# Quantum creation of topological defects during inflation

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Circular loops of string and spherical domain-wall bubbles of radius equal to the horizon can spontaneously nucleate in de Sitter space. These objects are expanded by the subsequent inflation, and by the end of the inflationary era they have a spectrum of sizes extending well beyond the present Hubble length. Monopole-antimonopole pairs with an initial separation equal to the horizon can also be produced. The cosmological implications of these effects are briefly discussed.

Cosmological inflation is a period of exponential expansion in the early history of the Universe [1]. It increases the size of the Universe by an enormous factor, so that all the presently observable space comes from a tiny initial region. Topological defects such as monopoles, strings, and domain walls, if they were formed before (or at early stages of) inflation, are inflated away and are never seen again. This seems to lead to the inevitable conclusion that the only defects we can hope to see now are those arising from phase transitions which occurred after (or near the end [2, 3] of) inflation. Here it will be argued that monopoles, strings, and domain walls can be continuously formed during inflation by quantum-mechanical tunneling. These processes are similar to the quantum production of particles in de Sitter space, first described by Gibbons and Hawking [4], and they are also similar to the quantum production of bubbles of false vacuum, as described by Lee and Weinberg [5]. As a result, topological defects corresponding to preinflationary phase transitions can still be present after inflation with appreciable densities, as long as the energy scale of these topological defects is not too far above the energy scale of inflation. Under these circumstances defects can also be produced by the post-inflationary reheating, but there is an important difference: the defects produced by reheating are generally no larger than the Hubble length at that time, and they disappear quickly. The defects produced during inflation, however, are stretched by the inflation and have a spectrum of sizes that can extend well beyond the present Hubble length.

In the next section we shall explain the physics of quantum string, wall, and monopole nucleation in de Sitter space, ignoring at first the gravitational effects of the nucleated objects. In Sec. II we will describe the classical evolution of a nucleated defect, and in Sec. III we will discuss the cosmological implications of string, wall, and monopole production. In Sec. IV we will return to the physics of nucleation, this time including the gravitational effects.

## I. NUCLEATION OF TOPOLOGICAL DEFECTS

The metric of de Sitter space, which describes the inflating universe, can be represented in several different ways [6]. Here it will be convenient to use the static form of the metric,

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) , \tag{1.1}$$

where

$$f(r) = 1 - H^2r^2 . \tag{1.2}$$

Let us consider first a circular loop of string in de Sitter space. The string world sheet can be parametrized by time $t$ and angle $\theta$, and the world-sheet metric can be written as

$$ds_2^2 = [f(R) - f^{-1}(R)\dot{R}^2]dt^2 - R^2\, d\theta^2 , \tag{1.3}$$

where $R(t)$ is the loop radius. The string action is proportional to the world-sheet area [7]:

$$S = -\mu \int dt\, d\theta \sqrt{-\gamma} = -2\pi\mu \int dt\, R\sqrt{f - f^{-1}\dot{R}^2} , \tag{1.4}$$

where $\gamma$ is the determinant of the metric (1.3), and $\mu$ is the string tension.

The loop dynamics is most conveniently studied using the conserved energy

$$E = p\dot{R} - L , \tag{1.5}$$

where the Lagrangian $L$ can be read from the action (1.4) and

$$p = \frac{\partial L}{\partial \dot{R}} = \frac{2\pi\mu \dot{R} R f^{-1}}{\sqrt{f - f^{-1}\dot{R}^2}} \tag{1.6}$$

is the momentum conjugate to $R$. After some simple algebra, the conservation law (1.5) can be rewritten as

$$\dot{R}^2 - f^2 + \epsilon^{-2} R^2 f^3 \equiv \dot{R}^2 + V(R) = 0 \ , \tag{1.7}$$

where $\epsilon = E/2\pi\mu$. For $\epsilon < (2H)^{-1}$ the potential $V(R)$ has the form of a barrier, as shown in Fig. 1. It is clear from the figure that there is a loop trajectory that starts at $R = 0$, expands to $R = R_1$ and recollapses. Another type of trajectory describes a contracting loop that bounces at $R = R_2$ and reexpands towards the horizon. The underbarrier range $R_1 < R < R_2$ is classically forbidden. The turning points $R_1$ and $R_2$ can be found from $V(R) = 0$, with the result

$$R_{1,2}^2 = \frac{1}{2H^2}(1 \pm \sqrt{1 - 4\epsilon^2 H^2}). \tag{1.8}$$

Instead of bouncing at $R = R_1$, the loop can tunnel quantum mechanically through the barrier and start expanding from $R = R_2$. The tunneling probability can be estimated using the semiclassical approximation,

$$P \propto e^{-B} \ , \tag{1.9}$$

with

$$B = 2 \int |p| dR = 4\pi\mu \int_{R_1}^{R_2} dR\, f^{-1}\sqrt{R^2 f - \epsilon^2} \ , \tag{1.10}$$

where in the last step we have used Eqs. (1.6) and (1.7) to express $p$ in terms of $R$. The integral in (1.10) can be evaluated in terms of elliptic integrals.

Now, the crucial observation is that the tunneling action $B$ remains finite in the limit of vanishing loop energy, $\epsilon \to 0$. In this limit $R_1 = 0$, $R_2 = H^{-1}$, and the integral in (1.10) is easily evaluated:

$$B = 4\pi\mu H^{-2} \ . \tag{1.11}$$

The zero initial radius of the loop means that there was no loop at all, and thus the limit $\epsilon \to 0$ corresponds to the spontaneous nucleation of a circular loop of radius $R = H^{-1}$.

The loop nucleation can also be described using the instanton language [8]. The Euclidean version ($t \to it$) of Eq. (1.7) is

$$\dot{R}^2 + f^2 - \epsilon^{-2} R^2 f^2 = 0 \ , \tag{1.12}$$

and in the limit $\epsilon \to 0$ it gives $\dot{R} \to \infty$. This looks very singular, but we will see in a moment that this infinity indicates only that $t$ is not a suitable coordinate of the instanton. Indeed, if we chose $R$ as a world-sheet coordinate instead of $t$, then the Euclideanized metric (1.3) takes the form

$$ds_2^2 = (f^{-1} + f\dot{R}^{-2})dR^2 + R^2\, d\theta^2. \tag{1.13}$$

In the limit $\dot{R} \to \infty$ it turns into the metric on a sphere of radius $H^{-1}$. Hence, the instanton is a sphere, and its Euclidean action is $\mu$ times the surface area:

$$S_E = \mu \cdot 4\pi H^{-2}. \tag{1.14}$$

The tunneling probability is proportional to $\exp(-S_E)$, in agreement with (1.9) and (1.11).

It is well known [9] that Euclideanized de Sitter space is a four-sphere of radius $H^{-1}$, so the instanton described above is a two-sphere of maximal radius. If the four-sphere is described by embedding it in a five-dimensional Euclidean space,

$$\zeta^2 + w^2 + \tau^2 = H^{-2} \ , \tag{1.15}$$

where $\zeta$ is a three-vector, then the instanton two-sphere can be described by

$$\zeta_1^2 + \zeta_2^2 + \tau^2 = H^{-2}, \quad \zeta_3 = w = 0. \tag{1.16}$$

The use of the semiclassical approximation is justified if $S_E \gg 1$ or $\mu \gg H^2/4\pi$. We have also implicitly assumed that the string is adequately described by the Nambu action (1.4). This is justified only if the string thickness $\delta$ is much smaller than the loop radius, $\delta \ll H^{-1}$. For $\mu \approx H^2/4\pi$ or $\delta \approx H^{-1}$, our results can still be expected to apply qualitatively.

Given a particle theory with discrete vacua that are degenerate in energy, domain-wall bubbles can be nucleated in a very similar way. One finds that the instanton is a three-sphere of radius $H^{-1}$, and its Euclidean action is

$$S_E = 2\pi^2 \sigma H^{-3} \ , \tag{1.17}$$

where $\sigma$ is the wall tension, and $2\pi^2 H^{-3}$ is the volume of a three-sphere of radius $H^{-1}$.

Finally, if the theory admits pointlike defects, such as magnetic monopoles, then monopole-antimonopole pairs will be spontaneously produced. The corresponding instanton is a circle of radius $H^{-1}$, and the tunneling action is

$$S_E = 2\pi m H^{-1} \ , \tag{1.18}$$

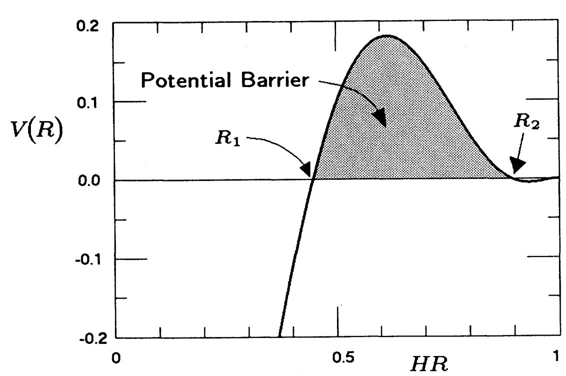


FIG. 1. A plot of the potential $V(R)$, showing the potential barrier of the tunneling problem. The graph was drawn for $\epsilon = 0.4\,H^{-1}$.

where $m$ is the monopole mass and $2\pi H^{-1}$ is the circumference of the circle. We note that $S_E = m/T_{\rm GH}$, where $T_{\rm GH} = H/2\pi$ is the Gibbons-Hawking [4] temperature of de Sitter space.

## II. CLASSICAL EVOLUTION OF A NUCLEATING TOPOLOGICAL DEFECT

We now turn to the classical evolution of nucleating strings, walls, and monopoles. Let us first consider the case of strings.

Analytically continuing Eqs. (1.15) and (1.16) back to the Minkowski signature physical space-time, one finds that the de Sitter space is described by the usual hyperboloid

$$\zeta^2 + w^2 - \tau^2 = H^{-2} , \tag{2.1}$$

and the string world sheet is given by

$$\zeta_1^2 + \zeta_2^2 = H^{-2} + \tau^2, \qquad \zeta_3 = w = 0. \tag{2.2}$$

To understand how the evolving loop would be viewed by an observer in an inflationary universe, it is useful to transform to Robertson-Walker flat coordinates $(t, \mathbf{x})$, with the metric

$$ds^2 = -dt^2 + e^{2Ht}\, d\mathbf{x}^2. \tag{2.3}$$

These coordinates are related to the hyperboloid coordinates by

$$t = H^{-1} \ln[H(w + \tau)], \qquad \mathbf{x} = \frac{H^{-1}\boldsymbol{\zeta}}{w+\tau}, \tag{2.4}$$

which can be inverted to give

$$\tau = H^{-1}\sinh(Ht) + \tfrac{1}{2}H\mathbf{x}^2 e^{Ht}, \qquad w = H^{-1}\cosh(Ht) - \tfrac{1}{2}H\mathbf{x}^2 e^{Ht}, \qquad \boldsymbol{\zeta} = \mathbf{x}\,e^{Ht}. \tag{2.5}$$

Inserting the transformations above into the string evolution equation (2.2), one finds

$$R_{\rm coord}^2 \equiv x^2 + y^2 = H^{-2}(1 + e^{-2Ht}), \qquad z = 0. \tag{2.6}$$

Thus, to an observer using the Robertson-Walker flat coordinates, the loop of string appears to have a stationary center, and at time $t$ it has a radius of physical size

$$R_{\rm phys} = H^{-1}\sqrt{e^{2Ht}+1} . \tag{2.7}$$

Note that although the purely Euclidean picture suggests that the string loop nucleates at radius $H^{-1}$, the loop actually has this radius only in the limit $t \to -\infty$. Thus, there seems to be no natural time at which one can say that the nucleation occurs. This is puzzling, but apparently we must remember that nucleation is a quantum-mechanical process, so the behavior of the string loop near the moment of nucleation cannot be described in purely classical terms. The evolution at late times, however, can be treated classically, and the string forms a circular loop with a physical radius given by Eq. (2.7). In the next section we will give a quantitative estimate of the time at which the classical description becomes valid.

Equation (1.16), and the resulting equations (2.2) and (2.6), describe a nucleating string in a particular configuration. To understand the other possible configurations, one can apply any five-dimensional rotation to the instanton in Eq. (1.16). Of the ten independent transformations in SO(5), there are four that leave the instanton invariant: three rotations in the $\zeta_1$-$\zeta_2$-$\tau$ space, and one rotation in the $\zeta_3$-$w$ plane. The remaining six generators of SO(5) have a nontrivial effect on the instanton, and thus there are six zero modes for small oscillations about the instanton solution. In general an $m$-sphere embedded in an $n$-sphere of equal radius is known [10] to have

$$N = (m + 1)(n - m) \tag{2.8}$$

zero modes.

Following through the effect of these transformations on Eqs. (2.2) and then (2.6), one finds that four of them correspond to translations in space and time, and two correspond to rotations of the orientation of the loop. There are no parameters, however, that correspond to a velocity for the loop. Like a bubble of true vacuum that nucleates within a false vacuum [11], a nucleating loop of string does not single out a rest frame, and hence there is no meaning to a velocity.

Since the loop is expanding, it is easy to see how the loop can be invariant under boosts in the plane of the loop. Like the bubbles discussed by Coleman [11], the expansion is along a boost-invariant hyperboloid. It is a little harder to see, however, how the loop can be invariant under a boost perpendicular to the plane of the loop. Eq. (2.6), for example, looks like it must obviously single out a rest frame for motion in the $z$ direction. To see how this paradox is resolved, one need only work out the relevant transformation equations. A de Sitter space possesses a ten-parameter symmetry group with structure O(4,1), and *locally* these transformations can be identified with those of the Poincaré group. In the vicinity of the origin of the Robertson-Walker flat coordinate system, the transformation that is locally identified with a $z$-boost corresponds to a $\zeta_3$-$\tau$ boost in the five-dimensional hyperboloid coordinates:

$$\zeta_3' = \zeta_3 + \epsilon\tau, \qquad \tau' = \tau + \epsilon\zeta_3, \qquad \zeta_1' = \zeta_1 , \quad \zeta_2' = \zeta_2 , \quad w' = w, \tag{2.9}$$

where $\epsilon$ is an infinitesimal parameter. Transforming to the $(t, \mathbf{x})$ coordinates with Eqs. (2.4) and (2.5), one finds (to first order in $\epsilon$) that

$$t' = t + \epsilon z, \qquad z' = z + \frac{\epsilon}{2}H^{-1}[1 - e^{-2Ht} + H^2(x^2 + y^2 - z^2)], \qquad x' = x - \epsilon Hzx , \quad y' = y - \epsilon Hzy. \tag{2.10}$$

Note that, when all the coordinates are small compared to $H^{-1}$, this transformation reduces to a standard Lorentz boost in the $z$ direction. The string loop, however, is always larger than $H^{-1}$, so the higher-order terms are very significant. When the transformation above is applied to the loop evolution equation (2.6), one finds

$$x'^2 + y'^2 = H^{-2}(1 + e^{-2Ht'}), \qquad z' = \epsilon H^{-1}. \tag{2.11}$$

So a boost in the $z$ direction does not impart a $z$ velocity to the loop, but instead is equivalent to a $z$ translation.

To summarize, an arbitrary configuration for the instanton leads to a nucleated loop of string that can have its center at any location, that can have any orientation, and that has a physical radius at time $t$ given by

$$R_{\rm phys} = H^{-1}\sqrt{e^{2H(t-t_0)} + 1} \ , \tag{2.12}$$

where $t_0$ can have any value. The quantity $t_0$ is related to the time of nucleation, but as discussed above there is no precise definition of the time of nucleation. Nonetheless, a change in the value of $t_0$ results in a time translation of the entire solution, and thus can be described intuitively as a shift in the time of nucleation.

The calculation for nucleating domain walls and monopoles is very similar, so the details need not be presented. The domain walls nucleate as spheres, with a radius again given by Eq. (2.12). In this case the instanton solution is just a three-sphere embedded in a four-sphere, so Eq. (2.8) indicates that there are four zero modes, and hence the instanton configuration is described by four parameters. In the five-dimensional Euclidean space the instanton three-sphere can be described as the intersection of a four-plane with the de Sitter four-sphere, and the four parameters can be taken as the independent components of a unit vector orthogonal to the four-plane. In the Robertson-Walker flat coordinates, the parameters correspond to spatial and time translations. The domain-wall sphere is invariant under boosts, so it is not meaningful to attribute a velocity to it.

For monopole-antimonopole pairs the instanton is a circle embedded in a four-sphere, and Eq. (2.8) tells us that it has six zero modes. These modes correspond to space and time translations and to the orientation of the pair (described, for example, by a unit vector pointing from the monopole to the antimonopole). The monopole-antimonopole separation as a function of time is given by $2R_{\rm phys}$, where $R_{\rm phys}$ is given by Eq. (2.12). A boost along the separation of the pair leaves the trajectories invariant, while a boost perpendicular to the separation is equivalent to a translation by $\epsilon H^{-1}$ in the direction of the boost.

## III. COSMOLOGICAL IMPLICATIONS OF NUCLEATING DEFECTS

Our results suggest that string loops can be continuously produced during inflation at a constant rate per unit volume per unit time. This rate can be written as

$$\lambda = H^4 A e^{-B} \ , \tag{3.1}$$

where $B$ is given by Eq. (1.11). Our description of string nucleation should be qualitatively accurate provided that the inflationary false-vacuum state admits string solutions with $\mu \gtrsim H^2$ and $\delta \lesssim H^{-1}$ ($\delta$ is the string thickness). If $\eta_s$ is the symmetry-breaking scale of strings, and the relevant coupling constants are not too small, then $\mu \sim \eta_s^2$, $\delta \sim \eta_s^{-1}$, and both conditions are satisfied for $\eta_s \gtrsim H$. The nucleation rate (3.1) is negligible if $\eta_s \gg H$, and so the most interesting case is when $\eta_s \sim H$. The prefactor $A$ is a function of the ratio $\mu/H^2$, and for $\mu \sim H^2$ we expect $A \sim 1$. After inflation the universe reheats to some temperature $T_r$, and the strings survive only if $T_r < \eta_s$. In most inflationary models the reheat temperature is low, so this condition is not difficult to satisfy.

Let us now find the size distribution of strings at any given time during the inflationary period. The number of loops that form in a coordinate interval $d^3\mathbf{x}$ and in an interval of the time parameter $dt_0$ is given by

$$dN = \lambda e^{3Ht_0}\, d^3\mathbf{x}\; dt_0 \ . \tag{3.2}$$

Note that this is a probability distribution in the parameter space of solutions, and therefore has no dependence on the time of observation $t$. To find the probability distribution for loops of a given physical radius $R$ that would be observed at time $t$, one uses Eq. (2.12) to replace the parameter $t_0$ by $R$. The result is given by

$$dN = \frac{\lambda H R}{(H^2R^2 - 1)^{5/2}}\, dR\, dV_{\rm phys} \ , \tag{3.3}$$

where

$$dV_{\rm phys} = e^{3Ht} d^3\mathbf{x} \tag{3.4}$$

represents the physical volume element. For large loops ($R \gg H^{-1}$) this reduces to the simple scale-invariant expression

$$dN = \frac{\lambda}{H^4}\frac{dR}{R^4}\, dV_{\rm phys} \ . \tag{3.5}$$

At the end of inflation, this distribution spans the range of scales from $R \sim H^{-1}$ up to comoving scales beyond the present horizon. [If $H$ changes during inflation, then the coefficient $\lambda$ in (3.5) becomes a function of $R$.]

Note that the distribution (3.3) has the peculiar property that an integration over $R$ would diverge at $R = H^{-1}$. This anomaly arose in the calculation because the solutions did not have a natural choice for the time of nucleation, so we have calculated under the oversimplified assumption that the classical trajectories have meaning back to time $-\infty$. In reality, there is some value $\Delta$ such that loops with radii $\lesssim (H^{-1} + \Delta)$ must be considered to have not yet nucleated, and thus the probability distribution is cut off. To estimate $\Delta$, one recalls that a WKB solution describes classical behavior except near the classical turning points. The nucleation of the string corresponds to the classical turning point, and the classical behavior can be expected to resume once the accumulated phase of the wave function becomes large. The phase of the wave function for $R > H^{-1}$ can be read off from Eq. (1.10), using $\epsilon = 0$ to correspond to a string loop nucleated in the vacuum. One has

$$\text{Phase}(R = H^{-1} + \Delta) = 2\pi\mu \int_{H^{-1}}^{H^{-1}+\Delta} \frac{R'\,dR'}{\sqrt{H^2R'^2 - 1}} = \frac{2\pi\mu}{H^2}\sqrt{H\Delta(H\Delta + 2)}. \quad (3.6)$$

Choosing $\Delta$, for example, as the point where the phase is about $2\pi$, one has

$$H\Delta \approx \sqrt{1 + \frac{H^4}{\mu^2}} - 1. \quad (3.7)$$

For the interesting case of $\mu \sim H^2$, this gives $\Delta \sim H^{-1}$.

It should be noted that the static coordinate system used in Eq. (3.6) is not valid for $R > H^{-1}$, and our calculation of the phase should be understood in the sense of analytic continuation.

After inflation, the probability distribution for loops with $R \gg t$ continues to be described by Eq. (3.5), with $H$ interpreted as the value of the Hubble parameter during inflation. When loops come within the horizon, they begin to oscillate and their mass and radius remain constant (disregarding the slow decay due to gravitational radiation). Hence, for $R < t$ we can write

$$dN(t) \approx \frac{\lambda}{H^4}\left(\frac{a(R)}{a(t)}\right)^3 \frac{dR}{R^4}\,dV_{\text{phys}}\ , \quad (3.8)$$

where $a(t)$ is the scale factor. This distribution has the same form as that in the standard scenario of string evolution, where loops are chopped off an infinite string network [7].

Note that the mass distribution $dm = 2\pi\mu R\,dN$ will diverge at small $R$ for loops that reenter the Hubble length during the radiation-dominated era (i.e., loops with $R < t_{\text{equality}}$), so the mass density will be dominated by loops near the cut-off at $R = H^{-1} + \Delta$. For loops that reenter the Hubble length during the matter-dominated era ($R > t_{\text{equality}}$) the mass distribution becomes logarithmically flat (i.e., $dm \propto dR/R$), so each decade of $R$ up to the Hubble scale makes a comparable contribution.

Since nucleating loops have the shape of a circle, one could expect that, instead of oscillating, they will all collapse to form black holes. However, there will be some deviations from circular shape due to quantum fluctuations. Equation (3.7) suggests that the ratio of the fluctuation amplitude to the loop radius is $H\Delta \sim H^4/\mu^2$. For $H \sim \eta_s$ this gives $H\Delta \sim 1$. On the other hand, for a black hole to be formed, we need (Ref. [12]) $H\Delta \lesssim G\mu$. Hence, for $G\mu \ll 1$ the probability of black-hole formation is small. We note in passing that spontaneous loop nucleation can provide an alternative (or additional) mechanism for string-driven inflation [13]. This would require $G\mu \sim 1$.

The cosmological evolution of domain-wall bubbles can be discussed in a similar way. In fact, it is easily understood that the size distribution of bubbles is described by the same equations (3.5) and (3.8). An important difference, however, is that walls of radius $R > (8\pi G\sigma)^{-1}$ inevitably collapse to form black holes. Another important difference is the mass distribution, which in this case is given by $dm = 4\pi R^2\sigma\,dN$. The extra power of $R$, when compared to strings, implies that for $R < t$ the distribution diverges at large $R$ in both the radiation- and matter-dominated eras. This means that the mass density is dominated by bubbles of size comparable to the Hubble length. (This is true for any power-law expansion $a \propto t^\nu$, with $\nu > \frac{1}{3}$.)

Although they dominate the domain-wall mass density when one averages over arbitrarily large scales, Hubble-size bubbles may still be very rare. The total number of bubbles of radius greater than $R$ within the visible Universe has an expectation value

$$N_{>R} = 36\pi\frac{\lambda}{H^4}\frac{t_0}{R}\ , \quad (3.9)$$

where we have used (3.8) with $V_{\text{phys}} = (4\pi/3)(3t_0)^3$, $t_0$ is the present cosmic time, and we have assumed that $R > t_{\text{equality}}$. It follows from Eq. (3.9) that the radius of the largest bubble one can expect to find at present in a typical Hubble size volume is

$$R_{\max} \approx 36\pi\frac{\lambda}{H^4}t_0\ . \quad (3.10)$$

If $\lambda$ is small, $R_{\max}$ can be much smaller than the Hubble length. [For $\lambda/H^4 < 10^{-8}$, Eq. (3.10) does not apply, since it gives $R_{\max} < t_{\text{equality}}$. An estimate of $R_{\max}$ in this case is easily obtained, however, along the same lines.]

To avoid conflict with the isotropy of the microwave background, the mass of the maximal bubble, $M_{\max} = 4\pi\sigma R_{\max}^2$, has to be less than $10^{-4}$ of the total mass of the visible Universe. From this one can derive a constraint on the possible values of the symmetry-breaking scale of the walls, $\eta_w$. On the other hand, gigantic black holes which can be produced by this mechanism can lead to interesting patterns of density fluctuations for the formation of cosmic structure.

If monopole-antimonopole pairs are produced, the monopole density at the end of inflation is of the order

$$n_m \sim H^3\exp(-2\pi m/H)\ . \quad (3.11)$$

Depending on the values of $H$ and $m$, this density can be too high (so that the corresponding model is ruled out), it can be negligible, or it can lie in the "interesting" range and be potentially observable.

## IV. INCLUDING STRING AND WALL GRAVITY

So far we neglected the gravitational effect of the nucleating strings and walls. In this section we shall find the instantons and calculate the corresponding actions with string and wall gravity taken into account.

### A. Strings

The gravitational field of a string is characterized by a conical deficit angle, $\delta\varphi = 8\pi G\mu$. Specifically, a space containing an infinitely thin string along the $z$ axis is described by the flat metric

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \ , \qquad (4.1)$$

where the range of the azimuthal angle $\varphi$ is given by

$$0 \le \varphi < 2\pi(1 - 4G\mu) \ , \qquad (4.2)$$

instead of the Euclidean range, $0 \le \varphi < 2\pi$. When a string is embedded in de Sitter space, the Einstein equations in the immediate vicinity of the string will be dominated by the $\delta$-function contribution to the energy-momentum tensor coming from the string. The implication of this $\delta$-function source will be the same as in the previous case: at any point on the string world sheet, an infinitesimal circle drawn around the string in the plane perpendicular to the world sheet will show a deficit angle $\delta\varphi = 8\pi G\mu$.

To see how the deficit angle affects the geometry of the Euclidean instanton, one can begin with Eq. (1.16), which describes the string instanton (without a deficit angle) embedded in a five-dimensional Euclidean space:

$$\zeta_1^2 + \zeta_2^2 + \tau^2 = H^{-2},$$
$$\zeta_3 = w = 0.$$

At each point on the string world sheet, it can be seen that the unit vectors in the $\zeta_3$ and $w$ directions are both tangent to the de Sitter sphere and perpendicular to the string world sheet. Thus, the proper conical singularity can be inserted by defining an azimuthal angle $\varphi$ in the $\zeta_3$-$w$ plane, and restricting its range by Eq. (4.2).

To continue, it is useful to introduce angular coordinates $(\psi, \chi, \theta, \phi)$ that are related to the five-dimensional Cartesian coordinates by

$$\begin{aligned} H\tau &= \sin\psi \ , \\ H\zeta_1 &= \cos\psi\, \cos\chi \ , \\ H\zeta_2 &= \cos\psi\, \sin\chi\, \cos\theta \ , \\ H\zeta_3 &= \cos\psi\, \sin\chi\, \sin\theta\, \sin\varphi \ , \\ Hw &= \cos\psi\, \sin\chi\, \sin\theta\, \cos\varphi. \end{aligned} \qquad (4.3)$$

(This transformation is reasonably standard, except that we have ordered the Cartesian coordinates in a peculiar manner in order to conform with other aspects of this paper.) The metric in terms of the angular coordinates is then

$$ds^2 = H^{-2}\{\, d\psi^2 + \cos^2\psi[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)]\}, \qquad (4.4)$$

where the range of $\varphi$ is given by Eq. (4.2). The other coordinates have the range

$$-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}, \ \ 0 \le \chi, \ \ \theta \le \pi. \qquad (4.5)$$

In these coordinates the string lies along the $\theta = 0$ and $\theta = \pi$ axes. In standard conventions, the locus of the string in the vicinity of the origin would be called the positive and negative $z$ axes. Recall that $\varphi$ is the azimuthal angle measured around the string, so it is a periodic variable with a period defined by Eq. (4.2).

To understand the evolution of the string loop in physical space-time, we analytically continue the above solution to imaginary values of $\tau$. By analytically continuing Eq. (4.4), one can write the Lorentzian signature metric as

$$ds^2 = dt^2 - H^{-2}\cosh^2 Ht \times [d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\varphi^2)], \quad (4.6)$$

where $Ht$ is the analytic continuation of $\psi$. Equation (4.6) is the Robertson-Walker closed universe representation of de Sitter space, except that the azimuthal angle $\varphi$ has the restricted range of Eq. (4.2). The string lies along the $z$ axis $(\theta = 0, \pi)$, and in this coordinate system it makes a maximal circle around the closed universe.

The metric (4.6) gives a valid description for the evolution of a nucleated string, but the coordinate system has been chosen in a very special way to make the string as simple as possible. To visualize how a typical string loop would appear to an observer in an inflationary universe, it is useful to again transform the metric to the Robertson-Walker flat form by using Eqs. (2.4) and (2.5). This time, however, we must remember that a deficit angle of $\delta\varphi = 8\pi G\mu$ must be removed from the $\zeta_3$-$w$ plane. For convenience, we choose to remove a wedge that is centered on the positive $w$ axis. Thus, for $w > 0$, $\zeta_3$ is restricted to the range

$$|\zeta_3| \ge w \tan(4\pi G\mu) \ . \qquad (4.7)$$

Transforming to the Robertson-Walker flat coordinates, one finds that the condition $w > 0$ becomes $\mathbf{x}^2 < R^2_{\rm coord}(t)$, where $R_{\rm coord}(t)$ is the coordinate radius of the loop, given by Eq. (2.6). When this inequality is satisfied, then Eq. (4.7) implies that

$$|z| \ge \frac{1}{2H}(1 + e^{-2Ht} - H^2\mathbf{x}^2)\ \tan(4\pi G\mu) \ . \qquad (4.8)$$

Substituting $\mathbf{x}^2 \equiv x^2 + y^2 + z^2$ and then solving the quadratic equation, one finds

$$|z| \ge \frac{\sqrt{1 + H^2(R^2_{\rm coord} - x^2 - y^2)\tan^2(4\pi G\mu)} - 1}{H\tan(4\pi G\mu)} \ . \qquad (4.9)$$

Thus, in the Robertson-Walker flat coordinate system the effect of the string gravity is to remove from the space a region within the loop that is shaped roughly like a convex lens. The upper and lower surfaces of this lens are then identified with each other. The form of this excised lens-shaped region is shown in Fig. 2. For very early times (presumably well before a classical description is applicable), the excised region approaches a sphere. For late times the apex angle, where the lens meets the string, approaches the deficit angle for an infinite string, $8\pi G\mu$.

We would also like to calculate the effect of string gravity on the action of the instanton solution. The Euclidean action is given by

$$S_E = \mu\int d^2x\,\sqrt{\gamma} + \int d^4x\,\sqrt{g}\,\left(\rho_V - \frac{\mathcal{R}}{16\pi G}\right) \ , \qquad (4.10)$$

where $\gamma$ is the determinant of the two-dimensional metric on the string world sheet, $\mathcal{R}$ is the four-dimensional scalar curvature, and $\rho_V$ is the false-vacuum energy density of de Sitter space. The horizon radius $H^{-1}$ is related to $\rho_V$ by

$$H^2 = \frac{8\pi}{3} G\rho_V \quad . \tag{4.11}$$

The quantity $B$ in Eq. (2.10) for the tunneling probability is given by the difference

$$B = \tilde{S}_E - S_E \quad , \tag{4.12}$$

where $\tilde{S}_E$ and $S_E$ are the Euclidean actions of de Sitter space with and without a string, respectively.

The scalar curvature $\mathcal{R}$ in (4.10) can be written as

$$\mathcal{R} = 8\pi G T^\nu_\nu = \mathcal{R}_V + \mathcal{R}_S \quad , \tag{4.13}$$

where $\mathcal{R}_V$ and $\mathcal{R}_S$ are the contributions of the false vacuum and the string, respectively. The false vacuum and string energy-momentum tensors are given by

$$T^{\mu\nu}_V \sqrt{g} = \rho_V \sqrt{g}\, g^{\mu\nu}, \tag{4.14}$$

$$T^{\mu\nu}_S \sqrt{g} = \mu \int d^2\xi \sqrt{\gamma}\, \gamma^{ab} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \delta^{(4)}(x - x(\xi)). \tag{4.15}$$

Here, $x^\mu(\xi)$ is the string world sheet parametrized using two arbitrary parameters $\xi^a$, $a = 1, 2$;

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \tag{4.16}$$

is the metric tensor on the world sheet, $\gamma^{ab}$ is its inverse and $\gamma = \det(\gamma_{ab})$. Now it is easily seen that

$$-\frac{1}{16\pi G} \int d^4x \sqrt{g}\, \mathcal{R}_S = -\mu \int d^2\xi \sqrt{\gamma} = -4\pi\mu H^{-2}. \tag{4.17}$$

Note that this curvature contribution cancels the Nambu action term in (4.10). To find the false-vacuum contribution to $\tilde{S}_E$, we use $\mathcal{R}_V = 32\pi G\rho_V$ and do the four-dimensional integration in (4.10) with a restricted range of $\varphi$, Eq. (4.2). This gives

$$\tilde{S}_E = -\frac{\pi}{GH^2}(1 - 4G\mu). \tag{4.18}$$

The action of a four-sphere without a string is [14]

$$S_E = -\frac{\pi}{GH^2}, \tag{4.19}$$

and thus

$$B = 4\pi\mu H^{-2}. \tag{4.20}$$

Surprisingly, inclusion of gravity does not change the tunneling action in this case.

### B. Domain walls

The instanton in this case includes two identical pieces of a four-sphere matched together at a three-sphere of a smaller radius, which represents the world sheet of the wall (see Fig. 3). This instanton belongs to a more general family of instantons studied by Berezin, Kuzmin, and

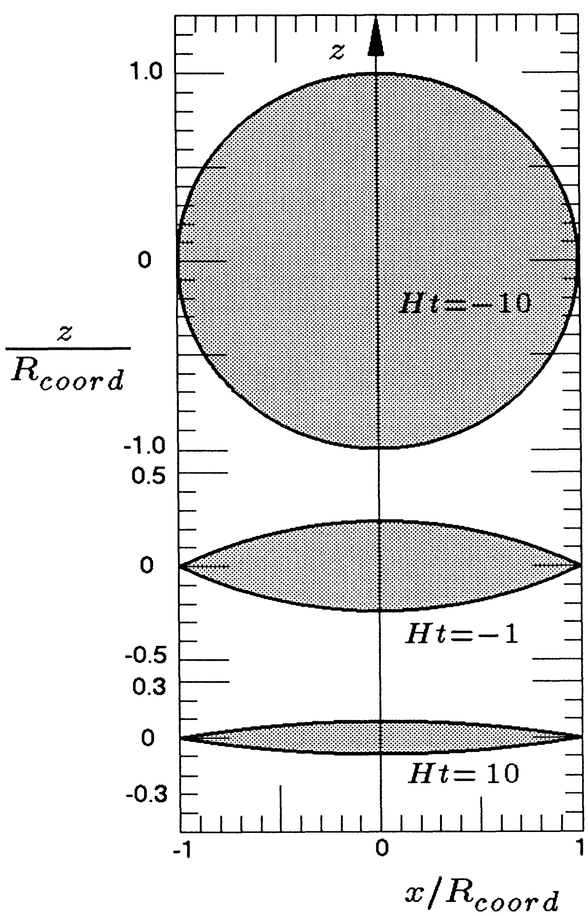


FIG. 2. A nucleated string loop in Robertson-Walker flat coordinates. In these coordinates the string deficit angle is implemented by excising from the manifold a lens-shaped region within the string loop. The diagram shows a cross section of this excised region, for three values of $Ht$. The figure is drawn for a very large deficit angle, 20°.

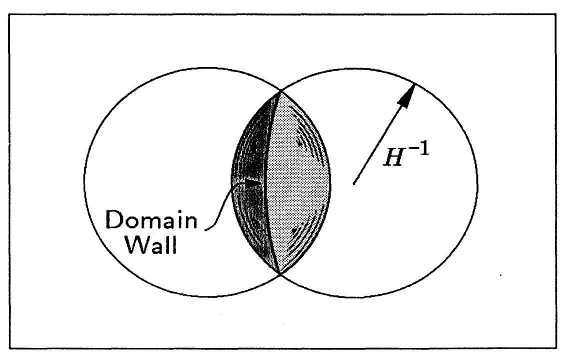


FIG. 3. Schematic diagram for the instanton describing domain-wall nucleation. The diagram shows two fewer dimensions than the real instanton, which is composed of pieces of two four-spheres that intersect at a three-sphere.

Tkachev [15]. We could obtain the tunneling action for our instanton from their formalism by setting the vacuum energies on the two sides of the wall equal to each other. Here we shall obtain the same result in a different, and perhaps somewhat more straightforward way.

The four-sphere metric can be chosen in the form (4.4), where in this case $\varphi$ has the usual range $0 \le \varphi < 2\pi$. The three-sphere is then a surface of constant $\psi$, $\psi = \psi_0$. The junction conditions at this surface are [16]

$$[K_{ab}] = -4\pi G\sigma\gamma_{ab} \ , \tag{4.21}$$

where $[K_{ab}]$ is the difference of extrinsic curvature on the two sides of the surface, $\gamma_{ab}$ is the world-sheet metric tensor and $a, b = 1, 2, 3$. In our case the extrinsic curvatures on the two sides of the wall world sheet are equal and opposite, and so $[K_{ab}] = 2K_{ab}$. $K_{ab}$ is the derivative of the three-dimensional metric with respect to a normalized coordinate orthogonal to the three-surface, so

$$K_{ab} = H\frac{\partial g_{ab}}{\partial\psi} \ , \tag{4.22}$$

and Eqs. (4.21) reduce to a single condition

$$H \tan\psi_0 = 2\pi G\sigma \ . \tag{4.23}$$

The radius of the domain-wall three-sphere (at $\psi = \psi_0$) is then given by

$$R_0 = H^{-1}\cos\psi_0 = \frac{1}{\sqrt{H^2 + (2\pi G\sigma)^2}} \ . \tag{4.24}$$

The four-geometry describing the Lorentzian evolution of the Universe with a domain wall consists of two pieces of de Sitter space matched together at a three-dimensional hyperboloid representing the world sheet of the wall. To discuss the analytic continuation in detail, it will be useful to adopt a new set of coordinate conventions. We will continue to embed the four-dimensional Euclidean system in a five-dimensional Euclidean space with coordinates $(\boldsymbol{\zeta}, w, \tau)$, and we will center one of the two de Sitter spheres at the origin. The second sphere will have its center along the positive $w$ axis, as shown in Fig. 4. The world sheet of the wall lies at the intersection of the two spheres, which occurs in the four-plane

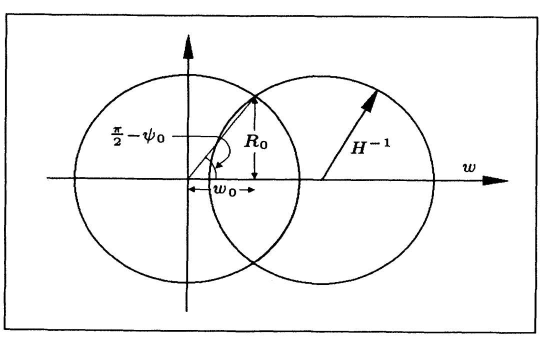


FIG. 4. Configuration of de Sitter four-spheres for the domain wall instanton. One sphere is at the center of the five-dimensional coordinate system, and the other sphere lies along the positive $w$ axis.

$$w = w_0 = H^{-1}\sin\psi_0 = \frac{2\pi G\sigma H^{-1}}{\sqrt{H^2 + (2\pi G\sigma)^2}} \ . \tag{4.25}$$

We will relate the Cartesian coordinates to angular coordinates as in Eq. (4.3), but this time we will use a more standard ordering of the axes:

$$\begin{aligned} H\tau &= \sin\psi \ , \\ Hw &= \cos\psi\,\cos\chi \ , \\ H\zeta_3 &= \cos\psi\,\sin\chi\,\cos\theta \ , \\ H\zeta_2 &= \cos\psi\,\sin\chi\,\sin\theta\,\sin\varphi \ , \\ H\zeta_1 &= \cos\psi\,\sin\chi\,\sin\theta\,\cos\varphi. \end{aligned} \tag{4.26}$$

By analytically continuing $\tau$ to $i\tau$ the metric acquires the form of Eq. (4.6), and Eq. (4.25) describing the position of the wall becomes $H^{-1}\cosh Ht\cos\chi = H^{-1}\sin\psi_0$. Thus, the wall is located at

$$\chi = \chi_0(t) = \arccos\left(\frac{\sin\psi_0}{\cosh(Ht)}\right) \ . \tag{4.27}$$

In these closed universe coordinates the physical radius of the domain wall at time $t$ is given by

$$\begin{aligned} R_{\rm phys}(t) &= H^{-1}\sin\chi_0(t)\cosh(Ht) \\ &= H^{-1}\sqrt{\cosh^2(Ht) - \sin^2\psi_0}. \end{aligned} \tag{4.28}$$

The radius of the wall at $t = 0$ is $R_{\rm phys}(0) = R_0$, where $R_0$ is from Eq. (4.24). It can be interpreted loosely as the wall radius at nucleation.

In the limit $\sigma \to 0$, Eqs. (4.25) and (4.27) give $\psi_0 = 0$, $\chi_0 = \pi/2$, and we recover the instanton without gravity. In the opposite limit, when $2\pi G\sigma \gg H$, we have $\psi_0 \to \pi/2$ and $R_0 \to 0$. The fact that gravity reduces the initial size of the wall has a simple physical explanation. Without gravity the wall radius at nucleation is such that the force of tension is balanced by the stretching force due to the expansion of the universe. The gravitational field of domain walls is repulsive [17], and therefore with gravity the balance is achieved at a smaller radius.

The closed-universe description above is somewhat restricted, since we chose a special coordinate system in which the bubble wall divides the universe precisely in two equal halves. It is easy, however, to use the five-dimensional hyperboloid description to obtain a description in Robertson-Walker flat coordinates. Using Eq. (4.25) with the transformation equations (2.4) and (2.5), one finds that in these coordinates the domain wall lies at

$$\mathbf{x}^2 = H^{-2}\left(1 + e^{-2Ht} - 2Hw_0e^{-Ht}\right) . \tag{4.29}$$

The most general nucleating bubble solution can be obtained by translating this solution in position and time. In this coordinate system the physical radius evolves as

$$R_{\rm phys} = H^{-1}\sqrt{e^{2Ht} + 1 - 2Hw_0e^{Ht}}. \tag{4.30}$$

Let us now calculate the Euclidean action of the instanton describing the wall nucleation. It is given by

$$\tilde{S}_E = \sigma \int d^3\xi \sqrt{\gamma} + \int d^4x \sqrt{g} \left( \rho_V - \frac{\mathcal{R}}{16\pi G} \right), \quad (4.31)$$

and as before we are interested in the difference $B = \tilde{S}_E - S_E$, where $S_E$ is the Euclidean action of de Sitter space without a wall. The calculation of $B$ is very similar to that in the case of a string. The energy-momentum tensor of the wall is

$$T_W^{\mu\nu} \sqrt{g} = \sigma \int d^3\xi \sqrt{\gamma}\, \gamma^{ab} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} \delta^{(4)}(x - x(\xi)), \quad (4.32)$$

and the scalar curvature is given by

$$\mathcal{R} \sqrt{g} = 32\pi G \rho_V \sqrt{g} + 24\pi G \sigma \int d^3\xi \sqrt{\gamma}\, \delta^{(4)}(x - x(\xi)). \quad (4.33)$$

Substitution of (4.33) into (4.31) gives

$$\tilde{S}_E = -\frac{\sigma}{2} \int d^3\xi \sqrt{\gamma} - \rho_V \int d^4x \sqrt{g}. \quad (4.34)$$

The first integral in (4.34) is just the volume of a three-sphere of radius (4.24):

$$\int d^3\xi \sqrt{\gamma} = 2\pi^2 R_0^3. \quad (4.35)$$

The second integral is given by

$$\int d^4x \sqrt{g} = 4\pi^2 H^{-4} \int_0^{\psi_0} d\psi \sin^3\psi = \frac{4\pi^2}{3H^4} \left(2 - 3\cos\psi_0 + \cos^3\psi_0\right), \quad (4.36)$$

and after some simple algebra we obtain

$$\tilde{S}_E = -\frac{\pi}{GH^2} + \frac{2\pi^2\sigma}{H^2\sqrt{H^2 + (2\pi G\sigma)^2}} \quad (4.37)$$

and

$$B = \frac{2\pi^2\sigma}{H^2\sqrt{H^2 + (2\pi G\sigma)^2}}. \quad (4.38)$$

We see that in the case of domain walls the tunneling action does get modified by wall gravity. An intriguing property of Eq. (4.38) is its behavior in the limit of large $\sigma$:

$$B \approx \frac{\pi}{GH^2} \qquad (G\sigma \gg H). \quad (4.39)$$

Surprisingly, the tunneling action in this limit is independent of the wall tension. Geometrically, this behavior is easily understood: as $\sigma$ gets larger, the three-sphere radius (4.24) becomes smaller and the instanton action $\tilde{S}_E$ vanishes in the limit $\sigma \to \infty$. Hence, in this limit $B \to -S_E = \pi/GH^2$. Physically one might expect $B$ to grow as $\sigma$ becomes large. The reason this does not happen is that the radius of the instanton $R_0$ decreases simultaneously, so that the product $\sigma R_0^3$ vanishes in the limit.

## V. NEGATIVE EIGENMODES OF THE INSTANTON SOLUTIONS

In standard tunneling problems solved by instanton methods, the instanton has exactly one negative eigenmode. That is, the expression for the second variation of the action about the instanton has exactly one negative eigenvalue. Coleman [18] has shown that, for a wide class of systems, the solution to the WKB formulation of tunneling necessarily corresponds to an instanton with exactly one negative eigenmode. In this section we will analyze the negative eigenmodes of the instanton solutions introduced in Sec. I, and we will find that the instanton for string nucleation has two negative eigenmodes, while the instanton for monopole pair nucleation has three. In general an $m$-sphere embedded in an $n$-sphere of equal radius is known [10] to have

$$N = n - m \quad (5.1)$$

negative eigenmodes. We will show how these instantons avoid the implications of Coleman's theorem, but the physical implications of the multiple negative eigenmodes will remain a topic for future work.

We begin by describing the eigenvalue problem for the string nucleation instanton. For simplicity we will ignore the gravitational effects of the string, so we will treat the space-time as a fixed background de Sitter space. The instanton is then a maximal two-sphere on the de Sitter four-sphere. The two-sphere surface will be parametrized by $\xi^a \equiv (\theta, \varphi)$, where $a = 1, 2$. An arbitrary perturbation of the two-sphere can be characterized by introducing two functions $\eta_i(\xi^a)$, where $i = 1, 2$, and the two-sphere is described in the five-dimensional Euclidean coordinates by

$$\begin{aligned} H\zeta_5 &= \eta_1(\theta, \phi)\ , \\ H\zeta_4 &= \eta_2(\theta, \phi)\ , \\ H\zeta_3 &= \sqrt{1 - \eta_i^2}\, \cos\theta\ , \\ H\zeta_2 &= \sqrt{1 - \eta_i^2}\, \sin\theta\, \sin\varphi\ , \\ H\zeta_1 &= \sqrt{1 - \eta_i^2}\, \sin\theta\, \cos\varphi. \end{aligned} \quad (5.2)$$

Using the Euclidean metric of the five-dimensional space to determine the induced metric $\gamma_{ab}$ on the two-sphere, one has

$$\gamma_{ab} = \sum_{\alpha=1}^{5} \frac{\partial \zeta_\alpha}{\partial \xi^a} \frac{\partial \zeta_\alpha}{\partial \xi^b} = (1 - \eta_i^2)\gamma_{ab}^{(0)} + H^{-2} \left( \delta_{ij} + \frac{\eta_i \eta_j}{1 - \eta_k^2} \right) \frac{\partial \eta_i}{\partial \xi^a} \frac{\partial \eta_j}{\partial \xi^b}, \quad (5.3)$$

where

$$\gamma_{ab}^{(0)} = H^{-2} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \quad (5.4)$$

is the unperturbed metric on the two-sphere.

Following the notation of Sec. IV, the action is given by

$$S_E = \mu \int d^2\xi \sqrt{\gamma}, \quad (5.5)$$

where $\gamma = \det(\gamma_{ab})$. Expanding to quadratic order, one has

$$S_E = S_E^{(0)} + \frac{1}{2}\mu \int d^2\xi \sqrt{\gamma^{(0)}} \left( H^{-2}\gamma^{(0)ab} \frac{\partial \eta_i}{\partial \xi^a} \frac{\partial \eta_i}{\partial \xi^b} - 2\eta_i^2 \right) + O(\eta^3), \qquad (5.6)$$

where $\gamma^{(0)ab}$ is the inverse of $\gamma^{(0)}_{ab}$. This can be put in a more recognizable form by an integration by parts, yielding

$$S_E = S_E^{(0)} + \frac{1}{2}\mu \int d^2\xi \sqrt{\gamma^{(0)}} \left( -H^{-2}\eta_i \nabla^2 \eta_i - 2\eta_i^2 \right) + O(\eta^3), \qquad (5.7)$$

where

$$\nabla^2 \eta \equiv \frac{1}{\sqrt{\gamma^{(0)}}} \frac{\partial}{\partial \xi^a} \left( \sqrt{\gamma^{(0)}}\, \gamma^{(0)ab} \frac{\partial \eta}{\partial \xi^b} \right) \qquad (5.8)$$

is the Laplacian operator defined on the unperturbed two-sphere. The Laplacian is diagonalized by a standard spherical harmonic expansion

$$\eta_i(\theta, \varphi) = \sum_{\ell m} c^i_{\ell m} Y_{\ell m}(\theta, \varphi), \qquad (5.9)$$

and so

$$S_E = S_E^{(0)} + \frac{\mu}{2H^2} \sum_{\ell m, i} \left[ \ell(\ell+1) - 2 \right] \left( c^i_{\ell m} \right)^2 + O(\eta^3). \qquad (5.10)$$

Thus, we can see that the eigenvalue problem has two negative modes ($\ell = m = 0$; $i = 1, 2$) and six zero modes ($\ell = 1; m = 0, \pm 1$; $i = 1, 2$), and all the other eigenvalues are positive. The two negative modes correspond to moving the entire instanton two-sphere uniformly in the $\zeta_4$ or $\zeta_5$ direction. The six independent zero modes, as discussed in Sec. II, are rotations that carry any of the 1, 2, or 3 directions into either the 4 or 5 direction.

A similar analysis for the monopole pair instanton shows that it has three negative eigenmodes, since there are three directions orthogonal to the loop. A domain-wall instanton, on the other hand, has only one negative eigenmode, since there is only one direction perpendicular to it.

While we do not claim to understand the physical consequences of the multiple negative eigenmodes, we can at least see that there is no direct contradiction with the arguments of Coleman [18]. In his introduction Coleman excludes the case of scalar fields evolving in de Sitter space, but for pedagogical purposes it is useful to pursue the arguments further to see where they break down. We begin by giving a crude summary of the argument, which rests on the fact that the minimal barrier penetration path of the WKB method is a true minimum of the action, while the instanton is not. The apparent discrepancy is resolved, however, if the instanton has a single negative eigenmode. The point is that not all small deviations from the instanton can be mapped into a barrier penetration path. The barrier penetration path is required to end on a surface with potential energy equal to the initial potential energy, while an arbitrary instanton configuration is likely to lead to a path that overshoots or undershoots. Thus one cannot use the negative eigenmode to find a barrier penetration path of lower action, since the addition of this mode will alter the potential energy of the end point. If there are two negative eigenmodes, on the other hand, then one can generally superimpose them to obtain a path that has a lower action *and* finishes on the correct equipotential surface. For the case of the string instanton, however, the two negative eigenmodes are closely related—one can be obtained from the other by a rotation in the 4-5 plane. Any linear combination of the two modes is just a rotation of the original, and so the final value of the potential energy cannot be adjusted by modifying the coefficients of the linear superposition.

## VI. CONCLUSIONS

The main conclusion of this paper is that spherical domain walls, circular loops of string, and monopole-antimonopole pairs can be spontaneously created during inflation by quantum-mechanical tunneling. The initial radii of strings and walls and the initial separations of monopole-antimonopole pairs are equal to the de Sitter horizon, $H^{-1}$. When viewed in Robertson-Walker flat coordinates, the initial radius (or separation) of $H^{-1}$ is approached asymptotically as $t \to -\infty$, and there is no sharply defined time at which one would naturally say that nucleation occurs. The time at which the evolution becomes classical, however, can nonetheless be estimated (see Sec. III) by finding when the accumulated phase of the wave function becomes large.

Assuming that the transverse dimensions of the defects are small compared to the horizon, we have found the corresponding instantons and determined the WKB tunneling amplitude and the classical evolution of the defects after nucleation. The results are particularly simple when the gravitational effects of the nucleating defects are neglected. In this case the background space-time is an unperturbed de Sitter space, which in the Euclidean formulation is analytically continued into a four-sphere. For walls, strings, and monopoles, the world surfaces of the instanton solutions are the maximal three-sphere, two-sphere, and circle, respectively. (The word "maximal" means that the radius of the world-surface of the defect is equal to the radius, $H^{-1}$, of the four-sphere in which it is embedded.)

For the cases of strings and walls, the gravitational field of the nucleating defects can be taken into account by rather simple constructions. For the string, the well-known conical deficit angle can be incorporated by excluding a segment from the de Sitter four-sphere and then identifying the boundaries of the excluded region. When this solution is analytically continued and expressed in Robertson-Walker flat coordinates, it is found that the conical deficit angle can be implemented by excluding from each constant-time hypersurface a region of space shaped like a convex lens, with the string running around the edge of the lens (see Fig. 2). For the case of walls, the discontinuity implied by the wall surface stress is achieved by taking the polar cap of a four-sphere and joining it at its boundary to a copy of itself (see Fig. 3). The boundary, which is a three-sphere, represents the world sheet of the domain wall.

A significant limitation of our analysis is the assumption that the defects are "thin," that is, that their transverse size is much smaller than the de Sitter horizon. This will typically be the case if the symmetry-breaking scale of the defects, $\eta$, is large compared to $H$. However, Eqs. (1.11), (1.17), and (1.18) indicate that for $\eta \gg H$ the tunneling action is large and the nucleation probability is very low. In the most interesting case, when $\eta$ is comparable to $H$, the "thin-defect" approximation is not valid and our results are meaningful only in a qualitative way. An accurate result would require a full field-theoretic treatment. The instantons can then be found by solving Euclidean Higgs and gauge field equations on a four-sphere. This remains a problem for future work.

A somewhat puzzling aspect of our results is the fact that the string and monopole instantons have more than one negative eigenmode, in apparent contradiction with Coleman's [18] theorem. We explained in Sec. V that there is no direct contradiction, since the theorem does not apply to this case, but the physical significance of the "wrong" number of negative eigenmodes remains obscure.

We briefly discussed possible cosmological implications of the nucleating defects. After nucleation, strings and walls are stretched by the exponential expansion of the universe, and by the end of inflation they have a wide spectrum of sizes which can extend well beyond the present Hubble radius. Somewhat surprisingly, we found that the size distribution of string loops has the same scale-invariant form as in the more familiar scenario where the strings are formed at a phase transition. The main difference is that the overall factor in this distribution depends on the nucleation probability and is very model dependent. With an appropriate choice of parameters the loops can serve as seeds for galaxies and clusters. Since nucleating loops are nearly circular, many of them can collapse to form black holes. The cosmological consequences of this, and the resulting constraints on the parameters of the model, will be discussed elsewhere. In the case of domain wall bubbles, all bubbles greater than a certain size inevitably collapse to form black holes. Gigantic black holes produced by this mechanism can also be cosmologically significant.

Finally, we would like to mention some related work in which formation of topological defects during inflation has been studied. Vishniac, Olive, and Seckel [3], Linde and Lyth [19], and Hodges and Primack [20] considered a situation in which the scalar field $\varphi$ responsible for the formation of defects has a very flat potential $V(\varphi)$. In this case quantum fluctuations of $\varphi$ in de Sitter space can be pictured as a random walk [21]: the field fluctuates by $\delta\varphi \sim H/2\pi$ on space and time scales $\sim H^{-1}$. These fluctuations can take $\varphi$ back and forth over the barrier between the degenerate minima of $V(\phi)$. Since fluctuations are statistically independent in regions separated by more than $H^{-1}$, the field $\varphi$ ends up on different sides of the barrier in different regions of space, leading to the formation of topological defects.

This work is closely related to ours, and is in a sense complimentary. The WKB method that we employed does not apply when the tunneling action is small, and in this case the random-walk approach is certainly more appropriate. On the other hand, when the potential $V(\varphi)$ is steep, the Brownian picture of fluctuations breaks down, and our method should be preferred. It would be interesting to investigate the relation between the two approaches in more detail and to find, in particular, if there is any overlap in their areas of applicability.

## ACKNOWLEDGMENTS

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