Finite-temperature gauge field propagator in the early Universe

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We consider a model of the early Universe where a Higgs-scalar electrodynamics describes its contents. The finite-temperature gauge field propagator is obtained in the real-time formalism during an interval of time when thermal equilibrium is maintained making effective masses large compared to curvature terms.

I. INTRODUCTION

One of the important areas of application of quantum field theory on curved space-time background [1] is provided by the problems of the early Universe [2]. Since the time when the average energy of the particles was less than the Planck energy by at least an order of magnitude, the gravitational field can be considered classical, justifying such an approach. But because of the time dependence of the background geometry, it is difficult to incorporate the effect of temperature through the so-called imaginary-time formalism [3], particularly for problems where time plays an essential role.

The formulation of a real-time finite-temperature scalar field theory on flat space-time [4] has been extended to that on a Robertson-Walker space-time by several authors [5], of which the one by Semenoff and Weiss [6] appears to be more satisfactory in that it does not involve the continuation of the scale factor to imaginary time. The technical difficulty in the original formulation associated with the short-distance singularity of the propagator can be easily removed [7], leading to a greatly simplified expression for the latter.

Here we try to extend this formulation to a Higgs-scalar electrodynamics. The main task is to find the expression for the gauge-field propagator. It appears more convenient to start with equations for the mode functions, from which those for the components of the propagator can be obtained. The former are written in the renormalizable R_ξ gauge [8], assuming the vacuum expectation value of a component of the Higgs field to vary slowly enough with time. The equations for the transverse components are uncoupled, as in flat space-time, while those for the longitudinal and scalar components are coupled through gravity, in addition to those already existing on flat space-time.

In the real-time formulation, one starts describing the system from a time t_0 when an effective thermal equilibrium prevails [6,7]. Such a condition is guaranteed if the collision rates among particles contained in the system far exceed the expansion rate of the Universe. The formalism then continues to describe the system at later times, even though thermal equilibrium has long ceased to hold. The expression for the gauge-field propagator we find here is valid when both the collision rates and effective masses are large compared with the expansion rate. In the radiation-dominated era, both are satisfied, provided the temperature is low enough.

In Sec. II we review scalar electrodynamics in a general space-time background. The equations for mode functions and for components of the gauge-field propagator are written in Robertson-Walker metrics in Sec. III. Our solution for the finite-temperature propagator is described in Sec. IV. In the concluding Sec. V, we discuss how to obtain the propagator at later times.

II. SCALAR ELECTRODYNAMICS

The action for the complex matter field $\phi(x)$ interacting with a U(1) gauge field $A_\mu(x)$ in an external gravitational field $g_{\mu\nu}$ is [9]

$$S=\int dx\sqrt{(-g)}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+g^{\mu\nu}(D_\mu\phi)^\dagger(D_\nu\phi)+m^2\phi^\dagger\phi-\frac{\lambda}{3!}(\phi^\dagger\phi)^2\right]+\text{gauge-fixing terms}\ , \tag{2.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\ ,\qquad D_\mu\phi=\partial_\mu\phi-ieA_\mu\phi\ . \tag{2.2}$$

Here m^2 is chosen positive to correspond to spontaneous symmetry breaking. In terms of real components $\phi=(1/\sqrt{2})(\phi_1+i\phi_2)$ the above action may be written as

$$S=\int dx\sqrt{(-g)}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}g^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_a-eA^\mu\epsilon_{ab}\phi_a\partial_\mu\phi_b+\frac{e^2}{2}A_\mu A^\mu(\phi_1^2+\phi_2^2)+\frac{m^2}{2}(\phi_1^2+\phi_2^2)-\frac{\lambda}{4!}(\phi_1^2+\phi_2^2)^2\right]\ ,$$
$$a,b=1,2,\quad \epsilon_{12}=+1\ . \tag{2.3}$$

Let ϕ_1 develop a vacuum expectation value $\bar{\phi}(x)$. It depends on x in general, because of the explicit x dependence brought about by the metric $g_{\mu\nu}$. We shift the field

$$\phi_1(x)=\bar{\phi}(x)+\phi_1'(x) , \tag{2.4}$$

so that the new field ϕ_1' satisfies

$$\langle 0|\phi_1'(x)|0\rangle=0 . \tag{2.5}$$

Omitting the prime on ϕ_1' henceforth, we get

$$S=S_a+S_b+S_c+S_g , \tag{2.6}$$

where

$$S_a=\int dx\sqrt{(-g)}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}e^2\bar{\phi}^2A_\mu A^\mu -eA^\mu(\bar{\phi}\partial_\mu\phi_2-\phi_2\partial_\mu\bar{\phi})+\frac{1}{2}(\partial_\mu\phi_1)^2 -\frac{1}{2}\left[\frac{\lambda\bar{\phi}^2}{2}-m^2\right]\phi_1^2+\frac{1}{2}(\partial_\mu\phi_2)^2 -\frac{1}{2}\left[\frac{\lambda\bar{\phi}^2}{6}-m^2\right]\phi_2^2\right] , \tag{2.7}$$

$$S_b=\int dx\sqrt{(-g)}\left[(\partial_\mu\bar{\phi})\partial^\mu\phi_1-\bar{\phi}\left[\frac{\lambda\bar{\phi}^2}{6}-m^2\right]\phi_1\right] , \tag{2.8}$$

$$S_c=\int dx\sqrt{(-g)}\left[-eA^\mu(\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1) +e^2\bar{\phi}A_\mu A^\mu\phi_1-\frac{\lambda\bar{\phi}}{6}\phi_1(\phi_1^2+\phi_2^2) +\frac{e^2}{2}A_\mu A^\mu(\phi_1^2+\phi_2^2)-\frac{\lambda}{4!}(\phi_1^2+\phi_2^2)^2\right] . \tag{2.9}$$

We now choose the gauge-fixing term in the R_ξ gauge:

$$S_g=-\frac{1}{2\xi}\int dx\sqrt{(-g)}(\nabla_\mu A^\mu+\xi e\bar{\phi}\phi_2)^2 . \tag{2.10}$$

Carrying out a partial integration and omitting the boundary terms [10],

$$S_g=\int dx\sqrt{(-g)}\left[-\frac{1}{2\xi}(\nabla_\mu A^\mu)^2 +A^\mu\partial_\mu(\bar{\phi}\phi_2)-\xi\frac{e^2}{2}\bar{\phi}^2\phi_2^2\right] . \tag{2.11}$$

The free or quadratic part of the complete action now becomes

$$S_0=S_a+S_g =\int dx\sqrt{(-g)}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\xi}(\nabla_\mu A^\mu)^2+\frac{1}{2}e^2\bar{\phi}^2A_\mu A^\mu+2e(\partial_\mu\bar{\phi})A^\mu\phi_2 +\frac{1}{2}(\partial_\mu\phi_1)^2-\frac{1}{2}\left[\frac{\lambda\bar{\phi}^2}{2}-m^2\right]\phi_1^2+\frac{1}{2}(\partial_\mu\phi_2)^2-\frac{1}{2}\left[\frac{\lambda\bar{\phi}^2}{6}-m^2+\xi e^2\bar{\phi}^2\right]\phi_2^2\right] . \tag{2.12}$$

It will be observed that, unlike the case of flat space-time where $\bar{\phi}$ is a constant, the choice of the R_ξ gauge does not eliminate, in general, the mixing of A_μ and ϕ_2 fields in the quadratic part of the action. In the following we assume that either the system is in the symmetric phase (restoration of symmetry at high temperature), $\bar{\phi}=0$, or $\bar{\phi}$ varies slowly enough to justify dropping the term in (2.12). Carrying out another partial integration and again ignoring the boundary terms [10], we write

$$S_0=\frac{i}{2}\int dx\sqrt{(-g)}(-A^\mu D_{\mu\nu}A^\nu+\phi_1D_1\phi_1+\phi_2D_2\phi_2) , \tag{2.13}$$

$$D_{\mu\nu}=i\left[g_{\mu\nu}\nabla_\lambda\nabla^\lambda-R_{\mu\nu}-\left[1-\frac{1}{\xi}\right]\nabla_\mu\nabla_\nu+M^2g_{\mu\nu}\right] , \tag{2.14}$$

$$D_{1,2}=i(\nabla_\mu\nabla^\mu+M_{1,2}^2) .$$

Anticipating mass generation at finite temperature, we have introduced effective masses [7,11] M, M_1, and M_2 for the gauge and scalar fields. The compensating terms are included in the interaction. The full set of interaction terms is now given by

$$S_1=\bar{S}+S_c , \tag{2.15}$$

where S_c denotes the cubic and quartic interaction terms (2.9) and $\bar{S}$ consists of linear terms (2.8) and quadratic mass terms of (2.7) and the compensating ones:

$$\bar{S}=\int dx\sqrt{(-g)}\left\{\left[\nabla_\mu\nabla^\mu\bar{\phi}+\left[\frac{\lambda\bar{\phi}^2}{6}-m^2\right]\bar{\phi}\right]\phi_1 +\frac{1}{2}(e^2\bar{\phi}^2-M^2)A_\mu A^\mu -\frac{1}{2}\left[\frac{\lambda\bar{\phi}^2}{2}-m^2-M_1^2\right]\phi_1^2 -\frac{1}{2}\left[\frac{\lambda\bar{\phi}^2}{6}-m^2+\xi e^2\bar{\phi}^2-M_2^2\right]\phi_2^2\right\} . \tag{2.16}$$

The unknown masses M, M_1, and M_2 and the classical field $\bar{\phi}$ are determined order by order in perturbation theory.

III. PROPAGATOR EQUATIONS

The propagators are defined formally as the inverse of the differential operators (2.14):

$$D_{\nu\lambda}(x)G^{\lambda\sigma}(x,x')=\delta_\nu^\sigma\delta(x-x')/\sqrt{-g} ,$$
$$D_a(x)G_a(x,x')=\delta(x-x')/\sqrt{-g} ,\quad a=1,2 . \tag{3.1}$$

With appropriate boundary conditions on the space of functions, the differential operators $D_{\mu\nu}$, D_1, and D_2 are clearly self-adjoint. So their Green's functions satisfy the symmetry relations

$$G_{\mu\nu}(x,x')=G_{\nu\mu}(x',x) ,$$
$$G_{1,2}(x,x')=G_{1,2}(x',x) . \tag{3.2}$$

The propagator for a scalar field is given in papers [6,7] developing the real-time finite-temperature formulation. In the following we discuss the case of the vector field only.

Instead of working directly with the equations for the gauge-field propagator, it is simpler first to consider those for the field modes:

$$D_{\nu\lambda}(x)A^\lambda(x)=ij_\nu(x) , \tag{3.3}$$

where the form of the current $j_\mu(x)$ is not needed for our purpose. The Green's function may then be obtained from the relation

$$A^\mu(x)=i\int dx'\sqrt{(-g')}G^{\mu\nu}(x,x')j_\nu(x') . \tag{3.4}$$

In the homogeneous, isotropic, and spatially flat metric of standard cosmology [12],

$$ds^2=dt^2-a^2(t)d\mathbf{x}^2 , \tag{3.5}$$

the equations for the vector field $A^\mu(x)=\{A^0(x),\mathbf{A}(x)\}$ become

$$\ddot{A}^0+3\frac{\dot a}{a}\dot{A}^0+\left[M^2-\frac{\nabla^2}{a^2}+3\left[\frac{\dot a}{a}\right]^{\cdot}\right]A^0-2\frac{\dot a}{a}\nabla\cdot\mathbf{A} -\left[1-\frac{1}{\xi}\right]\left[\ddot{A}^0+3\frac{\dot a}{a}\dot{A}^0+3\left[\frac{\dot a}{a}\right]^{\cdot}A^0+\nabla\cdot\dot{\mathbf{A}}\right]=j^0 ,$$
$$\ddot{\mathbf{A}}+5\frac{\dot a}{a}\dot{\mathbf{A}}+\left[M^2-\frac{\nabla^2}{a^2}+4\left[\frac{\dot a}{a}\right]^2+2\frac{\ddot a}{a}\right]\mathbf{A}-2\frac{\dot a}{a^3}\nabla A^0 +\left[1-\frac{1}{\xi}\right]\left[\frac{\nabla}{a^2}(\nabla\cdot\mathbf{A})+\frac{\nabla}{a^2}\dot{A}^0+3\frac{\dot a}{a^3}\nabla A^0\right]=\mathbf{j} , \tag{3.6}$$

the overdot indicating a derivative with respect to time t.

Equations (3.6) may be put in a more transparent form. First, define new amplitudes $(B^0,\mathbf{B})$ to remove first-order time derivatives by extracting appropriate powers of the scale factor from the old ones:

$$B^0=a^{3/2}A^0 ,\quad \mathbf{B}=a^{5/2}\mathbf{A} . \tag{3.7}$$

The source may also be similarly redefined:

$$J^0=a^{3/2}j^0 ,\quad \mathbf{J}=a^{5/2}\mathbf{j} . \tag{3.8}$$

Next, it is convenient to work with the spatial three-dimensional Fourier transforms, for example,

$$B^0(x,t)=\int\frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{B}^0(k,t) . \tag{3.9}$$

Finally, we split $\tilde{\mathbf{B}}$ into longitudinal and transverse parts

$$\tilde{\mathbf{B}}=\frac{\tilde{\mathbf{B}}\cdot\mathbf{k}}{k^2}\mathbf{k}+\left[\tilde{\mathbf{B}}-\frac{\tilde{\mathbf{B}}\cdot\mathbf{k}}{k^2}\mathbf{k}\right]\equiv\tilde{B}_L\hat{\mathbf{k}}+\tilde{B}_T\hat{\mathbf{t}} , \tag{3.10}$$

where $\hat{\mathbf{k}}$ and $\hat{\mathbf{t}}$ are unit vectors along and perpendicular to $\mathbf{k}$, respectively. Equations (3.6) then lead to the three equations

$$\ddot{\tilde{B}}^0+\left[M^2+\frac{k^2}{a^2}-\alpha\right]\tilde{B}^0-2i\frac{\dot a}{a}\frac{k}{a}\tilde{B}_L -\left[1-\frac{1}{\xi}\right]\left[\ddot{\tilde{B}}^0-\alpha\tilde{B}^0+i\frac{k}{a}\dot{\tilde{B}}_L-\frac{5ik\dot a}{2a^2}\tilde{B}_L\right]=\tilde{J}^0 ,$$
$$\ddot{\tilde{B}}_L+\left[M^2+\frac{k^2}{a^2}-\beta\right]\tilde{B}_L-2i\frac{\dot a}{a}\frac{k}{a}\tilde{B}^0 +\left[1-\frac{1}{\xi}\right]\left[-\frac{k^2}{a^2}\tilde{B}_L+i\frac{k}{a}\dot{\tilde{B}}^0+\frac{3ik\dot a}{2a^2}\tilde{B}^0\right]=\tilde{J}_L , \tag{3.11}$$
$$\ddot{\tilde{B}}_T+\left[M^2+\frac{k^2}{a^2}-\beta\right]\tilde{B}_T=\tilde{J}_T ,$$

where α and β involve the scale factor and its derivatives,

$$\alpha=\frac{15}{4}\left[\frac{\dot a^2}{a^2}-\frac{2}{5}\frac{\ddot a}{a}\right] ,\quad \beta=\frac{1}{4}\left[2\frac{\ddot a}{a}-\frac{\dot a^2}{a^2}\right] . \tag{3.12}$$

With the metric (3.5), the relations (3.4) connecting the amplitudes with the Green's functions become

$$\tilde{A}^0(t)=i\int dt'\,a^3(t')[\tilde{G}^{00}(t,t')\tilde{j}^0(t')-a^2(t')\tilde{G}^{0i}(t,t')\tilde{j}^i(t')]\,,$$
$$\tilde{A}^i(t)=i\int dt'\,a^3(t')[\tilde{G}^{i0}(t,t')\tilde{j}^0(t')-a^2(t')\tilde{G}^{ij}(t,t')\tilde{j}^j(t')]\,. \tag{3.13}$$

Here and in the following we suppress for brevity the k dependence of the amplitudes and Green's functions.

It now proves very convenient to express the components of $\tilde{G}^{\mu\nu}$ in terms of O(3) scalars:

$$\tilde{G}^{00}(t,t')=\frac{g(t,t')}{a^{3/2}(t)a^{3/2}(t')}\,,$$
$$\tilde{G}^{0i}(t,t')=ik^i\frac{h(t,t')}{a^{3/2}(t)a^{5/2}(t')}\,,$$
$$\tilde{G}^{i0}(t,t')=ik^i\frac{\bar{h}(t,t')}{a^{5/2}(t)a^{3/2}(t')}\,, \tag{3.14}$$
$$\tilde{G}^{ij}(t,t')=\frac{1}{a^{5/2}(t)a^{5/2}(t')}[\delta^{ij}u(t,t')+k^ik^jv(t,t')]\,.$$

From (3.2) we immediately see that the scalar functions g, u, and v are unchanged under interchange of its argument, while h and $\bar{h}$ are related as

$$\bar{h}(t,t')=-h(t',t)\,. \tag{3.15}$$

Extracting the scale factor in (3.14) has the effect of simplifying relations (3.13) when written for B amplitudes:

$$\tilde{B}^0(t)=i\int dt'[g(t,t')\tilde{J}^0(t')-ih(t,t')\tilde{J}_L(t')]\,,$$
$$\tilde{B}_L(t)=i\int dt'\{ih(t,t')\tilde{J}^0(t')-[u(t,t')+v(t,t')]\tilde{J}_L\}\,,$$
$$\tilde{B}_T(t)=-i\int dt'\,u(t,t')\tilde{J}_T(t')\,. \tag{3.16}$$

It is now easy to obtain equations satisfied by the scalar Green's functions. We set in turn each of the source functions $\tilde{J}^0$, $\tilde{J}_L$, and $\tilde{J}_T$ equal to a δ function and the remaining two equal to zero. Then Eqs. (3.16) relate the $\tilde{B}$ amplitudes to the scalar Green's functions, turning Eqs. (3.11) for the former into those for the latter:

$$\ddot{g}+\left[M^2+\frac{k^2}{a^2}-\alpha\right]g+2\frac{\dot{a}}{a}\frac{k}{a}h-\left[1-\frac{1}{\xi}\right]\left[\ddot{g}-\alpha g-\frac{k}{a}\dot{h}+\frac{5\dot{a}k}{2a^2}h\right]=-i\delta(t-t')\,,$$
$$\ddot{h}+\left[M^2+\frac{k^2}{a^2}-\beta\right]h-2\frac{\dot{a}}{a}\frac{k}{a}g+\left[1-\frac{1}{\xi}\right]\left[-\frac{k^2}{a^2}h+\frac{k}{a}\dot{g}+\frac{3\dot{a}k}{2a^2}g\right]=0\,,$$
$$\ddot{\bar{h}}+\left[M^2+\frac{k^2}{a^2}-\alpha\right]\bar{h}-2\frac{\dot{a}}{a}\frac{k}{a}(u+v)-\left[1-\frac{1}{\xi}\right]\left[\ddot{\bar{h}}-\alpha\bar{h}-\frac{k}{a}(\dot{u}+\dot{v})-\frac{5\dot{a}k}{2a^2}(u+v)\right]=0\,, \tag{3.17}$$
$$\ddot{v}+\left[M^2+\frac{k^2}{a^2}-\beta\right]v+2\frac{\dot{a}}{a}\frac{k}{a}\bar{h}-\left[1-\frac{1}{\xi}\right]\left[\frac{k^2}{a^2}(u+v)+\frac{k}{a}\dot{\bar{h}}+\frac{3\dot{a}k}{2a^2}\bar{h}\right]=0\,,$$
$$\ddot{u}+\left[M^2+\frac{k^2}{a^2}-\beta\right]u=i\delta(t-t')\,.$$

IV. FINITE TEMPERATURE

Consider scalar electrodynamics to describe the contents of the early Universe. In a region well within the causal horizon, collisions establish an effective thermal equilibrium if they are frequent enough compared to the expansion rate of the Universe. The collision rate is given by [13]

$$\Gamma_{\rm coll}\sim e^4T\,, \tag{4.1}$$

while the expansion rate is

$$\Gamma_{\rm exp}=\frac{\dot{a}}{a}\sim\frac{\sqrt{\rho}}{m_P}\,, \tag{4.2}$$

ρ being the energy density and m_P the Planck mass. In the radiation-dominated phase (i.e., $T>M, M_1, M_2$) with N effective number of degrees of freedom [14],

$$\rho\sim NT^4\,. \tag{4.3}$$

So the condition that thermal equilibrium is established through collisions,

$$\Gamma_{\rm coll}\gg\Gamma_{\rm exp}\,, \tag{4.4}$$

gives rise to

$$T\ll\left[\frac{e^4}{\sqrt{N}}\right]m_P\,. \tag{4.5}$$

Thus, provided the temperature is less than the Planck mass by the factor $e^4/\sqrt{N}$, thermal equilibrium is maintained in the radiation-dominated, expanding universe.

The important point to note here [15] is that the condition (4.5) of thermal equilibrium leads naturally to an adiabatic condition. Remembering that, at high enough temperatures [16],

$$M^2 \sim e^2 T^2 \ , \tag{4.6}$$

and using (4.2) and (4.3), this condition can be expressed as

$$M^2 e^6 \gg \left[\frac{\dot a}{a}\right]^2 . \tag{4.7}$$

For $e < 1$, the adiabatic condition

$$M^2 \gg \left[\frac{\dot a}{a}\right]^2 , \frac{\ddot a}{a} \tag{4.8}$$

is thus very comfortably satisfied during the radiation-dominated era, allowing positive- and negative-frequency mode functions to be defined even for large wavelengths.

At a time t_0 during the thermal equilibrium, when the temperature is $T_0 = 1/\beta_0$, the thermal state of the system is described by the density matrix

$$\rho = \frac{e^{-\beta_0 H(t_0)}}{\operatorname{tr} e^{-\beta_0 H(t_0)}} \ , \tag{4.9}$$

where $H(t)$ is the Hamiltonian of the system. In the Feynman path-integral formulation, the expectation value of the field $A_\mu(x)$, for example, is given by

$$\operatorname{tr} e^{-\beta_0 H(t_0)} A_\mu(\mathbf{x},t) = \int [d\phi_1][d\phi_2][dA_\mu] \exp\left[i \int_c d\tau \int d^3\mathbf{x} (\mathcal{L}_0 + \mathcal{L}_1)\right] A_\mu(\mathbf{x},t) \ , \tag{4.10}$$

where the contour C in the complex time plane (Fig. 1) is obtained by deforming the Euclidean periodicity interval of the imaginary-time formulation so as to include the real-time axis. On the segments C_1 and C_2, $\tau = t$, while on C_3, $\tau = t_0 - it$. Note that on the third segment the scale factor is fixed at $t = t_0$.

As discussed in detail in Ref. [7], the instantaneous thermalization associated with the density matrix (4.9) causes additional, nonrenormalizable short-distance singularities to be present in the propagator. A simple remedy to this problem suggested there is to thermalize the system in a fictitious, static geometry prior to t_0 and then connect it smoothly $[\dot a(t_0) = \ddot a(t_0) = 0]$ to the actual geometry around the time $t = t_0$ and show later that this deformation of the metric has no effect on the subsequent dynamics of the system. As a bonus, the mode functions and their derivatives belonging to the second and third segments can be chosen to be continuous at the junction of these segments, allowing us to construct extended mode functions on the entire contour C [17]. The fields in the path integral are also assumed to have the same properties.

Furthermore, the trace in (4.10) requires

$$\begin{aligned} A_\mu^{(1)}(\mathbf{x}, \tau = t_0) &= A_\mu^{(3)}(\mathbf{x}, \tau = t_0 - i\beta_0) \ , \\ \dot A_\mu^{(1)}(\mathbf{x}, \tau = t_0) &= i \dot A_\mu^{(3)}(\mathbf{x}, \tau = t_0 - i\beta_0) \ , \end{aligned} \tag{4.11}$$

and similar conditions on $\phi_{1,2}$ fields. The superscripts in (4.11) denote the segments to which the field belongs. Because of the these continuity conditions, the boundary terms mentioned in Sec. II vanish and the operator $D_{\mu\nu}$ is indeed self-adjoint.

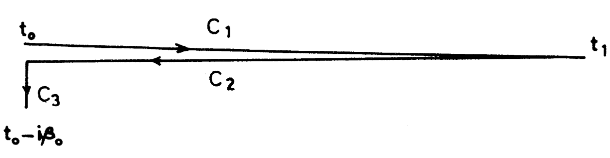

FIG. 1. Complex time contour.

The first of Eqs. (3.1) for the gauge-field Green's function is now replaced by

$$D_{\mu\nu}(\mathbf{x},\tau) G^{\nu\lambda}(\mathbf{x},\tau,\tau') = \delta_\mu^\lambda \delta^3(\mathbf{x}) \delta(\tau - \tau') \ , \tag{4.12}$$

which again gives rise to Eqs. (3.17) with t replaced by τ. The finite-temperature boundary conditions on $G^{\nu\lambda}$ follow from those on A_μ in (4.11).

We now obtain the gauge-field propagator under the condition (4.5) that thermal equilibrium holds. It will be observed that the condition (4.8) is equivalent to ignoring all gravitational couplings. The set of equations (3.17) then reduces to those on flat space-time, with k replaced by $k/a(t)$. These are $[\omega^2 = M^2(t) + k^2/a^2(t)]$,

$$\ddot g + \omega^2 g - \left[1 - \frac{1}{\xi}\right]\left[\ddot g - \frac{k}{a}\dot h\right] = -i\delta(\tau - \tau') \ , \tag{4.13}$$

$$\ddot h + \omega^2 h + \left[1 - \frac{1}{\xi}\right]\frac{k}{a}\left[\dot g - \frac{k}{a} h\right] = 0 \ , \tag{4.14}$$

$$\ddot{\bar h} + \omega^2 \bar h - \left[1 - \frac{1}{\xi}\right]\left[\ddot{\bar h} - \frac{k}{a}(\dot u + \dot v)\right] = 0 \ , \tag{4.15}$$

$$\ddot v + \omega^2 v - \left[1 - \frac{1}{\xi}\right]\frac{k}{a}\left[\dot{\bar h} + \frac{k}{a}(u + v)\right] = 0 \ , \tag{4.16}$$

$$\ddot u + \omega^2 u = i\delta(\tau - \tau') \ . \tag{4.17}$$

Here and below the overdot denotes a derivative with respect to complex time.

To solve this set of equations, we first construct, following Ref. [7], a set of mode functions extended over all the three segments of the time contour C $[\omega_0^2 = M^2(t_0) + k^2/a^2(t_0)]$:

$$f^{\pm}(\tau)\equiv f_a^{\pm}(t)=\begin{cases}\left[\dfrac{\omega_0}{\omega(t)}\right]^{1/2}\exp\left[i\displaystyle\int_{t_0}^{t}\frac{dt'}{\omega(t')}\right], & a=1,2\ ,\\ e^{\mp\omega_0 t}\ , & a=3\ .\end{cases} \tag{4.18}$$

We then introduce the basic Green's function

$$\Delta(\tau,\tau')=f^{+}(\tau)f^{-}(\tau')[\theta(\tau-\tau')+B] + f^{+}(\tau')f^{-}(\tau)[\theta(\tau'-\tau)+B]\ , \tag{4.19}$$

where

$$B=(e^{\beta_0\omega_0}-1)^{-1}\ . \tag{4.20}$$

We also introduce another set of two extended mode functions, denoted by $f^{\pm}(\tau)$, which are identical to (4.18) except for ω replaced by $\varpi=(\xi M^2+k^2/a^2)^{1/2}$, with its corresponding Green's function $\bar{\Delta}(\tau,\tau')$. They satisfy

$$\ddot{\Delta}(\tau,\tau')+\omega^2\Delta(\tau,\tau')\simeq -i\delta(\tau-\tau')\ , \tag{4.21}$$

and a similar one for $\bar{\Delta}$. Here and in the following the $\simeq$ sign implies equality to leading order under condition (4.8); i.e., we ignore $\dot{a}/a$ in comparison to ω.

By construction, Δ and $\bar{\Delta}$ satisfy thermal boundary conditions [7]. We now show that g, h, $\bar{h}$, u, and v can be solved in terms of them without adding any further homogeneous solutions. Let

$$g=\Delta+\gamma\ . \tag{4.22}$$

Then Eq. (4.13) gives an equation for γ:

$$\ddot{\gamma}+\omega^2\gamma-\left[1-\frac{1}{\xi}\right]\left[\ddot{g}-\frac{k}{a}\dot{h}\right]\simeq 0\ . \tag{4.23}$$

Differentiating (4.14) once and combining it with (4.23), we get

$$\left[\frac{k}{a}\gamma+\dot{h}\right]^{\cdot\cdot}+\omega^2\left[\frac{k}{a}\gamma+\dot{h}\right]\simeq 0\ . \tag{4.24}$$

If we do not add homogeneous solutions, we get

$$\gamma\simeq-\frac{a}{k}\dot{h}\ . \tag{4.25}$$

Inserting this relation in (4.14), we get an inhomogeneous equation for h:

$$\ddot{h}+\varpi^2 h\simeq-(\xi-1)\frac{k}{a}\dot{\Delta}\ . \tag{4.26}$$

Comparing it with the identity

$$(\Delta-\bar{\Delta})^{\cdot\cdot}+\varpi^2(\Delta-\bar{\Delta})\simeq(\xi-1)M^2\Delta\ , \tag{4.27}$$

we find that

$$h\simeq-\frac{1}{M^2}\frac{k}{a}(\Delta-\bar{\Delta})^{\cdot}\ . \tag{4.28}$$

Equations (4.15) and (4.16) may be treated in a similar way, obtaining

$$\bar{h}\simeq\frac{a}{k}\dot{v} \tag{4.29}$$

and

$$v\simeq-\frac{1}{M^2}\frac{k^2}{a^2}(\Delta-\bar{\Delta})\ . \tag{4.30}$$

Collecting the solutions, we have

$$\begin{aligned} g&\simeq\Delta+\frac{1}{M^2}(\Delta-\bar{\Delta})^{\cdot\cdot}\ ,\\ h&\simeq\bar{h}\simeq-\frac{1}{M^2}\frac{k}{a}(\Delta-\bar{\Delta})^{\cdot}\ ,\\ v&\simeq-\frac{1}{M^2}\frac{k^2}{a^2}(\Delta-\bar{\Delta})\ ,\\ u&\simeq-\Delta\ . \end{aligned} \tag{4.31}$$

With relations (3.14), this completes our expression for the real-time thermal gauge-field propagator during thermal equilibrium in a radiation-dominated early universe.

V. CONCLUSION

In this work we set up a perturbation expansion for a U(1) gauge field theory with Higgs scalars in the early Universe and calculate the finite-temperature free gauge-field propagator in the real-time formalism. We also write out all the interaction terms in the action which may be taken into account by a perturbation expansion.

Our derivation of the gauge-field propagator is valid from a time when the temperature starts satisfying the condition (4.5) for thermal equilibrium in the era of radiation domination. Its form reduces effectively to that on flat space-time. We emphasize that the adiabatic condition (4.8) used in the derivation is a direct consequence of thermal equilibrium condition (4.5). There is a physical inaccuracy inherent in the formulation of our problem in terms of the approximate notion of thermal equilibrium of the contents of the universe on a time-dependent geometry. The higher-order adiabatic terms which one encounters in solving our equations exactly are of the same order as this inaccuracy. Clearly, it does not make sense to "correct" our result by such terms.

In the course of time, the universe may enter a different era. If thermal equilibrium holds in the new era, we may construct the gauge-field propagator afresh without reference to our present construction. But the possibility of obtaining an adiabatic condition need be examined. If, on the other hand, thermal equilibrium is lost in the new era, we may write the general form of the propagator involving arbitrary constants, which may be fixed by matching it with the present form in a region where both forms retain their (approximate) validity.

The immediate use of the propagator we have found here is to evaluate the components of the energy-momentum tensor to lowest order in the couplings. The next simplest task is to evaluate the one-loop diagrams contributing to the effective masses.

[1] See, e.g., N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1984).

[2] See, e.g., E. Kolb and M. S. Turner, *The Early Universe,* Frontiers in Physics Vol. 69 (Addison-Wesley, Readwood City, CA, 1990).

[3] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955); R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965); D. A. Kirzhnits and A. D. Linde, Phys. Lett. **42B**, 471 (1972); L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974); S. Weinberg, *ibid.* **9**, 3357 (1974).

[4] A. J. Niemi and G. W. Semenoff, Ann. Phys. (N.Y.) **152**, 105 (1984); Nucl. Phys. **B230** [FS10], 191 (1984); G. W. Semenoff and H. Umezawa, *ibid.* **B220** [FS8], 196 (1983); H. Matsumoto, Y. Nakano, and H. Umezawa, J. Math. Phys. **25**, 3076 (1984), and reference cited therein.

[5] I. T. Drummond, Nucl. Phys. **B190** [FS3], 93 (1981); B. L. Hu, Phys. Lett. **B108**, 19 (1982); A. Chowdhury and S. Mallik, Phys. Rev. D **36**, 1259 (1987).

[6] G. Semenoff and N. Weiss, Phys. Rev D **31**, 689 (1985); 699 (1985).

[7] H. Leutwyler and S. Mallik, Ann. Phys. (N.Y.) **205**, 1 (1991).

[8] K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D **6**, 2923 (1972).

[9] A term $\xi R\phi^\dagger\phi$ coupling the fields ϕ to the scalar curvature R with strength ξ is also allowed in the action by renormalizability. Its incorporation presents no difficulty.

[10] The question of boundary conditions at finite temperature is discussed in Sec. IV.

[11] N. Banerjee and S. Mallik, Phys. Rev. D **43**, 3368 (1991).

[12] $\bar{\phi}$ can now depend on time only.

[13] We assume λ is small compared to g.

[14] N may count all the fields in a realistic theory.

[15] We wish to point out a difference in dealing with the equilibrium and adiabatic conditions here compared to what is done in Ref. [7]. Here we assume that the equilibrium condition is satisfied and then derive the latter, while in [7] the main discussion is based on the adiabatic condition only.

[16] The masses at finite temperature are determined by the loop diagrams in the perturbation expansion, which we do not calculate in this work. It would, however, be evident in the following that the usual flat-space-time results for these masses should hold.

[17] Such continuity conditions at the junction of C_1 and C_2 are trivially satisfied since the differential operators belonging to these segments are the same.