Tilted Universe and other remnants of the preinflationary Universe

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If the inflationary epoch lasted only ~ 10 *e*-foldings longer than required to solve the horizon problem, observable "remnants" of the preinflationary Universe may exist. They include dipole and quadrupole anisotropies of the cosmic microwave background radiation (CMBR). These "remnants" arise due to preinflationary fluctuations in scalar fields such as the inflaton, the axion, and the ilion, or due to preinflationary density perturbations. The dipole anisotropy can lead to the illusion of a "tilted universe": Viewed from the rest frame of the CMBR galaxies throughout the entire observable Universe would have a uniform streaming velocity. A dipole CMBR anisotropy could provide a very unconventional explanation for the large peculiar velocities measured for galaxies in our $\sim 50h^{-1}$ Mpc neighborhood. Among things very unlikely to be a "remnant" of inflation is a value of Ω today that is significantly different from unity.

I. INTRODUCTION

Inflation is very appealing because it makes the current state of the Universe relatively insensitive to its initial state and provides a "blueprint" (baryon asymmetry, density perturbations, total matter content) for the subsequent evolution of the Universe [1]. Inflation accomplishes the former by "inflating" a small, smooth patch of the preinflationary Universe to a size that encompasses all that we see today. To make this point more concrete, consider the *current* size of a preinflationary Hubble-sized patch:

$$d_{\rm patch}=\left[\frac{R_0}{R_{\rm start}}\right]H_I^{-1}\simeq e^N\left[\frac{M}{T_{\rm RH}}\right]^{4/3}\left[\frac{T_{\rm RH}}{3K}\right]H_I^{-1}$$
$$\equiv \exp(N-N_{\rm min})H_0^{-1}\;; \qquad (1a)$$

where $H_I=\sqrt{8\pi M^4/3m_{\rm Pl}^2}$ is the Hubble constant during inflation, $H_0\simeq(10^{28}h^{-1}\ {\rm cm})^{-1}$ is the present value of the Hubble parameter, the Planck mass $m_{\rm Pl}=1.22\times10^{19}$ GeV, $T_{\rm RH}$ is the temperature to which the Universe is reheated after inflation, and M^4 is the vacuum energy that "drives" inflation. (In slow-rollover inflation the inflationary era is usually followed by a matter-dominated epoch, where the energy density is dominated by the coherent oscillations of the inflaton field, and then by the usual radiation-dominated epoch; thus $T_{\rm RH}^4<M^4$. In inflationary models where reheating takes place through bubble nucleation and collisions, e.g., extended inflation, there is no matter-dominated era just after inflation and $T_{\rm RH}=M$.)

The value of the cosmic-scale factor $R(t)$ is $R_{\rm start}$ at the beginning of inflation, $R_{\rm end}$ at the end of inflation, and $R_0=1$ today, and $N\equiv\ln(R_{\rm end}/R_{\rm start})$ is the number of *e*-foldings of the scale factor during inflation. If N exceeds

$$N_{\rm min}=53+2\ln(M/10^{14}\ {\rm GeV})/3+\ln(T_{\rm RH}/10^{10}\ {\rm GeV})/3\;,$$

then today the preinflationary patch is large enough to encompass the current Hubble volume, thereby solving the so-called "horizon problem [1]." (Note, we have taken the smooth, preinflationary patch to be Hubble sized because this is the largest size region that could have become smooth due to causal physical processes; for more discussion see the Appendix.)

A similar amount of growth in the cosmic-scale factor is required to solve the "flatness" problem: namely, the fact that the value of Ω today ($\equiv\Omega_0$) is still of order unity; or equivalently, that the curvature radius is comparable to the Hubble radius. The curvature radius of a Friedmann-Robertson-Walker (FRW) model can be related to either the three-curvature k or the ratio of the total energy density to the critical density ($\equiv\Omega$):

$$R_{\rm curv}\equiv R(t)|k|^{-1/2}=\frac{H^{-1}}{|\Omega-1|^{1/2}}\;,$$

where $\Omega=8\pi G\rho_{\rm tot}/3H^2$, and $\rho_{\rm tot}$ includes *all* forms of energy density. The radius of curvature of the Universe today is equal to that at the beginning of inflation times $R_0/R_{\rm start}$. Using this, the relationship between $R_{\rm curv}$ and Ω, and Eq. (1a), we can relate Ω_0 to $\Omega_{\rm start}$:

$$|\Omega_0-1|=\left[\frac{H_0^{-1}}{d_{\rm patch}}\right]^2|\Omega_{\rm start}-1|$$
$$=\exp(2N_{\rm min}-2N)|\Omega_{\rm start}-1|\;. \qquad (1b)$$

The essence of Eq. (1b) is clear: Unless the size of the smooth, preinflationary region is much larger than H_I^{-1}, the amount of inflation required to solve the flatness problem is *comparable to or less than* that required to solve the horizon problem. (See the Appendix for details concerning the kinematics of the transition to the inflationary epoch.)

Provided that the amount of inflation exceeds that required to solve the horizon/flatness problems, i.e., $N > N_{\min}$, a scale that had a preinflationary size larger than H_I^{-1} will today still be outside the horizon. [When we refer to a scale crossing inside (or outside) the horizon, we mean more precisely that its physical size becomes less than (or greater than) the Hubble radius H^{-1}. In the standard cosmology, the distance to the particle horizon, $d_H \equiv R(t)\int_0^t dt'/R(t')$, is up to factors of order unity equal to the Hubble radius. Of course, this is not true in inflationary universes, and in fact d_H is much greater than H^{-1}.] In particular, if the scale was of physical size $l \equiv e^p H_I^{-1}$ at the beginning of inflation then its size today is $L \equiv R_0 l / R_{\text{start}} = e^P H_0^{-1}$, where $P = p + N - N_{\min}$. The content of the kinematic relationship $P = p + N - N_{\min}$ is perhaps more clearly expressed in words: A scale that was a factor of e^p larger than the Hubble radius at the beginning of inflation is today a factor of e^P larger than the current Hubble radius, where the difference between P and p is simply the logarithm of the size of our inflationary patch divided by the current Hubble radius. If the duration of inflation exceeds the minimum amount required to solve the horizon and/or flatness problems by a large amount, that is $N \gg N_{\min} \sim 50$, then all superhorizon-sized scales at the beginning of inflation are "exponentially superhorizon sized" today. *On the other hand, if the duration of inflation does not exceed the minimum by a large amount, then scales that were superhorizon sized at the onset of inflation are not exponentially far outside the horizon today. It is this possibility that interests us here.*

Scales that were superhorizon sized at the onset of inflation are the ones that we will be concerned with in this paper. These scales cannot be affected by events during inflation or the postinflationary epoch, and thus contain information about the preinflationary Universe. We will show that provided $N \lesssim N_{\min} + O(10)$, fluctuations on such scales can lead to observable consequences: quadrupole and dipole anisotropies of the cosmic microwave background radiation (CMBR). The dipole anisotropy can create the illusion that the Universe is tilted: If an intrinsic dipole anisotropy exists, the rest frame defined by the CMBR does not coincide with the cosmic rest frame, and viewed from the CMBR rest frame all the matter in the Universe will be seen to be moving with a uniform velocity. Such an effect *could* be the explanation for the large peculiar motions (relative to the CMBR) measured for almost 1000 galaxies in our $50h^{-1}$ Mpc neighborhood. Consideration of preinflationary inhomogeneity on superhorizon-sized scales also provides the basis for a strong argument that the value of Ω must be very close to unity in an inflationary Universe.

Before we begin, it is only fair to warn the reader that in many, if not most, models of inflation $N \gg N_{\min} \sim 50$, in which case the issues discussed here are moot. However, there are models of inflation in which N can be ~ 50 [2]; moreover, the question of whether or not the Universe even inflated, let alone the details of inflation, has yet to be answered. Thus, we feel justified in considering the possibility that the amount of inflation is not too different from that required to solve the horizon and flatness problems.

II. PRELIMINARIES: SCALAR-FIELD FLUCTUATIONS

The behavior of curvature fluctuations that are superhorizon sized at the onset of inflation is well known: Ordinary (scalar) density perturbations enter the horizon in the postinflationary epoch with the same amplitude that they would have in the absence of inflation, albeit at a time well after the present epoch [3]; the same holds true for anisotropic curvature perturbations (that is, growing modes of anisotropy) [4]. Inflation does not solve the woes of an anisotropic or inhomogeneous universe permanently; it merely postpones the epoch that we become aware of the inhomogeneity and anisotropy.

Inhomogeneities can also arise due to fluctuations in various scalar fields including the scalar field responsible for inflation, often referred to as the *inflaton,* the (complex) scalar field responsible for Peccei-Quinn symmetry breaking (whose phase is the axion field), and the "ilion" field, the field responsible for producing the baryon asymmetry in an unconventional and interesting model of baryogenesis [5]. We will use ϕ to denote the scalar field whose fluctuations we are considering at the time.

On length scales greater than the preinflationary Hubble radius there is no reason to expect ϕ to be homogeneous. As a simple ansatz for the spatial configuration of ϕ at the start of inflation, we take the mean value of ϕ to be ϕ_0 and consider one superhorizon-sized fluctuation mode:

$$\phi_{\text{start}}(\mathbf{r}) \equiv \phi_0 + \delta_0(\mathbf{r}) = \phi_0 + \frac{\delta\phi_k e^{-i\mathbf{k}\cdot\mathbf{r}}}{(2\pi)^3} . \tag{2}$$

Here $\delta\phi_k$ is the amplitude of the superhorizon-sized fluctuation, $l = R_{\text{start}}/|\mathbf{k}| = e^p H_I^{-1}$ is the physical size of the fluctuation at the start of inflation, which is greater than H_I^{-1}, and $\mathbf{r}$ are comoving coordinates. Since we have normalized the cosmic-scale factor so that its value today is unity, the current length scale of this fluctuation, $L \equiv |\mathbf{k}|^{-1}$, is related to the present horizon scale by $L = e^P H_0^{-1}$ where as before $P = p + N - N_{\min}$. Provided that the scalar field ϕ is minimally coupled (vanishing coupling to the scalar curvature $\mathcal{R}$) and any potential term for ϕ is unimportant ($V'' \ll H_I^2$), both are true for the examples of interest, the evolution of the scalar-field fluctuation while the scale L is outside the horizon ($k/RH \ll 1$) is extremely simple: $\delta\phi_k = \text{const}$ [6].

While we have chosen a particularly simple and specific form for the preinflationary fluctuation, any superhorizon-sized, preinflationary fluctuation in ϕ can be expanded in a Fourier integral involving only modes that are superhorizon sized; by writing $\delta\phi = \delta_k e^{-i\mathbf{k}\cdot\mathbf{r}}/(2\pi)^3$ we have made it convenient to do so.

III. CURVATURE PERTURBATIONS

Inflaton fluctuations

To begin, consider the case where the scalar field ϕ is the field responsible for inflation. For example, in the chaotic inflation model [7] the inflation is a very weakly coupled scalar field with scalar potential $V(\phi)=\lambda\phi^4$ ($\lambda\sim 10^{-15}$). The number of e-foldings of inflation is $N=\pi(\phi_0/m_{\rm Pl})^2-1/2$, and for $\phi_0\sim 4m_{\rm Pl}$, $N\sim 50$. As is well appreciated fluctuations in the inflaton field eventually lead to curvature fluctuations [8]. On length scales that are subhorizon sized at the onset of inflation fluctuations in ϕ arise as de Sitter (zero-point) quantum fluctuations, and their amplitude as they cross outside the horizon is $\Delta\phi=k^{3/2}|\delta\phi_k|/\sqrt{2\pi^2}=H_I/2\pi$. These quantum fluctuations lead to curvature fluctuations that cross back inside the horizon in the postinflationary Universe with amplitude $(\Delta\rho/\rho)_{\rm hor}\equiv k^{3/2}|\delta_k|/\sqrt{2\pi^2}\simeq 10^{-5}(\lambda/10^{-15})^{1/2}$, where $\delta\rho(\mathbf{r})/\rho=\int\delta_k e^{-i\mathbf{k}\cdot\mathbf{r}}d^3k/(2\pi)^3$. The value $(\Delta\rho/\rho)_{\rm hor}\simeq 10^{-5}$ is both consistent with the isotropy of the CMBR and suitable for structure formation [9].

The amplitudes of fluctuations that are superhorizon sizes at the beginning of inflation have nothing to do with de Sitter quantum fluctuations; rather, they reflect the initial configuration of the ϕ field, and thereby the preinflationary state of the Universe. We can estimate the horizon-crossing amplitude of the density perturbation produced by such a fluctuation on the scale L by a simple scaling argument: Since $(\delta\rho/\rho)_{\rm hor}\propto\Delta\phi$,

$$\left[\frac{\delta\rho}{\rho}\right]_{{\rm hor},L}\sim\left[\frac{\delta\phi}{H/2\pi}\right]10^{-5}\sim 30\frac{\delta\phi_k/(2\pi)^3}{\phi_0}\ ;$$

where subscript "hor,L" refers to the epoch when the scale L crosses back inside the horizon and the final expression is specific to the chaotic inflation model. Provided the Universe remains matter dominated until that epoch, $T_{{\rm hor},L}\sim 2.7$ K $(H_0^{-1}/L)^2$ and $t_{{\rm hor},L}\sim 10^{10}$ yr $(L/H_0^{-1})^3$. [If the initial fluctuation in ϕ is of order unity, that is $\delta\phi_k/(2\pi)^3\sim\phi_0$, the resulting curvature perturbations enter the horizon with amplitudes greater than unity and cannot be treated as *small perturbations*.]

CMBR anisotropy: The Sachs-Wolfe effect

Superhorizon-sized fluctuations, like their subhorizon-sized counterparts, can affect the isotropy of the CMBR [10]. To calculate the amplitude of the temperature anisotropies that arise we use the formalism developed by Sachs and Wolfe [11]. The relative deviation of the CMBR temperature seen in the direction $\hat{\mathbf{n}}$ by an observer today at position $\mathbf{r}=0$ is

$$\frac{\delta T(\mathbf{r}=\mathbf{0};\hat{\mathbf{n}})}{T}=\frac{1}{2}\left[\sqrt{R}\,(\mathbf{x}\cdot\nabla)\left[\frac{R^2H^2}{k^2}\frac{\delta\rho}{\rho}\right]+\left[\frac{R^2H^2}{k^2}\frac{\delta\rho}{\rho}\right]\right]\Bigg|_E^R\ , \qquad (3)$$

where R denotes the reception event at our spatial position ($\mathbf{r}=\mathbf{0}$), E denotes the emission event (last scattering), scale factor $R_E\simeq(1+z_{\rm dec})^{-1}\simeq 10^{-3}$ and spatial position $\mathbf{r}_e=(1-\sqrt{R_E})\mathbf{x}$ ($\mathbf{x}=2H_0^{-1}\hat{\mathbf{n}}$)—and $\delta\rho(\mathbf{r})/\rho=(2\pi)^{-3}\delta_k(t)\exp(-i\mathbf{k}\cdot\mathbf{r})|_{\mathbf{r}=0}$ is evaluated at our position (by integrating over wave number $\mathbf{k}$ this expression can be generalized to any density field).

The quantity $\delta\rho/\rho$ is not gauge invariant and here is to be computed in synchronous gauge. During the matter-dominated epoch $\delta\rho/\rho\propto R(t)$. Since $R^2H^2\propto R^{-1}$, the quantity $(R^2H^2/k^2)(\delta\rho/\rho)$ is *time independent.* Moreover, if we define the horizon-crossing epoch ($RL\sim H^{-1}$) to be precisely when $k/HR=1$, this time-independent quantity is just equal to the value of $(\delta\rho/\rho)$ at horizon crossing, $(\delta\rho/\rho)_{\rm hor}$, a fact which will prove useful since $(\delta\rho/\rho)_{\rm hor}$ is the quantity most easily specified in inflation models. (For reference, for the modes whose amplitude is determined by de Sitter quantum fluctuations, $(\Delta\rho/\rho)_{\rm hor}\simeq k^{3/2}|\delta_k|_{\rm hor}/\sqrt{2\pi^2}\simeq 2H_I^2/\dot\phi\simeq$ const [12].)

Before going on, we wish to remind the reader of three assumptions underlying the analysis of Sachs and Wolfe [11]: (i) flat universe; (ii) matter domination at the epoch of decoupling and after; and (iii) pure curvature-mode perturbations. Given that our interest is inflationary cosmology, assumption (i) is quite appropriate. Since we may wish to consider a universe that at present is not matter dominated, we may wish to relax assumption (ii). And of course, we do plan to discuss isocurvature perturbations, so we will certainly relax assumption (iii).

The two terms in Eq. (3) that contribute to the CMBR anisotropy have simple physical interpretations: (1) Owing to the gradient operator, the first term leads to a dipole anisotropy about the direction $\hat{\mathbf{k}}$; this dipole anisotropy arises due to the relative peculiar motion between the observer and the last-scattering surface. Peculiar velocities come about because of the inhomogeneous distribution of matter, and in the linear regime the Fourier expansion of the peculiar-velocity field is related to that of the density field, $\delta\mathbf{v}_{\mathbf{k}}=-i\hat{\mathbf{k}}(RH/k)\delta_{\mathbf{k}}$. Thus we see that the first term corresponds to a Doppler shift caused by the velocity of the observer relative to the last-scattering surface. (2) From a Newtonian perspective, the second term in the Sachs-Wolfe formula corresponds to the gravitational-potential difference between the last-scattering surface and the observer [13].

Subhorizon-sized modes

While here we are not interested in subhorizon-sized modes, as a warm up let us review quickly the CMBR anisotropies that arise due to these modes. Since the dipole anisotropy cannot be distinguished from the effect of our own peculiar motion, some of which arises due to large-scale modes that are still in the linear regime and most of which arises due to small-scale modes that are already nonlinear, it is useful to separate out the dipole anisotropy when discussing subhorizon-sized modes. Considering only the second term in Eq. (3) it follows that the relative temperature fluctuation seen in the direction $\hat{\mathbf{n}}$ by an observer at position $\mathbf{r}$ is

$$\frac{\delta T(\mathbf{r};\hat{\mathbf{n}})}{T}=-\frac{1}{2}\frac{1}{(2\pi)^3}\int\left[\frac{R^2H^2}{k^2}\right]\delta_k e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}\cdot\mathbf{r}}d^3k\ ; \qquad (4)$$

where here we have expanded the density field in a Fourier integral, $\delta\rho(\mathbf{r},t)/\rho=\int d^3k\,\delta_k(t)e^{-i\mathbf{k}\cdot\mathbf{r}}/(2\pi)^3$. Note that the time-independent quantity $R^2H^2\delta_k(t)/k^2$ is equal to its value at horizon crossing, so that the contributions of the various subhorizon-sized modes to the CMBR temperature fluctuation are of order $(\Delta\rho/\rho)_{\rm hor}$. For a realistic model of inflation $(\Delta\rho/\rho)_{\rm hor}\sim 10^{-5}$, and so temperature fluctuations of a similar amplitude are predicted.

If we expand $\delta T/T$ in spherical harmonics,

$$\frac{\delta T(\mathbf{r};\hat{\mathbf{n}})}{T}\equiv\sum_{lm}a_{lm}(\mathbf{r})Y_{lm}(\hat{\mathbf{n}})\ ,$$

and calculate the ensemble average of $|a_{lm}|^2$ over all observation positions $\mathbf{r}$ we obtain the standard result

$$\langle|a_{lm}|^2\rangle=\int\left[\frac{R^4H^4}{2\pi k^2}\right]|\delta_k|^2 j_l(kx)^2dk\ , \tag{5}$$

where $x=|\mathbf{x}|=2H_0^{-1}$, $k=|\mathbf{k}|$, and j_l is the spherical Bessel function of order l. This expression is valid for $l\geq 2$, and the Y_{lm} are the usual spherical harmonics, normalized such that $\int Y_{lm}Y_{pq}^*d\Omega=\delta_{lp}\delta_{mq}$.

Superhorizon-sized modes

Now let us move on to the effects of the preinflationary superhorizon-sized modes. The present wavelengths of these modes are larger than the Hubble radius, $L=e^PH_0^{-1}\gg H_0^{-1}$; or put another way $k/H_0R_0=kH_0^{-1}=e^{-P}\ll 1$. Because of this, the density perturbation seen within our present horizon takes the appearance of a linear density gradient in the direction $\hat{\mathbf{k}}$. One *might* expect this density gradient to lead to a peculiar velocity for *all* the matter within our horizon volume: The Universe is "tilted" (in a gravitational sense), so everything should slide from one side to the other.

A "tilted universe"?

Such a "tilting of the Universe" could provide an interesting and unconventional explanation for the peculiar-velocity field in our neighborhood: The peculiar-velocity measurements for the local volume $(50h^{-1}\ \mathrm{Mpc})^3$ made by the authors of Ref. [14] are consistent with a *uniform* bulk flow, relative to the CMBR, of about 700 km s^{-1} toward Hydra-Centaurus with a smaller, incoherent "noise" component, of about 200 km s^{-1} [15,16]. In the tilted universe interpretation the uniform flow would arise because of the linear density gradient associated with the superhorizon-sized mode, while the lesser noise component would be due to small-scale density inhomogeneities. (Of course, these observations are also consistent with more conventional explanations such as the existence of a "great attractor [17]," or even just the gravitational effects of the inhomogeneous distribution of galaxies within our local neighborhood [18].)

The peculiar velocity that arises due to the "tilting of the Universe" corresponds to the first term in Eq. (3):

$$\frac{\delta\mathbf{v}}{c}=\hat{\mathbf{k}}\left[\frac{k}{H_0}\right]\left[\frac{\delta\rho}{\rho}\right]_{{\rm hor},L}\ . \tag{6}$$

Since this peculiar velocity is uniform across our Hubble volume the question arises as to how one might infer its existence, or if indeed it has physical meaning. Since the CMBR is used to define the local frame of rest, a dipole temperature anisotropy, of amplitude given by Eq. (6), would be the physical manifestation. (In the previous case, where we were dealing with subhorizon-scale peculiar velocities, peculiar motions can be measured relative to other, nearby galaxies.) By use of the Sachs-Wolfe result, Eq. (3), we can compute $\delta T/T$ and look for a dipole $(l=1)$ term. For superhorizon-sized modes $\mathbf{k}\cdot\mathbf{x}$ is small, and we expand the $\exp(-i\mathbf{k}\cdot\mathbf{x})$ factor in $(\delta\rho/\rho)$:

$$\exp(-i\mathbf{k}\cdot\mathbf{x})=1-i\mathbf{k}\cdot\mathbf{x}-(\mathbf{k}\cdot\mathbf{x})^2/2!+\cdots\ .$$

In evaluating Eq. (3) the lowest-order term cancels (as expected); however, the order kx term in the expansion of the second term *cancels* exactly the gradient term (first term). *The dipole anisotropy associated with the tilting of the Universe is not observable because it is canceled by a corresponding dipole anisotropy from the potential term (at order kx).* Said another way, in spite of the existence of the density gradient associated with the superhorizon-sized curvature perturbation, the spatial hypersurfaces defined by the isotropy of the CMBR coincide with those defined by the isotropy of the expansion.

Quadrupole anisotropy: The Grishchuk-Zel'dovich effect

The lowest-order, nonvanishing temperature anisotropy is $O((\mathbf{k}\cdot\mathbf{x})^2)$ and quadrupole in form:

$$\frac{\delta T(\mathbf{r};\hat{\mathbf{n}})}{T}=\frac{-(\mathbf{k}\cdot\hat{\mathbf{n}})^2}{(2\pi)^3}\left[\frac{k^2}{H_0^2}\right]\left[\frac{k^2\delta_k}{R^2H^2}\right]e^{-i\mathbf{k}\cdot\mathbf{r}}+O(k^4/H_0^4)\ . \tag{7a}$$

Expanding $\delta T/T$ in spherical harmonics, we find

$$a_{2m}(\mathbf{r})=\frac{-1}{15\pi^2}\left[\frac{k^2}{H_0^2}\right]\left[\frac{k^2\delta_k}{R^2H^2}\right]Y_{2m}(\hat{\mathbf{k}})\ ; \tag{7b}$$

$$a_{00}(\mathbf{r})=\frac{-1}{6\pi^2}\left[\frac{k^2}{H_0^2}\right]\left[\frac{k^2\delta_k}{R^2H^2}\right]Y_{00}(\hat{\mathbf{k}})\ ; \tag{7c}$$

where we have used the addition theorem for spherical harmonics to express $(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})^2$ in terms of Y_{2m} and Y_{00}. Recall that $k^2\delta_k(t)/R^2H^2$ is independent of time and is equal to the $(\delta_k)_{\rm hor}$. This means that the lowest-order temperature anisotropy is $O(k^2/H_0^2)\sim O((H_0^{-1}/L^2))$ times the horizon-crossing amplitude of the density perturbation. By way of contrast, the temperature anisotropy that arises due to a subhorizon-sized mode is of the order of the horizon-crossing amplitude, cf. Eq. (4). Superhorizon-sized curvature modes can indeed affect the isotropy of the CMBR, but their effect is suppressed by a factor of $O((k/H_0)^2)$. This was first pointed out by Grishchuk and Zel'dovich [10].

The lowest-order, nonvanishing temperature anisotropy has both a monopole component and a quadrupole component. The origin of the monopole component is simple to understand: Because the density wave is superhorizon sized, unless our Hubble volume happens by chance to be located near the node of the density wave, there will be a net overdensity or underdensity "locally."

Taking the ensemble average over all observation positions $\mathbf{r}$,

$$\langle |a_{00}|^2\rangle = \frac{2}{9\pi}\int \left[\frac{k^4}{H_0^4}\right]|\delta_k|^2_{\rm hor}k^2dk\ ; \tag{8a}$$

$$\langle |a_{2m}|^2\rangle = \frac{8}{225\pi}\int \left[\frac{k^4}{H_0^4}\right]|\delta_k|^2_{\rm hor}k^2dk\ ; \tag{8b}$$

where we have integrated over d^3k and assumed that $|\delta_k|^2$ is independent of $\hat{\mathbf{k}}$ to simplify these expressions.

Since we do not have the means of carrying out the ensemble average, the monopole term cannot be distinguished from a small shift in the CMBR temperature in our Hubble volume, relative to its value averaged over a larger volume. On the other hand, the quadrupole anisotropy is observable, and current limits to the quadrupole anisotropy can be used to constrain the amplitude of superhorizon-sized fluctuations. The current upper limit to the quadrupole anisotropy [19], $|a_{2m}|\lesssim 3\times 10^{-5}$, constrains the horizon-crossing amplitude of superhorizon-sized modes,

$$\left[\frac{\Delta\rho}{\rho}\right]_{{\rm hor},L} \lesssim 3\times 10^{-5}\left[\frac{L}{H_0^{-1}}\right]^2 , \tag{9a}$$

and allows us to infer that the Universe is homogeneous ($\delta\rho/\rho\lesssim 1$) out to scales as large as 200 times the present Hubble scale.

Inflation and $\Omega_0\neq 1$

The sensitivity of the CMBR to superhorizon-sized scales provides the basis of a strong argument that inflation cannot accommodate $\Omega_0\neq 1$, and thus that $\Omega_0=1.0$ is a very robust inflationary prediction [20]. Achieving $\Omega_0\neq 1$ today requires $N\lesssim N_{\rm min}$, cf. Eqs. (1) and the Appendix. This fact implies that scales that were just outside the Hubble radius at the onset of inflation are today just larger than the Hubble radius ($P\lesssim p$); as a result, if $\Omega_0\neq 1$, the CMBR is very sensitive to the preinflationary state of the Universe. Perturbations on scales that were superhorizon-sized at the beginning of inflation have "every right" to be of significant size. However, if we require $\Omega_0\neq 1$ we can conclude that such perturbations must have been quite small, $(\Delta\rho/\rho)\lesssim 3\times 10^{-5}e^{2P}$; $(\Delta\rho/\rho)\lesssim 3\times 10^{-5}$ on the present-horizon scale and $(\Delta\rho/\rho)\lesssim 1$ on the scale $L\sim 200H_0^{-1}$. Since $N\lesssim N_{\rm min}$, the scale $L\sim 200H_0^{-1}$ was at least a factor of 200 larger than the horizon at the onset of inflation. *Thus in order to have Ω_0 comparable to, but not equal to, one, two conditions must be satisfied: (1) the amount of inflation, quantified by N, must be precisely equal to a number that is slightly less than $N_{\rm min}$; and (2) the inflationary patch in which we happen to find ourselves must have been very smooth on scales much larger than the Hubble radius ($\gtrsim 200H_I^{-1}$) at the onset of inflation.* While meeting these conditions is not logically impossible, it seems very contrary to the spirit of inflation. Finally, we note that our argument is kinematic in nature, and thus does not depend upon the details of inflation.

Probing the preinflationary Universe

We can also use isotropy of the CMBR to constrain the amplitudes of preinflationary fluctuations in the inflation field:

$$\delta\phi_k/(2\pi)^3 \lesssim 3\left[\frac{H}{2\pi}\right]\left[\frac{L}{H_0^{-1}}\right]^2 \lesssim 10^{-7}e^{2P}\phi_0\ , \tag{9b}$$

where the final expression applies to the chaotic inflation model. Recall that P is related to the number of e-foldings of inflation above that required to solve the horizon problem and the size of preinflationary fluctuation: $P=p+N-N_{\rm min}$, where $l=e^pH_I^{-1}$. It then follows that preinflationary modes characterized by $p\lesssim 8-(N-N_{\rm min})$ must have had amplitudes that were less than order unity.

Because of the cancellation that takes places between the gradient term in Eq. (3) and the potential term in Eq. (3) for superhorizon-sized curvature fluctuations, the "tilting of the Universe" is not observable. However, this cancellation depends crucially on the relationship between the first and second terms in Eq. (3), which in turn depends upon the three aforementioned assumptions made by Sachs and Wolfe [11]. In the next section we show that this cancellation does not occur for isocurvature fluctuations. In this case the "tilting of the Universe" is an observable effect.

Before we go on to consider isocurvature perturbations, let us mention a scenario where the tilting of the Universe is observable even for curvature perturbations. If the energy density of the Universe is today dominated by a component other than nonrelativistic matter, e.g., a cosmological constant or relativistic particles produced by the recent decay of a massive relic, then $\delta_k(t)$ has not simply increased as $R(t)$ since decoupling and the cancellation that renders the tilting of the Universe unobservable does not take place. A dipole anisotropy will result, even for curvature perturbations. Its amplitude will be proportional to $(1-\Omega_{\rm NR})$, where $\Omega_{\rm NR}$ is the fractional contribution of nonrelativistic particles to the critical density today.

Having the bulk of the present energy density in relativistic particles or a cosmological constant has been advocated by some to reconcile the flat Universe predicted by inflation with dynamical determinations of Ω_0 that indicate $\Omega_0\sim 0.1-0.3$ [21]. The energy density contributed by a "smooth" component of mass density would not show up in dynamical determinations of Ω_0, and thus in these scenarios $\Omega_{\rm NR}\sim 0.1-0.3$ and Ω_R (or Ω_Λ)$\sim 0.7-0.9$. For a relic cosmological constant, linear density perturbations cease growing at a redshift of about $1+z_\Lambda\sim(\Omega_{\rm NR}^{-1}-1)^{1/3}$, while for relativistic particles they cease growing at the decay epoch, $z_D\sim 2-5$.

IV. ISOCURVATURE FLUCTUATIONS

Isocurvature axion perturbations

Isocurvature perturbations arise in inflatonary models due to fluctuations in fields other than the inflaton, fields whose contribution to the energy density is subdominant. Unlike fluctuations in the inflaton field, fluctuations in these fields do not lead to significant perturbations in the energy density. A very simple and relevant example is provided by axions. We will begin by reviewing the cosmological production of axions and the origin of isocurvature axion perturbations. In the discussion that follows ϕ will refer to the axion (angular) degree of freedom. The field that breaks Peccei-Quinn (PQ) symmetry is a complex scalar field σ that acquires a nonzero vacuum expectation value: $\langle\sigma\rangle = f_a \exp(i\phi)/\sqrt{2}$; the axion field ϕ corresponds to the phase degree of freedom; see Refs. [22].

The primary cosmological production mechanism for axions is the misalignment of the axion field with the minimum of its potential [22]. The potential for the axion field, which arises due to instanton effects, is "flat" (i.e., vanishes) at high temperatures because instanton effects are suppressed for temperatures $T \gg 1$ GeV. At the epoch of Peccei-Quinn symmetry breaking ($T \sim f_a \sim 10^{13}$ GeV), when the initial value of the axion field is set, dynamics do not dictate the value of ϕ, and so ϕ takes on a random value which, in general, is misaligned with the minimum of the potential. When the potential does develop, the axion field "discovers" that it is misaligned with the minimum of its potential and begins to oscillate about the minimum. These oscillations correspond to nonrelativistic axions, with a number density proportional to the square of the initial misalignment angle.

Because the number density of axions is ultimately proportional to the misalignment angle squared, fluctuations in the initial misalignment of the axion field lead to fluctuations in the local axion-number density:

$$\frac{\delta n_a}{n_a} \simeq \frac{2\delta\phi}{\phi_0} , \tag{10}$$

where ϕ_0 denotes the average value of the misalignment angle within our inflationary patch. Since the energy density contained in the axion field during inflation is negligible these fluctuations do not lead to significant curvature ("true energy-density") fluctuations; thus they are referred to as "isocurvature" fluctuations.

While isocurvature fluctuations are superhorizon sized they are characterized by $\delta\rho = 0$ (in synchronous gauge). The reason is simple: Causality precludes transporting energy on scales larger than the horizon. Isocurvature perturbations correspond to spatial variations in the equation of state; in the present example, the fraction of the total energy density in axions varies from place to place. When an isocurvature perturbation becomes subhorizon sized, the initial perturbation in the equation of state develops into a density perturbation of similar size.

"Compensating" perturbations in the energy density of radiation must develop in order to maintain $\delta\rho = 0$ in the face $\delta\rho_a \neq 0$. (For simplicity we will assume a two-component universe, axions and radiation, and neglect the minor role played by baryons.) At early times when $\rho_a \ll \rho_R$ these fluctuations are very small (explaining why isocurvature perturbations used to be referred to as "isothermal" perturbations). As the Universe evolves, the ratio of axion-energy density to radiation-energy density increases, $\rho_a/\rho_R \propto R$, and the "compensating" fluctuations in the radiation-energy density become significant. The compensating fluctuations in the radiation lead to fluctuations in the radiation temperature beyond those that develop due to the metric perturbation which arise [and are described by Eq. (3)].

Once the Universe becomes axion dominated, isocurvature axion fluctuations lead to compensating temperature fluctuations of amplitude:

$$\delta_T(k) = -\tfrac{1}{3}\delta_A(k) , \tag{11}$$

for scales that are superhorizon sized ($k/HR \ll 1$). Here $\delta_T(k) \equiv (\delta T/T)_k$ and $\delta_A(k) \equiv (\delta n_a/n_a)_k = 2\delta\phi_k/\phi_0$. Equation (11) quantifies the crucial difference between isocurvature and curvature perturbations: For isocurvature fluctuations there is an extra perturbation to CMBR temperature. We refer the reader interested in a complete treatment of isocurvature perturbations to Efstathiou and Bond [23].

Our discussion of fluctuations in the axion misalignment angle parallels that of the inflaton field, beginning with a brief review of scales that were subhorizon sized at the start of inflation, and then going on to the scales of interest, those that were superhorizon sized. On the subhorizon-sized scales the axion-field fluctuations are those associated with de Sitter space quantum (zero-point) fluctuations, and $\Delta\phi/\phi_0 \simeq H_I/\pi\phi_0 f_a$, where f_a is the scale of Peccei-Quinn symmetry breaking. When these perturbations cross the horizon in the postinflation, matter-dominated epoch they lead to density perturbations of amplitude $(\Delta\rho/\rho)_{\rm hor} \simeq k^{3/2}|\delta_A(k)|/\sqrt{2\pi^2} \sim H_I/\phi_0 f_a$ that are independent of scale (like their curvature perturbation counterparts). Isocurvature axion fluctuations on scales that were smaller than H_I^{-1} at the onset of inflaton are treated in detail in Refs. [23] and [24].

Our interest is in isocurvature perturbations on scales that were larger than H_I^{-1} at the onset of inflation; fluctuations in the axion field on these scales are unaffected by inflation and reflect the initial configuration of the axion field. We describe the superhorizon-sized fluctuation in the axion misalignment as we did in the inflaton case, considering one superhorizon-sized fluctuation about the mean; cf. Eq. (2). The temperature anisotropy seen by observers today arises due to two effects. (1) The metric fluctuations associated with the density perturbation that develops from the initial isocurvature perturbation. The resulting temperature fluctuation due to this effect is described by Eq. (3). As we saw in the previous section the dipole component vanishes. The lowest-order effect is quadrupole and proportional to $(k/H_0)^2\delta_A$. (2) The intrinsic fluctuations in the radiation field that arise to compensate the axion-energy density fluctuation; cf. Eq. (11).

Compensating temperature fluctuations lead to a tilted universe

The important new twist is the additional temperature fluctuation associated with the compensating perturbations in the radiation: $\delta_T(k)=-\delta_A(k)/3$. They lead to a dipole anisotropy in the CMBR temperature that is uniform across our present Hubble volume and is of $O(k/H_0)$:

$$\frac{\delta t(\mathbf{0};\hat{\mathbf{n}})}{T}=\frac{-2(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}})}{3}\frac{k}{H_0}\frac{\delta\phi_k/(2\pi)^3}{\phi_0} . \tag{12a}$$

This dipole anisotropy is *intrinsic* to the CMBR, and any observer within our Hubble volume in the rest frame of the expansion will conclude that he is moving with respect to the CMBR rest frame in the $-\hat{\mathbf{k}}$ direction with a speed of order

$$\frac{\delta v}{c}\sim\frac{H_0^{-1}}{L}\frac{\delta\phi_k/(2\pi)^3}{\phi_0} . \tag{12b}$$

Stated another way, when viewed from the CMBR rest frame, the Universe appears to be "tilted." The existence of an isocurvature fluctuation breaks the connection between the peculiar-motion term and potential term that led to the cancellation of the dipole term for curvature fluctuations. Because of the compensating temperature fluctuations that develop, the rest frames defined by the isotropy of the expansion and the isotropy of the CMBR do not coincide. (It is simple to show that the peculiar velocity that arises due to the linear gradient in local axion number is subdominant.)

A superhorizon-sized isocurvature axion perturbation could tilt the Universe enough to explain the uniform bulk motion of about 700 km s$^{-1}\sim 2\times10^{-3}$ c that galaxies in our local neighborhood have with respect to the CMBR provided

$$\frac{\delta\phi_k/(2\pi)^3}{\phi_0}\sim 2\times10^{-3}e^{P} . \tag{13}$$

If we suppose that the fluctuation in the misalignment angle ϕ_0 is of order unity, then $N-N_{\min}$ must be less than $6-p$ to satisfy this requirement. On the other hand, if the average value of the initial misalignment angle ϕ_0 happened to be small in our inflationary patch, as it could well be [25], the left-hand side of Eq. (13) could be large, perhaps as large as 10^4. In this case $P=p+N-N_{\min}$ would be large, and $N-N_{\min}$ could be as large as $15-p$. Thus it is possible that a superhorizon-sized isocurvature axion fluctuation could have observable consequences even if inflation lasted ~ 10 e-foldings longer than the minimum needed to solve the horizon and flatness problems. It should also be clear that the tilted Universe scenario *requires* that the superhorizon-sized mode be superhorizon sized at the beginning of inflation also: If it were not, then its amplitude would be the same as those modes that are subhorizon sized today, and are constrained by the isotropy of the CMBR to have amplitudes of $\sim 10^{-5}$ or smaller.

As in the case of curvature fluctuations a quadrupole anisotropy of $O(k^2/H_0^2)$ also arises. The quadrupole anisotropy will be smaller than the dipole anisotropy by a factor of $O(k/H_0)$. Thus, the quadrupole anisotropy associated with the tilting of the Universe needed to explain the local bulk motion is $O(2\times10^{-3}e^{-P})$, which would be consistent with the current limits to the quadrupole anisotropy provided that $P\gtrsim 5$.

Isocurvature baryon-number fluctuations

If the baryon asymmetry of the Universe is produced in such a way that its value is proportional to the value of some scalar field, then isocurvature baryon-number fluctuations can arise in a similar way as isocurvature axion perturbations do [26]. Such occurs in an unconventional model of baryogenesis in which the baryon asymmetry is proportional to the "ilion" field [5], and the analysis above can be applied directly. In this case ϕ is the ilion field and the local baryon asymmetry that evolves is proportional to the initial value of ϕ. When baryons and antibaryons annihilate, the local baryon-number density will be proportional to the local baryon asymmetry, and hence the initial value of the ilion field; thus,

$$\frac{\delta n_B}{n_B}=\frac{\delta\phi}{\phi_0} ,$$

and the discussion above for isocurvature axion fluctuations carries over.

Testing for a "tilted universe"

How can we test for a "tilted universe?" If the explanation for the bulk of the dipole anisotropy of the CMBR is due to an intrinsic CMBR dipole, rather than a kinematic dipole, then we are indeed in the cosmic rest frame (as defined by the expansion). (Of course, we do expect that a portion of the local peculiar motions are due to the inhomogeneous distribution of matter nearby; for the moment we will assume that this portion is small and will neglect it.) We can infer that the CMBR dipole is intrinsic by measuring the anisotropy of another background radiation whose origin traces to objects at sufficiently high redshift that their distribution is isotropic and homogeneous in the cosmic rest frame. As an example consider the cosmic x-ray background, whose origin is believed to be discrete sources at high redshift [quasi-stellar objects (QSO's), active galactic nuclei (AGN's), hot gas, starburst galaxies, massive x-ray binaries, and the like]; if the Universe is tilted, the x-ray background should be isotropic in our local rest frame, rather than in the frame defined by the CMBR. It is possible that ROSAT (Roentgensatellit) will have sufficient sensitivity to determine the anisotropy of the x-ray background radiation and will settle this issue in the near future.

If the dipole anisotropy of the CMBR is kinematic, which is the conventional explanation, it arises because the observer is moving with respect to the cosmic rest frame and therefore measures a direction-dependent temperature:

$$T(\hat{\mathbf{n}})=T_0\frac{\sqrt{1-v^2}}{(1-\mathbf{v}\cdot\hat{\mathbf{n}})}$$
$$=T_0[1+v\cos\theta+v^2(-0.5+\cos^2\theta)+\cdots]\;; \tag{14}$$

where $\mathbf{v}$ is the observer's velocity and $\cos=\hat{\mathbf{n}}\cdot\hat{\mathbf{v}}$. It is simple to see that a kinematic quadrupole of $O(v^2)$, which is aligned with the dipole, must arise; in terms of spherical harmonics (oriented about $\mathbf{v}$), $T(\hat{\mathbf{n}})/T_0$ is specified by

$$a_{00}=\frac{\sqrt{\pi}}{3}v^2\,,\quad a_{10}=\left[\frac{4\pi}{3}\right]^{1/2}v\,,$$
$$a_{20}=\frac{4}{3}\left[\frac{\pi}{5}\right]^{1/2}v^2\,. \tag{15}$$

If the CMBR dipole anisotropy is intrinsic, there will also be a quadrupole anisotropy, which is $O(H_0^{-1}/L)$ smaller than the dipole anisotropy; however, its relationship to the dipole cannot be specified beyond this. Whether or not experiments can both achieve sufficient sensitivity and separate a kinematic quadrupole from other contributions remains to be seen. Conversely, if the universe is tilted and H_0^{-1}/L is greater than $v\sim10^{-3}$, one might be able to "detect" the existence of a quadrupole anisotropy associated with the tilting of the universe. For example, if the quadrupole anisotropy is found to be larger than is expected on the basis of the CMBR anisotropies detected on smaller angular scales, one could infer the presence of a superhorizon-sized fluctuation. Of course we must remind the reader that no CMBR anisotropy beyond that of the dipole has yet been detected.

Next, if the bulk of our *apparent* peculiar velocity is due to an intrinsic dipole, and not motion with respect to the cosmic rest frame, then there is no reason for the velocity vector computed from the local distribution of matter,

$$\mathbf{v}_P(\mathbf{r},t)=-\frac{H_0}{4\pi}\int\frac{\delta\rho(\mathbf{r}',t)(\mathbf{r}-\mathbf{r}')d^3r'}{|\mathbf{r}-\mathbf{r}'|^3\rho}\,, \tag{16}$$

to be aligned with the direction of the CMBR dipole. Present work indicates that the two vectors differ by an angle of $\sim10°$ [18]. Other tests of a tilted universe are discussed in Ref. [16].

Finally, we can use the measured peculiar motion in our neighborhood to limit superhorizon-sized isocurvature fluctuations, be they associated with axions, baryons, or whatever.

V. CONCLUDING REMARKS

If inflation lasted only ~10 *e*-foldings longer than required to solve the horizon and flatness problems, preinflationary fluctuations on superhorizon scales can have observable consequences today. Curvature perturbations on these scales, whose origin might trace to fluctuations in the inflation field or simply reflect the primeval inhomogeneity in the Universe, lead to a quadrupole anisotropy of $O((k/H_0)^2)$ times their horizon-crossing amplitude. This quadrupole anisotropy arises in addition to the usual CMBR anisotropies predicted by inflation, and its existence could be inferred if the measured quadrupole anisotropy were larger than that expected on the basis of anisotropies found on smaller scales. As previously emphasized [20], the sensitivity of the CMBR to superhorizon-sized fluctuations provides a very compelling argument against $\Omega_0\neq1$ in an inflationary Universe.

Isocurvature fluctuations in axions or baryons (or other particles) can lead to both a quadrupole anisotropy of $O(k^2/H_0^2)$ and a dipole anisotropy that is *intrinsic* to the CMBR and is of $O(k/H_0)$. If the bulk of the CMBR dipole is intrinsic, rather than kinematic, then viewed from the rest frame defined by the isotropy of the CMBR the Universe appears to be tilted; i.e., all the matter in the Universe will be found to be streaming at about 700 $\mathrm{km\,s^{-1}}$ in the direction of the CMBR dipole (in our neighborhood, toward Hydra-Centaurus). This could explain the major part of the peculiar-velocity field observed in our local neighborhood [14], which we might add is difficult to explain otherwise [15]. This unconventional hypothesis could be tested by measuring the isotropy of other background radiations and the alignment of the CMBR dipole with the acceleration vector that arises due to the inhomogeneous distribution of matter. If the "tilting of the Universe" is indeed the explanation for the bulk of the local peculiar-velocity field, the peculiar velocities in our local neighborhood associated with the inhomogeneous distribution of matter are significantly smaller than is presently believed (more like 200 $\mathrm{km\,s^{-1}}$), a fact with many consequences. For one thing, models of structure formation (including cold dark matter), which are currently in trouble because they cannot account for the large peculiar velocities that have been measured, would be in much better shape as they would only have to explain the much smaller "noise" component in the peculiar-velocity field. For another, determinations of Ω_0 based upon the local distribution of matter and the measured peculiar-velocity field [18] and which yield values close to the inflationary prediction of 1 would have to be reevaluated.

Note added in proof. This work was originally presented at the Workshop on Particle Astrophysics: Forefront Experimental Issues, Berkeley, California, 1988 (unpublished). An essay version of this paper will appear in Gen. Relativ. Gravit. (to be published). Discussions of a "tilt" between the rest frame defined by the matter flow and that defined by the surfaces of homogeneity extend back to the work of K. Godel, Proc. Intl. Math. Congress (Cambridge, MA) **1**, 175 (1952); other discussions of tilt include A. R. King and G. F. R. Ellis, Commun. Math Phys. **31**, 209 (1973) and G. F. R. Ellis and J. E. Baldwin, Mon. Not. R. Astr. Soc. **206**, 377 (1984).

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APPENDIX: KINEMATICS OF INFLATIONARY FRW MODELS

A simple model

Here we briefly review the kinematics of the transition to inflationary expansion in an FRW model with arbitrary curvature, and energy density comprised of thermal radiation, $\rho_R = g_* \pi^2 T^4/30$, and vacuum energy, $\rho_{\rm vac} \equiv M^4$. The quantity g_* (expected to be greater than 100 at the temperature of interest) counts the number of ultrarelativistic degrees of freedom, and the entropy density associated with the radiation is $s = 2\pi^2 g_* T^3/45$.

With a nonstandard, but useful, choice for the normalization of the cosmic-scale factor, the Friedmann equation can be written as

$$\frac{\dot{R}^2}{R^2} \equiv H^2 = H_I^2 \left[\frac{a}{R^4} + 1 \pm \frac{1}{R^2} \right] , \tag{A1}$$

where the plus sign applies to a positively curved model and the minus sign to a negatively curved model, $H_I \equiv \sqrt{8\pi M^4/3m_{\rm Pl}^2}$ is the Hubble parameter during inflation, and the curvature radius of the Universe is

$$R_{\rm curv} \equiv R(t) H_I^{-1} .$$

Note, the value of cosmic-scale factor is not one today; it achieves a value of order unity when the Universe begins to inflate. (More precisely, the Universe begins to inflate when the scale factor is of order $\max\{1, a^{1/4}\}$.)

The first term on the right-hand side of Eq. (A1) corresponds to the energy density in radiation, the second to the vacuum energy density, and the third to the curvature. The quantity a is a dimensionless parameter with important physical significance:

$$a = 6 \left[\frac{5}{g_*} \right]^{1/3} \left[\frac{M}{m_{\rm Pl}} \right]^4 S_{\rm curv}^{4/3} ,$$

$$S_{\rm curv} \equiv \left[\frac{4\pi R_{\rm curv}^3}{3} \right] s .$$

It is related to the entropy contained within a sphere whose radius is equal to the curvature radius ($\equiv S_{\rm curv}$) and the size of the vacuum energy relative to the Planck energy. When the expansion is adiabatic, $S_{\rm curv}$ is conserved; its value today is known to be greater than 10^{88}, which provides another way of characterizing the flatness problem: Avoiding recollapse or free expansion ($R \propto t$) until the present epoch would, in the absence of inflation, require that $S_{\rm curv} \gtrsim 10^{88}$. The reheating event at the end of inflation grossly violates adiabaticity; thus, in an inflationary Universe the present value of $S_{\rm curv}$ is not indicative of its initial value. We assume that prior to (and during) inflation the expansion is adiabatic, and that $g_* = $const. (Adiabaticity implies that $R^3 g_* T^3 = $const, or $\rho_R \propto g_*^{-1/3} R^{-4}$.)

In general a characterizes the relative importance of radiation compared to the curvature term. Its value determines whether or not a positively curved FRW model recollapses before it can inflate: If $a < 0.25$ it recollapses before it can begin to inflate; if $a > 0.25$ it does not. As $a \to 0.25$, the model approaches an Einstein-Lemaitre model and has a long static phase during which $R^2 \simeq 0.5$. (With $a = 0.25$, $H = \ddot{R} = 0$ for $R^2 = 0.5$.) Since $a \propto S_{\rm curv}^{4/3}$, positively curved models with $a < 0.25$, correspond to low-entropy models in which there is not enough radiation to prevent recollapse before inflation begins. To be specific, if we take $M \simeq 10^{15}$ GeV, then $a \sim 10^{-16} S_{\rm curv}^{4/3}$, and the requirement that $a \gtrsim 1$ implies that $S_{\rm curv} \gtrsim 10^{12}$. All negatively curved models ultimately inflate.

Scale factor

The evolution of the scale factor is straightforward to obtain. For the positively curved models,

$$R(t) = \{0.5 + \sqrt{a - 0.25} \sinh[2H_I(t - t_0)]\}^{1/2} ,$$
$$t_0 \equiv \frac{1}{2H_I} \operatorname{arcsinh} \left[\frac{0.5}{\sqrt{a - 0.25}} \right] . \tag{A2}$$

For $a \gg 0.25$, at early times, $t \lesssim H_I^{-1}/2$, the Universe is radiation dominated and the scale factor evolves as $R(t) \simeq a^{1/4} \sqrt{2H_I t}$; at late times, $t \gtrsim H_I^{-1}/2$, the Universe inflates and the scale factor evolves as $R(t) \simeq a^{1/4} \exp(H_I t)$. If a is very close to 0.25, at early times, $t \lesssim H_I^{-1}/2$, the scale factor evolves as $R(t) \simeq \sqrt{H_I t}$; at intermediate times, $|\ln(a - 0.5)| H_I^{-1}/2 \gtrsim t \gtrsim H_I^{-1}/2$, the scale factor remains roughly constant, $R(t) \simeq 1/\sqrt{2}$; and at late times, $t \gtrsim H_I^{-1} |\ln(a - 0.25)|/2$, the scale factor grows exponentially, $R(t) \simeq \sqrt{(a - 0.25)} \exp(H_I t)$.

For negatively curved models and $a > 0.25$,

$$R(t) = \{-0.5 + \sqrt{a - 0.25} \sinh[2H_I(t + t_0)]\}^{1/2} ,$$
$$t_0 \equiv \frac{1}{2H_I} \operatorname{arcsinh} \left[\frac{0.5}{\sqrt{a - 0.25}} \right] . \tag{A3}$$

The cosmic-scale factor has the same behavior as in the positively curved models.

For negatively curved models a can be less than 0.25; in this case

$$R(t) = \{-0.5 + \sqrt{0.25 - a} \cosh[2H_I(t + t_0)]\}^{1/2} ,$$
$$t_0 \equiv \frac{1}{2H_I} \operatorname{arccosh} \left[\frac{0.5}{\sqrt{0.25 - a}} \right] . \tag{A4}$$

At early times, $t \lesssim \sqrt{a} H_I^{-1}$, the universe is radiation dominated and $R \simeq a^{1/4} \sqrt{2H_I t}$; at intermediate times, $H_I^{-1}/2 \gtrsim t \gtrsim \sqrt{a} H_I^{-1}/2$, the universe is curvature dominated and $R \simeq H_I t$; and at late times, $t \gtrsim H_I^{-1}/2$, it is vacuum dominated and $R \simeq 0.5 \exp(H_I t)$.

The behavior of the cosmic-scale factor is shown in Fig. 1 for models with $a = 100$, $a = 0.250\,001$ ($k > 0$), and $a = 0.0001$.

Omega

The evolution of Ω in all models that inflate is qualitatively the same: As $R \to 0$ or $R \to \infty$, $\Omega \to 1$, and Ω,

$$\Omega \equiv \frac{a+R^4}{a+R^4 \pm R^2} , \tag{A5}$$

achieves its extremum for $R = a^{1/4}$ (a minimum for $k<0$ and a maximum for $k>0$):

$$\Omega_{\text{extremum}} = \frac{1}{1 \mp \sqrt{1/4a}} . \tag{A6}$$

The value of Ω at the onset of inflation, i.e., when the cosmic-scale factor begins to grow exponentially, is given by

$$\Omega_{\text{start}} = \begin{cases} 1 \pm \sqrt{1/2a} \ , & a \gg 0.25 \ , \\ 2 \ , & a \simeq 0.25, \quad k>0 \ , \\ 0.25 \ , & a \ll 0.25, \quad k<0 \ . \end{cases}$$

In all cases, the deviation of Ω_{start} from unity when inflation begins is at most of order a few. The evolution of $\Omega(t)$ is shown in Fig. 2.

Particle horizons

In discussing the amount of inflation required to solve the horizon and flatness problems we assumed that the smooth, inflationary patch that we find ourselves in today had an initial size of H_I^{-1}. For a model that is radiation dominated at the onset of inflation the distance to the horizon at the beginning of inflation,

$$d_H(t = t_{\text{start}} \simeq H_I^{-1}/2) \equiv R(t) \int_0^t \frac{dt'}{R(t')} ,$$

is comparable to H_I^{-1}. However, if the curvature term is important when the Universe begins to inflate, the distance to the horizon at the beginning of inflation can be somewhat greater than H_I^{-1}.

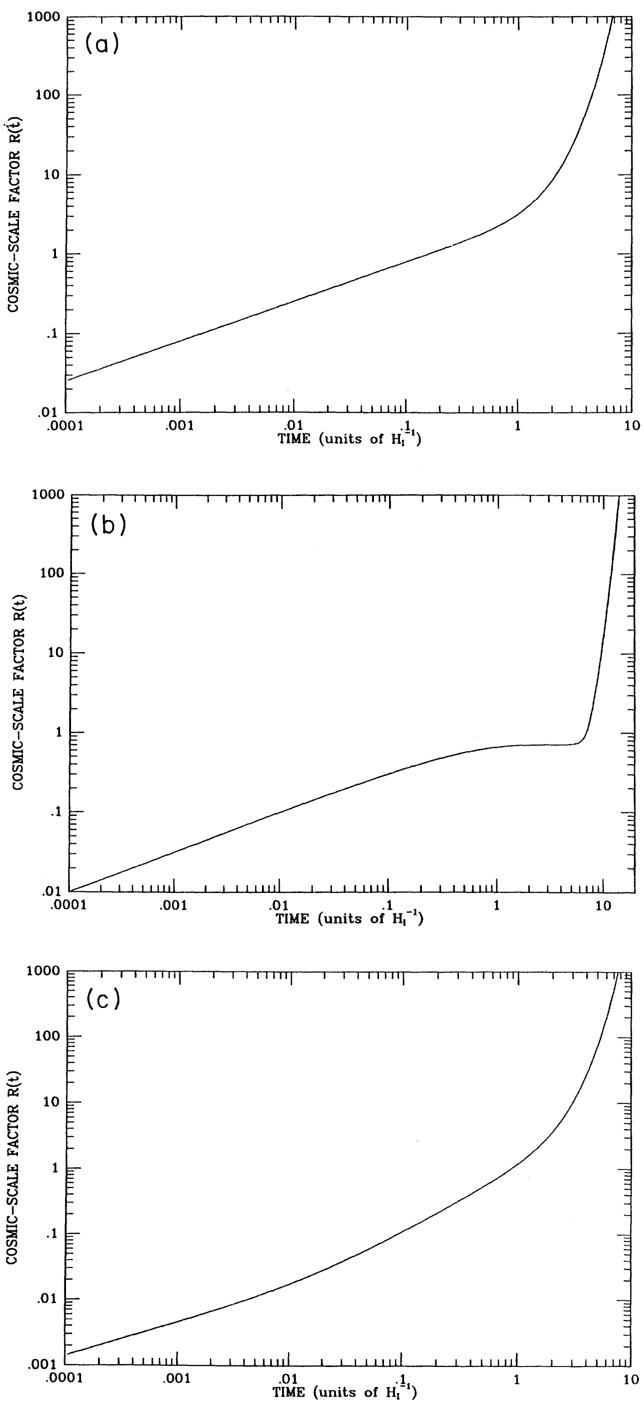

FIG. 1. Evolution of the cosmic-scale factor $R(t)$: (a) for $a=100$ and $k>0$ ($a=100$ and $k<0$ is indistinguishable); (b) for $a=0.250001$ and $k>0$; and (c) for $a=0.0001$ and $k<0$.

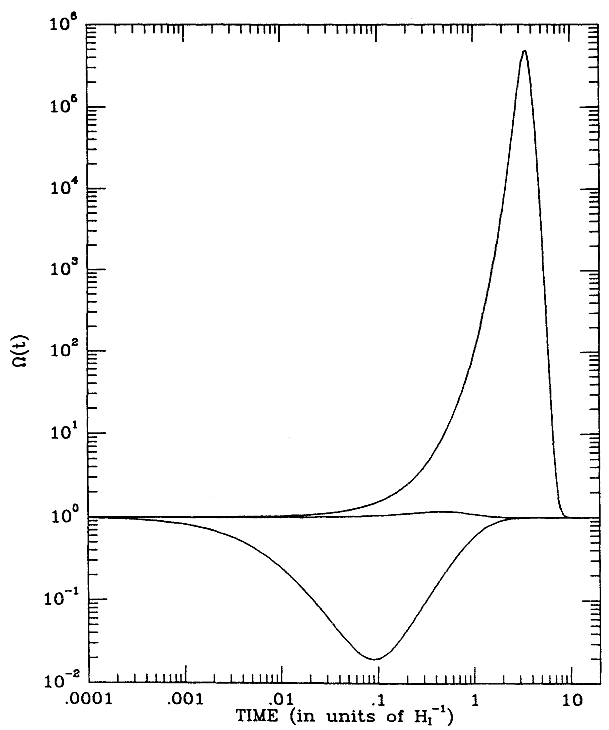

FIG. 2. Evolution of Ω as a function of time for the same three models as in Fig. 1. Top curve corresponds to $a=0.250001$, middle curve to $a=100$, and bottom curve to $a=0.0001$.

The distance to the particle horizon can be written as

$$d_H(t)=\frac{H_I^{-1}}{2}R(t)\int_{-u_0}^{u}\frac{du}{(0.5+b\sinh u)^{1/2}}$$
$$=\frac{H_I^{-1}}{2}\frac{R(t)}{a^{1/4}}F(\alpha,r)\quad(k>0)\ , \tag{A7a}$$
$$d_H(t)=\frac{H_I^{-1}}{2}R(t)\int_{u_0}^{u}\frac{du}{(-0.5+b\sinh u)^{1/2}}$$
$$(k<0\ ;\ a>0.25)\ , \tag{A7b}$$

where $u_0=\mathrm{arcsinh}(1/2b), b=\sqrt{a-0.25}$,

$$r=\frac{1}{2a^{1/4}}(1+2\sqrt{a})^{1/2}\ ,$$
$$\cos\alpha=\frac{1-(R/a^{1/4})^2}{1+(R/a^{1/4})^2}\ ,$$

and $F(\alpha,r)$ is the elliptic integral of the first kind:

$$F(\alpha,r)\equiv\int_0^{\alpha}\frac{d\phi}{(1-r^2\sin^2\phi)^{1/2}}$$
$$\to\begin{cases}\alpha \ \text{ for } \alpha\ll 1\ ,\\ \ln(4/\sqrt{1-r^2})\ \text{ for } \alpha=\pi/2, r\simeq 1\ .\end{cases}$$

For $k<0$ and $a<0.25$,

$$d_H(t)=\frac{H_I^{-1}}{2}R(t)\int_{u_0}^{u}\frac{du}{(-0.5+b\cosh u)^{1/2}}$$
$$=\frac{H_I^{-1}}{2}\frac{R(t)}{\sqrt{b+0.5}}F(\alpha,r)\ ; \tag{A7c}$$

where $u_0=\mathrm{arccosh}(1/2b), b=\sqrt{0.25-a}$,

$$r=\left[\frac{2b}{b+0.5}\right]^{1/2}\ ,$$
$$\sin\alpha=\left[\frac{R^2}{R^2+0.5-b}\right]^{1/2}\ .$$

Only when the curvature term plays an important role ($k>0$ and $a\simeq 0.25$) and ($k<0$ and $a\ll 1$), does the size of the particle horizon at the beginning of inflation differ significantly from H_I^{-1}; it can be logarithmically larger than H_I^{-1}:

$$d_H(t_{\rm start})\simeq H_I^{-1}\begin{cases}|\ln(a-0.25)|\ ,\quad k>0\ a\simeq 0.25,\\ |\ln a|\ ,\quad k<0\ a\ll 1, k<0\ ,\end{cases}$$

where in the first case $t_{\rm start}\simeq|\ln(a-0.25)|H_I^{-1}/2$ and in the second case $t_{\rm start}\simeq H_I^{-1}/2$. In these extreme cases it is *kinematically* possible for a patch that is larger in size than H_I^{-1} to have "smoothed itself" by the beginning of inflation; whether or not there are dynamical processes that can accomplish this is another matter (for $k<0$ the smoothing must occur over many e-foldings of the scale factor).

In order to have $\Omega\neq 1$ today as well as consistency with the isotropy of the CMBR our inflationary patch must have been ~ 200 times larger than H_I^{-1} at the beginning of inflation. If we are to take advantage of the fact that d_H can be larger than H_I^{-1} at the beginning of inflation to allow for its possible microphysical origin, rather than simply postulating that such a smooth patch existed, we must have

$$(a-0.25)\lesssim\exp(-200)\quad(k>0)$$

or

$$a\lesssim\exp(-200)\ (k<0)\ ,$$

in addition to the usual requirement that the amount of inflation is just right to give $\Omega\neq 1$ today. For $k<0$ and $M\simeq 10^{15}$ GeV, the requirement that $a\lesssim e^{-200}$ implies that $S_{\rm curv}\lesssim 10^{-53}$, which corresponds to a very low-entropy Universe.

[1] For a review of inflation, see, e.g., E. W. Kolb and M. S. Turner, *The Early Universe, Frontiers in Physics,* Vol. 69 (Addison-Wesley, Redwood City, CA, 1990), Chap. 8. For simplicity we have assumed that inflation is exponential; all results can be easily generalized to "power-law" inflation, i.e., $R\propto t^n$, $n\geq 1+0.5\{-[2+\ln(T_{\rm RH}/3K)]/\ln(T_{\rm start}/3K)\}$. In particular, in power-law inflation the relationship between $d_{\rm patch}, H_0^{-1}, \Omega_{\rm start}$, and Ω_0 is unaffected. The "insensitivity" of the present state of the Universe to the initial state in inflationary models is discussed by M. S. Turner and L. M. Widrow, Phys. Rev. Lett. **57**, 2237 (1986); L. Jensen and J. A. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987); A. A. Starobinskii, Pis'ma Zh. Eksp. Teor. Fiz. **37**, 55 (1983) [JETP Lett. **37**, 66 (1983)]. Extended inflation is discussed in D. La, Phys. Rev. Lett. **62**, 376 (1989); E. Weinberg, Phys. Rev. D **40**, 3950 (1989); E. W. Kolb, D. S. Salopek, and M. S. Turner, *ibid.* **42**, 3925 (1990).

[2] See, e.g., A. D. Linde, Phys. Lett. **108B**, 389 (1982); P. J. Steinhardt and M. S. Turner, Phys. Rev. D **29**, 2162 (1984); P. J. Steinhardt, Nature (London) **345**, 47 (1990); R. Holman *et al.*, Phys. Rev. D **43**, 995 (1991); **43**, 3833 (1991).

[3] J. A. Frieman and M. S. Turner, Phys. Rev. D **30**, 265 (1984).

[4] M. S. Turner and L. M. Widrow, Phys. Rev. Lett. **57**, 2237 (1986); L. Jensen and J. A. Stein-Schabes, Phys. Rev. D **34**, 931 (1986).

[5] A. Cohen and D. Kaplan, Phys. Lett. B **199**, 251 (1987).

[6] The equation of motion for ϕ is $\ddot{\phi}+3H\dot{\phi}-\nabla^2\phi/R(t)^2+\partial V/\partial\phi=0$ where $V(\phi)$ is the scalar potential for ϕ and ∇ is with respect to comoving coordinates. Provided that the potential term is unimportant ($V''\ll H_I^2$) and the scale of spatial variation for the ϕ is much greater than H^{-1}, the solution is $\phi=$const; see, e.g., Kolb and Turner [1].

[7] A. D. Linde, Phys. Lett. **108B**, 389 (1982); for a careful estimate of λ see Chap. 8 of Kolb and Turner [1].

[8] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983); A. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982); A. A. Starobinskii, Phys. Lett. **117B**, 175 (1982); S. W. Hawking, *ibid.* **115B**, 295 (1982); and Ref. [1]. How fluctuations on subhorizon-sized scales evolve to their de Sitter expectation values is discussed by A. Guth and S.-Y. Pi, Phys. Rev. D **32**, 1899 (1985); F. Graziani and G. Mathews (in preparation).

[9] See, e.g., Chap. 9 of Kolb and Turner [1]; G. Efstathiou, in *Physics of the Early Universe,* edited by J. A. Peacock, A. F. Heavens, and A. T. Davies (Higler, New York, 1990); G. Blumenthal, S. M. Faber, J. R. Primack, and M. J. Rees, Nature (London) **311**, 517 (1984); or C. D. M. Frenk, in *The Birth and Early Evolution of Our Universe,* Proceedings of the Nobel Symposium No. 79, edited by B. Gustafsson, Y. Nilsson, and B. Skagerstam [Phys. Scr. **T36**, 70 (1991)].

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[11] R. K. Sachs and A. M. Wolfe, Astrophys. J. **147**, 73 (1967).

[12] The density perturbation on a given scale is conventionally defined as the rms mass fluctuation, averaged over the Universe, on that scale. We will use $(\Delta\rho/\rho)_L$ to denote the rms mass fluctuation on the scale L to distinguish it from the Fourier component δ_k on the same scale. Using a sphere of radius L to define the volume over which the mass fluctuation is to be computed ("top-hat" window function),

$$(\Delta\rho/\rho)_L^2=(9/2\pi^2)\int k^2|\delta_k|^2[\sin kL/(kL)^3 - \cos kL/(kL)^2]^2 dk \sim (k^3|\delta_k|^2/2\pi^2)|_{k=L^{-1}} .$$

[13] In the Newtonian treatment of density perturbations (valid on scales $\ll H^{-1}$), the Fourier components of the perturbed Newtonian potential Φ are related to the density perturbation by $\delta\Phi_k=-(3H^2R^2/2k^2)\delta_k$. The contribution of the second term to the temperature fluctuation can be written as $\delta T/T=\delta\Phi_k/3$.

[14] V. G. Rubin *et al.*, Astron. J. **81**, 687 (1976); A. Dressler *et al.*, Astrophys. J. **313**, L37 (1987); D. Lynden-Bell *et al.*, *ibid.* **326**, 19 (1988); M. Aaronson *et al.*, *ibid.* **302**, 536 (1986); C. A. Collins *et al.*, Nature (London) **320**, 506 (1986). Very recent work of D. S. Mathewson, V. L. Ford, and M. Buchhorn (Mt. Stromolo Observatory report, 1991) indicates that galaxies even out to distances of $60h^{-1}$ Mpc are moving with respect to the CMBR rest frame with a velocity similar to that of our Galaxy.

[15] Gunn proposed the idea that the CMBR does not define the local frame of rest as an explanation for the large peculiar motion in our neighborhood; see J. E. Gunn, in *The Extragalactic Distance Scale,* Proceedings of the Astronomical Society of the Pacific 100th Anniversary Symposium, Victoria, Canada, 1988, edited by S. van der Bergh and C. J. Pritchet, ASP Conf. Ser. 4 (Brigham Young University, Provo, UT, 1988), p. 344. Ostriker has emphasized the difficulty that all conventional scenarios of structure formation have explaining the "high Mach number" (~ 3) implied by the local large bulk motion (~ 700 km s^{-1}) and small random velocities (~ 200 km s^{-1}); see J. P. Ostriker and Y. Suto, Astrophys. J. **348**, 378 (1990).

[16] Paczynski and Piran have recently discussed a cosmological toy model based upon the Tolman-Bondi cosmology in which the Universe is, in our terms, "tilted." In their model the tilting is provided by a spatially varying baryon-to-photon ratio; see B. Paczynski and T. Piran, Astrophys. J. **364**, 341 (1990).

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