\begin{aligned} G_c^{(6)}(x_1,\ldots,x_6) &= \int d^4y\sqrt{g(y)}\int d^4z\sqrt{g(z)} \prod_{i=1}^{3} G_0(x_i,y) \prod_{j=4}^{6} G_0(x_j,z)\delta^2 v_4^{(1)}v_4^{(1)}Z_0^2 G_0(y,z) + \cdots \\ &= \int d^4y\sqrt{g(y)}\int d^4z\sqrt{g(z)} \prod_{i=1}^{3} G_0(x_i,y) \prod_{j=4}^{6} G_0(x_j,z)G_0(y,z)\left(\delta^2\lambda^2\right) + \cdots, \end{aligned} \tag{3.54}$$

where the ellipsis represents nine other similar diagrams with permutations of the x's. Again all these diagrams are exactly the ones we have in the $\lambda\phi^4$ theory. This concludes our calculation of the n-point connected Green's functions up to order δ^2.

IV. CONCLUSIONS

In the last section we have obtained the renormalized connected n-point Green's functions in the δ expansion up to order δ^2. We stay in coordinate space throughout the calculation and adopt the covariant point-splitting regularization in which the geodesic distance s is used as the regularization parameter. There is no apparent

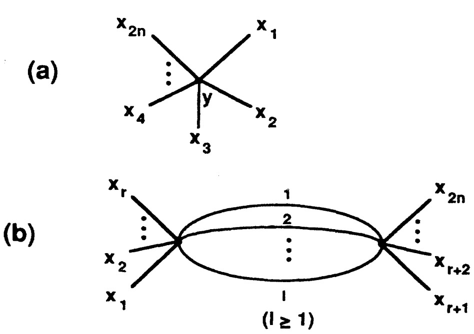


FIG. 5. Order-δ^2 diagrams for $G_c^{(2n)}$, $n \geq 3$. There are again similar diagrams with permutations of the x's in (b).

obstacle to carry out this renormalization procedure to higher orders in δ. This result leads us to believe that the δ expansion is renormalizable for a general space-time, at least for the self-interacting scalar theory discussed here. This extends the result obtained in Ref. [3] where flat space-time is considered.

It is curious to see [for example, from Eqs. (3.29) and (3.44)] that the contributions to the renormalized Green's functions come from diagrams which are the same ones present in the $\lambda\phi^4$ theory. It seems likely that for higher orders in δ, the same pattern will be followed. That is, solely those diagrams with only four-particle interactions will contribute to the connected n-point Green's functions. Therefore the class of theories with $\delta < 1$ all degenerate to the $\lambda\phi^4$ theory once the cutoff is removed upon renormalization. A similar conclusion has also been reached in the flat-space-time case.

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APPENDIX A

In this appendix we would like to give the asymptotic behavior of an integral which we come across frequently in Sec. III. This integral,

$$\mathcal{O}_{n,p}(x) \equiv \int d^4y \sqrt{g(y)}\,[2\sigma(x,y)]^{p/2}[G_0(x,y)]^n, \qquad \text{(A1)}$$

comes from Feynman diagrams such as the one in Fig. 3(e). To extract the ultraviolet-divergent part of this integral, we resort to using the Riemann normal coordinates around x. To simplify the notation we shall call this particular coordinate y again. Then we can expand the relevant quantities in a power series of the Riemann tensor:

$$\sigma^\mu(x,y) \to y^\mu, \qquad \text{(A2)}$$

$$\sqrt{g(y)} \to 1 - \frac{1}{6}R_{\mu\nu}y^\mu y^\nu + \cdots, \qquad \text{(A3)}$$

$$[2\sigma(x,y)]^{p/2} \to y^p, \qquad \text{(A4)}$$

$$\Big[G_0(x,y)\Big]^n \to (4\pi^2y^2)^{-n}\left\{1 + \frac{n}{2}y^2\left[\left(m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right)\left(\ln\frac{1}{2}m_0y - \psi(1)\right) - \frac{1}{2}m_0^2 + \frac{1}{6}R_{\mu\nu}\frac{y^\mu y^\nu}{y^2}\right] + \cdots\right\}, \qquad \text{(A5)}$$

where $y = (y^\mu y_\mu)^{1/2}$. Making these replacements in Eq. (A1), we have

$$\mathcal{O}_{n,p} = \frac{2\pi^2}{(4\pi^2)^2}\int_0^\infty dy\; y^{3+p-2n}\left\{1 + \frac{n}{2}\left[m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right]y^2\ln\frac{1}{2}m_0y\right.$$
$$\left. + \frac{1}{24}y^2R\left(\frac{n}{2} - 1\right) + \frac{n}{2}y^2\left[-m_0^2\left(\psi(1) + \frac{1}{2}\right) - \left(\xi_0 - \frac{1}{6}\right)\psi(1)R\right] + \cdots\right\}. \qquad \text{(A6)}$$

We have to consider this integral for three separate cases. For $n \geq 4 + p/2$,

$$\mathcal{O}_{n,p} = \frac{2\pi^2}{(4\pi^2)^n}\left(\frac{s^{4+p-2n}}{2n-4-p}\right)\left\{1 + s^2\left[\frac{n(2n-4-p)}{2(2n-6-p)}\right]\left[\left(m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right)\left(\ln\frac{1}{2}m_0s - \psi(1) + \frac{1}{2n-6-p}\right)\right.\right.$$
$$\left.\left. - \frac{1}{2}m_0^2 + \left(\frac{n-2}{24n}\right)R\right] + \cdots\right\}, \qquad \text{(A7)}$$

where we have shown only the two most ultraviolet-divergent terms. s is the geodesic distance which is used as the regularization parameter here. For $n = 3 + p/2$,

$$\mathcal{O}_{n,2n-6} = \frac{2\pi^2}{(4\pi^2)^n}\left\{\frac{1}{2s^2} - \frac{n}{4}\left(m_0^2 + \left(\xi_0 - \frac{1}{6}\right)\right)\left(\ln\frac{1}{2}m_0s\right)^2\right.$$
$$\left. + \left[\frac{n}{2}\psi(1)\left(m_0^2 + \left(\xi_0 - \frac{1}{6}\right)R\right) + \frac{n}{4}m_0^2 + \frac{2-n}{48}R\right]\ln\frac{1}{2}m_0s + \cdots\right\}. \qquad \text{(A8)}$$

The remaining terms are finite as $s \to 0$. Finally, for $n = 2 + p/2$,

$$\mathcal{O}_{n,2n-4} = \frac{2\pi^2}{(4\pi^2)^n}\left(-\ln\frac{1}{2}m_0 s + \cdots\right), \tag{A9}$$

where we have again only shown the divergent part of the integral.

APPENDIX B

In this appendix we evaluate the sums

$$f(a) \equiv \sum_{l=0}^{\infty} \frac{\Gamma(l+1)}{(l+a)\Gamma(l+3/2)} \tag{B1}$$

and

$$f'(a) \equiv -\sum_{l=0}^{\infty} \frac{\Gamma(l+1)}{(l+a)^2\Gamma(l+3/2)}. \tag{B2}$$

Using the integral representations of both the ratio of the Γ functions and $1/(l+a)$ in $f(a)$, we get

$$\begin{aligned} f(a) &= \sum_{l=0}^{\infty} \frac{1}{\Gamma(1/2)} \int_0^1 dt\, t^l (1-t)^{-1/2} \int_0^1 d\lambda\, \lambda^{l+a-1} \\ &= \frac{1}{\Gamma(3/2)} \int_0^1 d\lambda\, \lambda^{a-1}\, F(1,1;3/2;\lambda), \end{aligned} \tag{B3}$$

where $F(a,b;c;z)$ is the hypergeometric function. In arriving at the above expression, we have made use of the integral representation of $F(a,b;c;z)$,

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1}(1-t)^{c-b-1} \times (1-tz)^{-a}, \tag{B4}$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|\arg(1-z)| < \pi$. Now using the fact that

$$F(1,1;\tfrac{3}{2};\sin^2 z) = \frac{z}{\sin z \cos z}, \tag{B5}$$

we have

$$f(a) = \frac{4}{\sqrt{\pi}} \int_0^{\pi/2} dz\; z(\sin z)^{2a-2}$$

and

$$f'(a) = \frac{8}{\sqrt{\pi}} \int_0^{\pi/2} dz\; z(\sin z)^{2a-2}\ln(\sin z);$$

$f(a)$ can be evaluated easily for $a = \frac{1}{2}, 1, \frac{3}{2}, \ldots$, whereas the corresponding $f'(a)$ has to be solved numerically. The values of f and f' that we need in Sec. III are listed in Table II.

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