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$\pi\eta$ scattering in QCD

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We evaluate the $\pi\eta$ scattering amplitude at next-to-leading order in chiral perturbation theory. We discuss the conventional approach of the pseudoscalar Goldstone bosons (π,K,η) only and an extension including explicit resonance fields. The contributions of these resonances saturate the low-energy constants at order E^4. We present predictions for the scattering lengths and phase shifts of the low partial waves.

Green's functions of quark currents can be analyzed at low energies in the framework of chiral perturbation theory (CHPT) as dictated by chiral symmetry and its spontaneous breakdown. It is most economical to make use of an effective Lagrangian involving the pseudoscalar Goldstone bosons [1] eventually accompanied by explicit resonance degrees of freedom [2,3]. In the first version (conventional CHPT), one evaluates physical observables for a given loop order which is equivalent to a systematic expansion in external momenta and quark masses. In most cases, the one-loop approximation suffices (i.e., taking terms up to and including order E^4). Whenever resonances can contribute to a process under consideration, the one-loop approximation encounters a barrier. This naturally leads one to include the low-lying resonance multiplets. Chiral symmetry dictates the form of the interaction vertices between the Goldstone pseudoscalars and the resonances at leading order (E^2). At order E^4, it can be demonstrated that the momentum-independent parts of the resonance propagators generate polynomial terms whose coefficients (low-energy constants) are very close to the empirically determined values of these constants for the conventional approach (working at the resonance scale, $\mu=M_\rho$) [2]. Taking into account the full resonance propagators, one includes some terms of order E^6 and higher. In Ref. [4] it has been shown that with the inclusion of the resonances the chiral predictions for $\pi\pi$ [5–7] and πK [8] scattering are in fact improved. In the following, we will be concerned with the elastic $\pi\eta$ scattering amplitude in chiral $SU(3)_L\times SU(3)_R$. We will consider the conventional CHPT at next-to-leading order as well as the enlarged effective Lagrangian with explicit resonance degrees of freedom. In the absence of any phase-shift analysis in the $\pi\eta$ system, we will predict the scattering lengths and phase shifts of the low partial waves in what follows. Even if it will take a long time before accurate phases might be available, we believe that our results are useful for the model builders.

In the presence of external sources of scalar, pseudoscalar, vector, and axial-vector type (s,p,v,a), the QCD generating functional at next-to-leading order takes the form

$$\mathcal{Z}[U,s,p,v,a]=\mathcal{Z}_2+\mathcal{Z}_4^T+\mathcal{Z}_4^P+\mathcal{Z}_4^U+\mathcal{Z}_4^{\mathrm{WZ}} . \quad (1)$$

Here, $U=\exp[i\Phi/F_0]$ embodies the eight pseudoscalar Goldstone bosons with F_0 the pion decay constant in the chiral limit. The subscripts 2 and 4, respectively, refer to the leading (tree-level) and next-to-leading (one-loop) order. It is common (but not necessary) to split the one-loop contribution into terms of tadpole type $(\mathcal{Z}_4^T)$, the unitarity corrections $(\mathcal{Z}_4^U)$ and the polynomial terms of order E^4 $(\mathcal{Z}_4^P)$. $\mathcal{Z}_4^{\mathrm{WZ}}$ is the Wess-Zumino action which accounts for the chiral anomalies. The local action related to $\mathcal{Z}_4^P$ is accompanied by a set of *a priori* undetermined low-energy constants. They have to be determined from phenomenology [1,9]. The external scalar source $s(x)$ includes the quark mass matrix, $s(x)\sim\mathcal{M}=\mathrm{diag}(m_u,m_d,m_s)$. Considering elastic (on-shell) $\pi\eta$ scattering, one can omit the external sources and expand the generating functional to fourth order in the physical meson fields ϕ_P $(P=\pi,K,\eta)$ [10]. In this approach, one has to keep track of the wave-function renormalization of the external legs which is given by the physical meson decay constant F_P. In what follows, we will refer to the generating functional (1) as conventional CHPT at next-to-leading order. It is given in explicit form in Ref. [1].

As has been demonstrated in Ref. [2], to order E^4 there is a strict correspondence between $\mathcal{Z}[U,s,p,v,a]$ and the generating functional $\mathcal{Z}_R[U,R,s,p,v,a]$:

$$\mathcal{Z}_R[U,R,s,p,v,a]=\mathcal{Z}[U,s,p,v,a]-\mathcal{Z}_4^P[U,s,p,v,a]+\mathcal{Z}^{\mathrm{res}}[U,R,s,p,v,a] . \quad (2)$$

Here, R stands for resonance nonets (octets) of scalar (S), non-Goldstone pseudoscalar (P), vector (V), and axial-vector (A) type. To leading order (E^2), the interaction Lagrangian of the resonances with the Goldstones is fixed by chiral symmetry [2]. In fact, the resonance exchange contributes at order E^4 (and higher) to the effective action. The pertinent observation made in Ref. [2] was that at the typical resonance scale (say $\mu=M_\rho$) resonance exchange generates a polynomial piece of order E^4 with low-energy constants that agree to a high degree of accuracy with the phenomenologically determined ones in the conventional approach (resonance saturation). This means that there is a strict one-to-one correspondence between the generating functionals (1) and (2) at order E^4. One can then go further and take seriously the momentum-dependent part of the resonance propagator which starts out at order E^6. This was done in Ref. [4] and it was shown that the chiral predictions for the channels in which various resonances contribute are significantly improved. The prime examples are the P-wave phase shifts $\delta_1^1(s)$ and $\delta_1^{1/2}(s)$ for $\pi\pi$ and πK scattering, respectively, which are dominated by ρ and K^* exchange. In what follows, we will present results using the conventional approach at next-to-leading order (1) and also the effective action (2) including the resonances for the $\pi\eta$ system. While the latter has the advantages discussed before, in the conventional approach there is a cleaner way of organizing contributions according to their chiral power $E^{2(n+1)}$ (with n the number of loops) and estimating the error bars of the chiral prediction.

The $\pi\eta$ scattering amplitude takes a simple form in terms of one Lorentz-invariant amplitude depending on the Mandelstam variables since the η has isospin zero:

$$T^{ab}(s,t,u)=\delta^{ab}T_{\pi\eta}(s,t,u)\ . \tag{3}$$

The amplitude $T_{\pi\eta}(s,t,u)$ is furthermore crossing symmetric, $T_{\pi\eta}(s,t,u)=T_{\pi\eta}(u,t,s)$, since the η is its own antiparticle. Let us briefly comment on the effect of $\pi^0\eta$ mixing. The physical pion and eta are not eigenstates of the generators λ_3 and λ_8, respectively, but there is mixing, i.e., $\lambda_{\pi^0}=\lambda_3\cos\epsilon+\lambda_8\sin\epsilon$ and $\lambda_\eta=\lambda_8\cos\epsilon-\lambda_3\sin\epsilon$. The mixing angle ϵ can be expressed in terms of the quark masses [1]:

$$\tan 2\epsilon=\frac{\sqrt{3}}{2}\frac{m_d-m_u}{m_s-\hat{m}},\quad \hat{m}=\frac{1}{2}(m_u+m_d)\ . \tag{4}$$

Its numerical value is $\epsilon\simeq 8\times 10^{-3}$ [12]. By simple G-parity arguments, one can show that the induced isospin-breaking amplitude is of $O(\epsilon^2)$ and therefore (strongly) suppressed. We have checked this also by a direct calculation of the $O(\epsilon)$ contribution to the $\pi^0\eta$ scattering process.

Considering now the specific process $\pi^0(p_1)+\eta(p_2)\to\pi^0(p_3)+\eta(p_4)$ we calculate from the effective action (1)

$$\begin{aligned}T_{\pi\eta}(s,t,u)=\frac{1}{F_\pi^2F_\eta^2}\Bigg\{&\frac{F_0^2}{3}(1-3\mu_\pi-2\mu_K-\tfrac{1}{3}\mu_\eta)(M_\pi^0)^2+8(L_1^r+\tfrac{1}{6}L_3^r)(t-2M_\pi^2)(t-2M_\eta^2)\\&+4(L_2^r+\tfrac{1}{3}L_3^r)[(s-M_\pi^2-M_\eta^2)^2+(u-M_\pi^2-M_\eta^2)^2]+8L_4^r[(t-2M_\pi^2)(M_\eta^0)^2+(t-2M_\eta^2)(M_\pi^0)^2]\\&-\tfrac{8}{3}L_5^r(M_\pi^2+M_\eta^2)(M_\pi^0)^2+8L_6^r(M_\pi^0)^2[(M_\pi^0)^2+5(M_\eta^0)^2]+32L_7^r(M_\pi^0)^2[(M_\pi^0)^2-(M_\eta^0)^2]\\&+\tfrac{64}{3}L_8^r(M_\pi^0)^4+\frac{(M_\pi^0)^4}{9}[J_{\pi\eta}^r(s)+J_{\pi\eta}^r(u)]+\tfrac{3}{8}J_{KK}^r(s)[s-M_\pi^2-M_\eta^2+\tfrac{2}{3}(M_\pi^0)^2]^2\\&+\tfrac{3}{8}J_{KK}^r(u)[u-M_\pi^2-M_\eta^2+\tfrac{2}{3}(M_\pi^0)^2]^2+\frac{(M_\pi^0)^2}{3}J_{\pi\pi}^r(t)[t-2M_\pi^2+\tfrac{3}{2}(M_\pi^0)^2]\\&+\tfrac{2}{9}J_{\eta\eta}^r(t)(M_\pi^0)^2[(M_\eta^0)^2-\tfrac{1}{4}(M_\pi^0)^2]+\tfrac{1}{8}J_{KK}^r(t)[t-2M_\pi^2+2(M_\pi^0)^2]\\&\times[3t-6M_\eta^2+4(M_\eta^0)^2-\tfrac{2}{3}(M_\pi^0)^2]\Bigg\}\ .\end{aligned} \tag{5}$$

Here, M_P^0 denotes the meson masses at leading order E^2. The loop functions J_{PQ}^r and quantities μ_P ($P,Q=\pi,K,\eta$) are explicitly given in Ref. [1]. Notice the appearance of the low-energy constant L_7^r, which does not appear in either $\pi\pi$ or πK scattering. As a nontrivial check one can demonstrate that the amplitude (4) does not depend on the scale parameter μ (of dimensional regularization) entering in the functions J_{PQ}^r and μ_P as well as in the low-energy constants L_i^r. For the values of these constants, we take the central values of Refs. [1] and [10]. For the η decay constant, we use the chiral prediction $F_\eta=1.3F_\pi$ [1]. If one uses instead the effective action including the resonances, one has to substitute in Eq. (4) the polynomial piece proportional to the L_i^r ($i=1,\ldots,8$) by the resonance exchange contribution:

$$\begin{aligned}T_{\pi\eta}^{\rm res}(s,t,u)=\frac{1}{F_\pi^2F_\eta^2}\Bigg\{&\frac{4}{M_{S_1}^2-t}[\tilde{c}_d(t-2M_\pi^2)+2\tilde{c}_m(M_\pi^0)^2][\tilde{c}_d(t-2M_\eta^2)+2\tilde{c}_m(M_\eta^0)^2]\\&+\frac{4\tilde{c}_m^{\,2}}{M_{S_1}^2}(M_\pi^0)^2[(M_\pi^0)^2+(M_\eta^0)^2]+\frac{2}{3(M_S^2-s)}[c_d(s-M_\pi^2-M_\eta^2)+2c_m(M_\pi^0)^2]^2\\&+\frac{2}{3(M_S^2-u)}[c_d(u-M_\pi^2-M_\eta^2)+2c_m(M_\pi^0)^2]^2+\frac{4c_m^2}{3M_S^2}(M_\pi^0)^2[(M_\pi^0)^2-(M_\eta^0)^2]\\&-\frac{2}{3(M_S^2-t)}[c_d(t-2M_\pi^2)+2c_m(M_\pi^0)^2]\{c_d(t-2M_\eta^2)+2c_m[2(M_\eta^0)^2-(M_\pi^0)^2]\}\\&+\frac{16\tilde{d}_m^2}{M_{\eta_1}^2-M_\eta^2}(M_\pi^0)^2[(M_\eta^0)^2-(M_\pi^0)^2]\Bigg\}\ .\end{aligned} \tag{6}$$

Only scalar exchange and $\eta\eta_1$ mixing (with η_1 the flavor-singlet non-Goldstone pseudoscalar) contribute to the $\pi\eta$ scattering amplitude. The parameters c_d, $\tilde{c}_d$, c_m, $\tilde{c}_m$, and $\tilde{d}_m$ have been determined in Ref. [2]. We use these together with $M_S = M_{S_1} = 983$ MeV. To check the stability of our results, we will also vary the scalar mass in the chiral limit. Possible contributions from higher-lying tensor mesons will not be discussed (we only consider cms energies $\sqrt{s} \leq 1$ GeV). With Eqs. (5) and (6), the chiral expansion of the $\pi\eta$ scattering amplitude at next-to-leading order (and beyond for the resonance case) is completely determined. At leading order (E^2), one can identify $F_0 = F_\pi = F_\eta$ and $M_\pi = M_\pi^0$ and recover the current-algebra (CA) amplitude of order E^2:

$$T_{\pi\eta}^{(2)}(s,t,u) = \frac{M_\pi^2}{3F_\pi^2} \ . \tag{7}$$

To leading order, the $\pi\eta$ scattering amplitude is a constant which vanishes in the chiral limit $M_\pi = 0$ [11].

The pertinent kinematical relations for calculating partial-wave amplitudes and phase shifts in the $\pi\eta$ system are particularly simple since no isospin projection is involved. The total isospin (in the s channel) is always $I=1$. For a given relative angular momentum l the partial-wave amplitude reads

$$t_l(s) = \frac{1}{32\pi} \int_{-1}^{+1} dz \, P_l(z) T_{\pi\eta}(s,t,u) \ , \tag{8}$$

with

$$t = -2q^2(1-z) \ ,$$

$$q = \sqrt{[s-(M_\pi+M_\eta)^2][s-(M_\eta-M_\pi)^2]}/(2\sqrt{s}) \ ,$$

$$u = 2(M_\pi^2 + M_\eta^2) - s - t \ ,$$

and

$$z = \cos\theta$$

related to the scattering angle in the center-of-mass system. The corresponding phase shifts follow from

$$\delta_l(s) = \arctan\left[\frac{2q}{\sqrt{s}} \operatorname{Re} t_l(s)\right] \ . \tag{9}$$

In principle, for the conventional approach (to one-loop order) one could expand the arctan in Eq. (9) and get a simpler form for $\delta_l(s)$ which differs only at order E^6 from the one given. To properly unitarize the resonance exchange contributions, however, it is mandatory to use the full form, Eq. (9), for calculating the phase shifts (for a detailed discussion, see Ref. [4]). As $q \to 0$ ($\sqrt{s} \to M_\pi + M_\eta$), we define the scattering lengths by

$$\operatorname{Re} t_l^I(s) = \frac{\sqrt{s}}{2} q^{2l} [a_l^I + O(q^2)] \ . \tag{10}$$

Notice that the a_l are normalized such that they have di mension [mass $]^{-2l-1}$. Results will be given in appropriate powers of the inverse charged pion mass. One might be worried about the influence of inelasticities, here, the coupling to the $K\bar{K}$ channel in the S wave. In CHPT, this is a three-loop effect. Therefore, to the order we are working, we cannot consistently include it.

In Table I, we give the results for the scattering lengths

TABLE I. Scattering lengths. We give the current-algebra (CA) results in comparison with the one-loop CHPT predictions (CHPT) and the Lagrangian including explicit resonance fields (Res.). The low-energy constants and the resonance parameters are taken from Ref. [1] and Ref. [2] for the CHPT and Res. cases, respectively.

	CA	CHPT	Res.
a_0	6.0×10^{-3}	7.2×10^{-3}	4.9×10^{-3}
a_1	0	-5.2×10^{-4}	-1.5×10^{-4}
a_2	0	-2.1×10^{-5}	$+1.3\times10^{-5}$

in the S, P, and D waves, for the lowest order (current algebra), the next-to-leading order (CHPT) using the low-energy constants given in [1] and the resonance case using the parameters pinned down in Ref. [2]. All scattering lengths turn out to be rather sensitive to the low-energy constants. Of particular interest is the dependence on L_7^r since this constant does not appear in the $\pi\pi$ or πK scattering amplitudes. For the one-loop CHPT result, we find $a_0 = 9.0\times10^{-3}$ to 5.4×10^{-3} for $L_7^r = -5.5\times10^{-4}$ to -2.5×10^{-4} (these are the error bars on L_7^r given in Ref. [1]). If one keeps the low-energy constants on their central values and varies the scalar mass $M_S = M_{S_1}$ between 850 and 1100 MeV, the S-wave scattering length a_0 changes from 7.3×10^{-3} to 4.1×10^{-3}. This strong dependence is due to the fact that the position of the scalar resonance is only 300 MeV above the threshold. One should therefore not attach any deep physical significance to these strong variations. The P- and D-wave scattering lengths are much less affected by changes in the scalar mass. It is obvious from the resonance contribution (6) to the scattering amplitude $T_{\pi\eta}(s,t,u)$ that there is scalar exchange in the t channel which carries isospin zero. This resonance would have to be identified with the $f_0(975)$, but we have not treated mass splittings within the SU(3) resonance multiplets. These corrections are of order E^6 (and higher) and have not been investigated yet.

In Fig. 1, we show the S-wave phase shift $\delta_0(s)$. For the conventional case, it is a slowly increasing function as $\sqrt{s}$ increases, whereas for the resonance Lagrangian the explicit $a_0(983)$ exchange is of course visible. Up to $\sqrt{s} = 1$ GeV, the phase $\delta_0(s)$ is similar to a Breit-Wigner form with the parameters dictated by the mass and width of the $a_0(983)$ resonance. This is different to the calculation of Weinstein and Isgur [13] based on the $K\bar{K}$-molecule model for the scalars. For the P and D waves, we prefer to work with the functions

$$K_l(s) = \frac{1}{q^{2l+1}} \tan\delta_l(s) \ , \tag{11}$$

which approach the respective scattering lengths as $\sqrt{s} \to M_\pi + M_\eta$. These are shown in Fig. 2 for the conventional case. For the resonance case, the energy dependence is the same, but the absolute values are different. For the D wave, one might think of coupling explicit tensor mesons, but this goes beyond the scope of the present paper. For the average mass of these tensor mesons ($M_T \simeq 1500$ MeV), one does not expect a significant

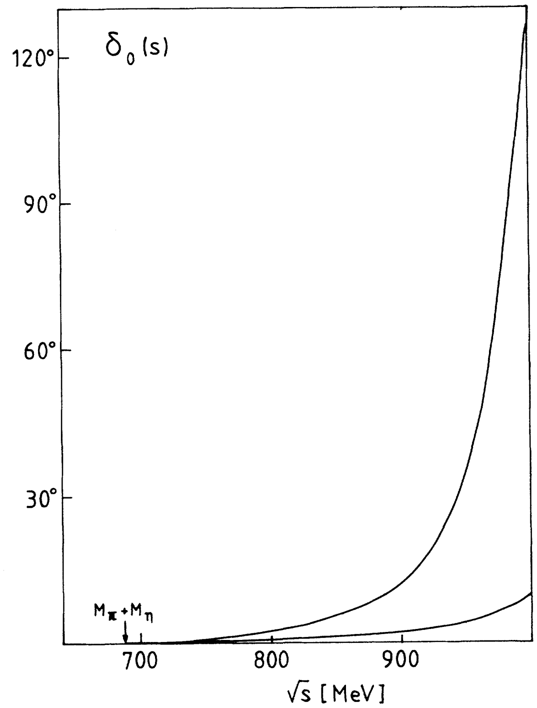

FIG. 1. S-wave phase shift $\delta_0(s)$ for $\sqrt{s} \leq 1$ GeV. The lower solid line gives the result for the one-loop CHPT and the upper solid line the one for the chiral Lagrangian with explicit resonance fields.

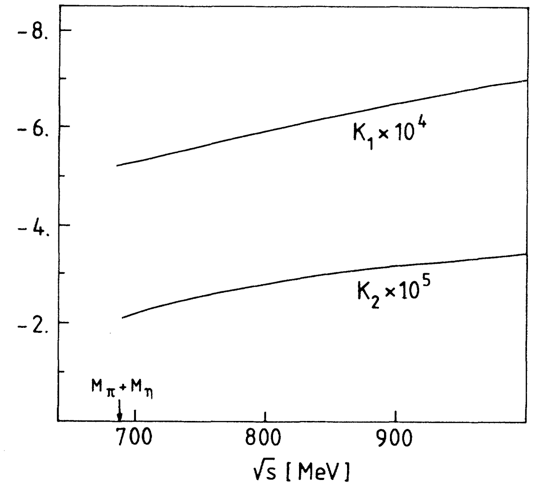

FIG. 2. The P- and D-wave K functions for the conventional one-loop CHPT.

change of $K_2(s)$ for $\sqrt{s} \leq 1$ GeV.

In this paper, we have considered elastic $\pi\eta$ scattering. It is a very interesting dynamical process involving mesons of unequal masses and strange-quark–antiquark pairs (similar to the case of πK scattering). At present, no phase-shift analysis based on the existing data in the $\pi\eta$ system exists. We therefore predict the scattering lengths and low-energy behavior ($\sqrt{s} \leq 1$ GeV) of the S, P, and D waves. We have demonstrated that isospin violation stemming from the quark mass difference $m_d - m_u$ is absent to linear order and can therefore be neglected. In the $I=1$ S wave, the presence of the $a_0(983)$ resonance strongly influences the elastic phase shift down to threshold, $\sqrt{s} = M_\pi + M_\eta \simeq 690$ MeV. The calculation presented here has been done in the same framework as the determination of the low-energy Green's functions of QCD for $\pi\pi$ and πK scattering with no free (adjustable) parameters. The experimental confirmation of these $\pi\eta$ threshold parameters would provide an interesting and necessary check on our understanding of the low-energy structure of QCD which is governed by the implications of chiral symmetry violation. Considering the present status of the data in the $\pi\eta$ system, we believe that our investigation is of use for the model builders to implement the pertinent constraints set by chiral symmetry.

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