Hidden variables and quantum-mechanical probabilities for generalized spin-s systems

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We derive some generalized spin-s Bell inequalities for a set of three distinct coplanar axes. Using a remarkable theorem due to Kronecker, we show that for at least 85.7% of the volume of the three-axis (coplanar) configuration space, the magnitude of violation of Bell inequalities vanishes approximately as $1/s^2$, far more slowly than the inequalities of Garg and Mermin.

I. INTRODUCTION

In 1980, Mermin [1] derived a set of generalized spin-s Bell [2] inequalities, showing explicitly that local realism is inconsistent with the numerical predictions of quantum mechanics for arbitrary values of spin s right up to the classical $s \to \infty$ limit. However, both the magnitude and the range of violation of his inequalities vanish linearly with spin in the classical limit. Subsequently, Garg and Mermin [3] showed that this vanishing of the range of violation is an artifact of the particular analytical trick used in the argument. They derived some generalized spin-s Bell inequalities with a nonvanishing range of violation as $s \to \infty$. The magnitude of violation of their inequalities, however, vanishes extremely rapidly, falling off to 0 approximately as $\cos^{4s}(\theta/2)$, with θ being the angle between axes $\hat{a}_i$ and $\hat{a}_j$. A question of considerable theoretical interest is how fast local realism emerges in the classical limit, i.e., how fast the magnitude of violation of Bell inequalities vanishes for a nonvanishing range of angles as $s \to \infty$. In this paper, we shall address this question. We derive a set of generalized spin-s Bell inequalities with a magnitude of violation that vanishes approximately as $1/s^2$, considerably more slowly than the inequalities of Garg and Mermin.

II. LOCAL REALISM VERSUS QUANTUM THEORY

We start by considering the spin-s generalization of Bohm's [4] version of the Einstein-Podolsky-Rosen paradox [5], in which two counterpropagating particles in a singlet-spin state ϕ are emitted by the decay of a zero angular momentum particle and thus have zero total spin. If the spin of the second particle along axis $\hat{a}_i$ is n_i, then by conservation of spin the spin of the first particle along the same axis is $m_i = -n_i$. It is therefore possible to determine the spin of the first particle along any axis by measuring the spin of the second particle, which is assumed to be very far away, along the same axis without disturbing the first particle. Einstein [6], Podolsky, and Rosen (EPR) account for this by introducing their famous criterion of local realism [5]. "*If, without in anyway disturbing a system, we can predict with certainty (i.e., with probability unity) the value of a physical quantity, then there exists an element of physical reality corresponding to that physical quantity.*" In 1964, Bell pointed out that the requirement of local realism, as postulated by EPR, essentially means that each separate particle should be characterized by an N-axis distribution function $p_{\hat{a}_1\hat{a}_2\cdots\hat{a}_n}(m_1,m_2,\ldots,m_n)$ which gives the probability that the spin components of the particle along axes $\hat{a}_1,\hat{a}_2,\ldots,\hat{a}_n$ are $m_1,m_2,\ldots,m_n$, with m_i taking any value in the set $\{-\frac{1}{2},\frac{1}{2}\}$ (throughout this paper we use the notation of Garg and Mermin [1]). Quantum theory, however, vehemently denies that such a probability distribution function has any meaning for a single particle, since it assigns simultaneous values to several components of the noncommuting spin operators. In his original paper, Bell showed that for $s=\frac{1}{2}$, the existence of a three-axis probability distribution function $p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)$ leads to the validity of an inequality that is sometimes grossly violated by quantum theory. In the following, we shall generalize Bell's result for an arbitrary value of the spin, and show that the magnitude of violation of these inequalities vanishes approximately as $1/s^2$ as $s \to \infty$.

Our results are based on the fact that if local realism holds, that is, if each particle is characterized by an N-axis probability distribution function $p_{\hat{a}_1\hat{a}_2\cdots\hat{a}_n}(m_1,m_2,\ldots,m_n)$, then the two-axis distribution of a single particle, $p_{\hat{a}_1\hat{a}_2}(m_1,m_2)$, can be expressed in terms of the quantum-theoretic joint distribution function, i.e.,

$$p_{\hat{a}_1\hat{a}_2}(m_1,m_2)=q_{\hat{a}_1\hat{a}_2}(m_1,n_2) \ . \tag{1}$$

According to the quantum theory, $q_{\hat{a}_1\hat{a}_2}(m_1,n_2)$, which gives the probability that the spin of the first particle along $\hat{a}_1$ is m_1 and that the spin of the second particle along $\hat{a}_2$ is $n_2=-m_2$, is a perfectly well-defined quantity, whereas $p_{\hat{a}_1\hat{a}_2}(m_1,m_2)$, which gives the probability that the spin components of the first particle along axes $\hat{a}_1$ and $\hat{a}_2$ are m_1 and m_2, has no meaning for a single particle. However, from a statistical point of view, there is absolutely nothing objectionable about $p_{\hat{a}_1\hat{a}_2}(m_1,m_2)$; it is non-negative, normalizable, and returns $p_{\hat{a}_1}(m_1)$ as marginal. The difficulty arises when one attempts to characterize a single particle by a three-axis distribution $p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)$ that would return $p_{\hat{a}_1\hat{a}_2}(m_1,m_2)$ as marginal. In the following, we shall show that the existence of such a three-axis distribution function is numerically inconsistent with the quantitative predictions of quantum theory.

Given any candidate for the three-axis distribution function $p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)$, we define the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)$ as the expected value of $(xm_1+ym_2+m_3)^2$, i.e.,

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)=\frac{1}{s^2}\langle (xm_1+ym_2+m_3)^2\rangle_{\hat{a}_1\hat{a}_2\hat{a}_3} \tag{2}$$

$$=\frac{1}{s^2}\sum_{m_1m_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3) \times (xm_1+ym_2+m_3)^2 \ . \tag{3}$$

Note that (2) is the correct average, since the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)$ remains of order 1 for any value of the spin, including the classical $s\to\infty$ limit because of the factor $1/s^2$ in front. We evaluate the minimum of the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$, using two different but equivalent techniques: in one technique, we calculate $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ by explicitly squaring the trinomial in Eq. (2); in the other technique, we evaluate $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ from its definition, i.e., from Eq. (3). If local realism holds, that is, if the three-axis probability distribution function $p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)$ exists, then both techniques should certainly give the same result.

III. VIOLATION OF BELL INEQUALITIES

First we calculate $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ from Eq. (2). By explicitly squaring the trinomial, we obtain

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)=\frac{1}{s^2}(x^2\langle m_1\rangle^2_{\hat{a}_1\hat{a}_1}+y^2\langle m_2\rangle^2_{\hat{a}_2\hat{a}_2} +\langle m_3\rangle^2_{\hat{a}_3\hat{a}_3}+2xy\langle m_1m_2\rangle_{\hat{a}_1\hat{a}_2} +2x\langle m_1m_3\rangle_{\hat{a}_1\hat{a}_3}+2y\langle m_2m_3\rangle_{\hat{a}_2\hat{a}_3}) \ , \tag{4}$$

where the two-axis correlation function $\langle m_im_j\rangle_{\hat{a}_i\hat{a}_j}$ is defined as [7]

$$\langle m_im_j\rangle_{\hat{a}_i\hat{a}_j}=\sum_{m_im_j} m_im_jp_{\hat{a}_i\hat{a}_j}(m_i,m_j) \tag{5}$$

$$=-\sum_{m_im_j} m_in_jp_{\hat{a}_i\hat{a}_j}(m_i,-n_j) \tag{6}$$

$$=-\sum_{m_in_j} m_in_jq_{\hat{a}_i\hat{a}_j}(m_i,n_j) \tag{7}$$

$$=-\langle m_in_j\rangle_{\hat{a}_i\hat{a}_j} \tag{8}$$

$$=-k_s\hat{a}_i\hat{a}_j \ , \tag{9}$$

with $ij=1, 2, 3$. Here (5) is true by definition; (6) follows from the conservation of spin, i.e., from the fact that $m_j=-n_j$; (7) follows from Eq. (1); (8) is true by definition; and (9) follows from the rotational invariance of the singlet spin-s state. [Note that $k_s=s(s+1)/3$, but this is not needed for any of our arguments.] By substituting Eq. (9) in Eq. (4), we obtain

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)=-\frac{k_s}{s^2}(x^2+y^2+1+2xy\cos\alpha +2x\cos\gamma+2y\cos\beta) \ , \tag{10}$$

where α is the angle between $\hat{a}_1$ and $\hat{a}_2$, γ is the angle between $\hat{a}_1$ and $\hat{a}_3$, and β is the angle between $\hat{a}_2$ and $\hat{a}_3$, with $\gamma=\alpha+\beta$ (note that the three axes are assumed to be coplanar). The minimum of $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)$, i.e., $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ occurs at

$$\frac{\partial f}{\partial x}=0 \quad \frac{\partial f}{\partial y}=0 \ , \tag{11}$$

or

$$\begin{aligned} &2x_{\min}+2y_{\min}\cos\alpha+2\cos\gamma=0 \ , \\ &2y_{\min}+2x_{\min}\cos\alpha+2\cos\beta=0 \ . \end{aligned} \tag{12}$$

Solving (12), we obtain

$$x_{\min}=\frac{\sin\beta}{\sin\alpha}, \quad y_{\min}=-\frac{\sin(\alpha+\beta)}{\sin\alpha} \ . \tag{13}$$

Substituting $(x_{\min},y_{\min})$ in $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x,y)$, it can easily be shown that $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})=0$. Thus, by explicitly squaring the trinomial in (2), we have shown that the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})=0$.

We now calculate $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ from its definition, i.e., from

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min}) =\frac{1}{s^2}\sum_{m_1m_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3) \times\left[\frac{\sin\beta}{\sin\alpha}m_1-\frac{\sin(\alpha+\beta)}{\sin\alpha}m_2+m_3\right]^2 . \tag{14}$$

If local realism holds, that is if $p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2m_3)$ exists, then $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ obtained from (14) should also be 0. The only property needed to establish a nonzero lower bound on the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ is the following: *m_1,m_2,m_3 are uniformly distributed in the interval* $\{-s,-s+1,\ldots,s-1,s\}$, i.e., $p_{\hat{a}_1}(m_1)=p_{\hat{a}_2}(m_2)=p_{\hat{a}_3}(m_3)=1/(2s+1)$. It should be noted that our result does not depend on the explicit form of the two-axis probability distribution function. It is perhaps quite remarkable that a nonzero lower bound on $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ can be obtained without exploiting any of the properties of $p_{\hat{a}_i\hat{a}_j}(m_i,m_j)$. In the following, we shall consider two cases: (1) $x_{\min}$ and $y_{\min}$ are rational, and (2) $x_{\min}$ and $y_{\min}$ are irrational. We shall show that in both cases the magnitude of violation vanishes approximately as $1/s^2$, much more slowly than the inequalities of Garg and Mermin.

A. Violation of Bell inequalities for rational values $x_{\min}$ and $y_{\min}$

First we consider the special case when $x_{\min}$ and $y_{\min}$ are rational, i.e., when $\sin\beta/\sin\alpha=p/q$ and $-\sin(\alpha+\beta)/\sin\alpha=r/t$, with $p,q,r,$ and t integers. We also assume that s is an integer (we shall consider half-integer spins later). Let $r|q|/t=u/v$, where the greatest common divisor of u and v is 1, i.e., u and v are relatively prime, and let $m_2\neq lv$, where l is an integer. Since the sum in Eq. (14) is term by term non-negative, it is bounded by any partial sum:

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}\left[\frac{p}{q},\frac{r}{t}\right]\geq\frac{1}{s^2}\sum_{m_1m_2\neq lvm_3}p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)\times\left[\frac{p}{q}m_1+\frac{r}{t}m_2+m_3\right]^2 . \tag{15}$$

Let μ^2 be

$$\mu^2=\left[\frac{p}{q}m_1+\frac{r}{t}m_2+m_3\right]^2 , \tag{16}$$

or

$$|\mu|=\left|\frac{p}{q}m_1+\frac{r}{t}m_2+m_3\right| . \tag{17}$$

By multiplying both sides by $|q|$, we obtain

$$|\mu q|=\left|\pm pm_1+\frac{r|q|}{t}m_2+m_3|q|\right| . \tag{18}$$

Since $\pm pm_1$ and $|q|m_3$ are integers, let $n=\pm pm_1+|q|m_3$, where n is an integer. Using the relation $r|q|/t=u/v$, we obtain

$$|\mu q|=\left|\frac{u}{v}m_2+n\right| . \tag{19}$$

Because $m_2\neq lv$ (where l is an integer), we can immediately conclude that

$$|\mu q|=\left|\frac{u}{v}m_2+n\right|\geq\left|\frac{1}{v}\right| , \tag{20}$$

or

$$\mu^2=\left[\frac{p}{q}m_1+\frac{r}{t}m_2+m_3\right]^2\geq\frac{1}{(qv)^2} . \tag{21}$$

Thus, for integer spins

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}\left[\frac{p}{q},\frac{r}{t}\right]\geq\frac{1}{s^2}\frac{1}{(qv)^2}\sum_{m_1,m_2\neq lv,m_3}p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3) . \tag{22}$$

We now obtain a lower bound on the summation $\sum_{m_1,m_2\neq lv,m_3}p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)$. Obviously

$$\sum_{m_1,m_2\neq lv,m_3}p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)=\sum_{m_2\neq lv}p_{\hat{a}_2}(m_2)=1-\sum_{m_2=lv}p_{\hat{a}_2}(m_2) , \tag{23}$$

since m_2 is uniformly distributed in the interval $\{-s,\ldots,s\}$, i.e., $p_{\hat{a}_2}(m_2)=1/(2s+1)$,

$$\sum_{m_2=lv}p_{\hat{a}_2}(m_2)\leq\frac{1}{2s+1}\frac{2s+1}{v} ; \tag{24}$$

hence

$$\sum_{m_1,m_2\neq lv,m_3}p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2m_3)\geq 1-\frac{1}{v} . \tag{25}$$

Substituting (25) in (22), we finally obtain a lower bound on the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ for integer spins:

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})\geq\frac{1}{s^2}\frac{1}{(qv)^2}\left[1-\frac{1}{v}\right] . \tag{26}$$

We now consider half-integer spins. Let

$$|\mu|=\left|\frac{p}{q}\frac{m_1}{2}+\frac{r}{t}\frac{m_2}{2}+\frac{m_3}{2}\right| . \tag{27}$$

By multiplying both sides of (27) by 2, and employing the same argument as before, we obtain

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})\geq\frac{1}{s^2}\frac{1}{(2qv)^2}\left[1-\frac{1}{v}\right] . \tag{28}$$

In fact, since m_1, m_2, and m_3 can only take odd integer values, we can obtain a stronger lower bound, but that is not necessary for our purpose.

Let us consider a concrete example. Let $\sin\alpha=\sin\beta=\frac{3}{5}$, then the magnitude of violation is larger than or equal to $4/125s^2$ for all values of the spin. As another example, let s be half-integer, and let $\alpha=\beta=\gamma=120°$; then $x_{\min}=1$ and $y_{\min}=1$. Thus

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(1,1)=\frac{1}{s^2}\sum_{m_1m_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)\times(m_1+m_2+m_3)^2 \ . \quad (29)$$

Since m_1, m_2 and m_3 are half integers, the smallest term in the sum is $(m_1+m_2+m_3)^2=\frac{1}{4}$. Therefore, for this kind of geometry, the magnitude of violation of the Bell inequalities is greater than or equal to $1/4s^2$ for all half-integer spins.

In fact, one can use this technique to show that the magnitude of violation vanishes approximately as $1/s^2$ for all rational values of $x_{\min}$ and $y_{\min}$ (except for the very special case when $x_{\min}=\pm y_{\min}=\pm 1$, and s an integer). Rational numbers, however, constitute a set of measure 0, and one might suspect that the range of angles over which the above results hold is a set of measure 0. It would be far more useful to have a lower bound on the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ for a set of irrational real numbers with a nonzero measure. This is particularly significant considering the errors in the orientation of the analyzer which, in general, correspond to the variations in $x_{\min}$ and $y_{\min}$ [8,9]. In the following, we shall generalize our results for irrational values of $x_{\min}$ and $y_{\min}$.

B. Violation of Bell inequalities for irrational values $x_{\min}$ and $y_{\min}$

1. $\alpha=\beta$

First we consider the special case when $\alpha=\beta$. Once this simple case is understood, we shall then generalize our results for $\alpha\neq\beta$. In this case $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ is defined as

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})=\frac{1}{s^2}\sum_{m_1m_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)\times[m_1-2\cos(\alpha)m_2+m_3]^2 \ . \quad (30)$$

Using a remarkable theorem due to Kronecker [10], we obtain a lower bound on $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$.

Theorem. If n is an integer, μ an irrational, and $\{n\mu\}=n\mu-[n\mu]$, with $[n\mu]$ being the greatest integer in $n\mu$, then the series of points $\{0\mu\},\{1\mu\},\{2\mu\},\{3\mu\},\{4\mu\},\ldots$ are uniformly distributed in the interval (0,1).

Since $2\cos(\alpha)m_2$ is assumed to be irrational, the fractional part of $2\cos(\alpha)m_2$, i.e., $\{2\cos(\alpha)m_2\}$, is uniformly distributed over the interval (0,1) as $s\to\infty$. We choose only those m_2, denoted by n_2, which satisfy $\{2\cos(\alpha)n_2\}\in(0.25,0.75)$. Since each term in the sum is non-negative,

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})\geq\frac{1}{s^2}\sum_{m_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,n_2,m_3)\times(m_1-2\cos(\alpha)n_2+m_3)^2 \ . \quad (31)$$

Because the smallest term in the sum is 0.25,

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})\geq\frac{1}{s^2}0.25\sum_{m_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,n_2,m_3)\ , \quad (32)$$

but

$$\sum_{m_1n_2m_3} a_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,n_2,m_3)=\sum_{n_2} p_{\hat{a}_2}(n_2)\ . \quad (33)$$

Now as $s\to\infty$, according to Kronecker's theorem,

$$\sum_{n_2} p_{\hat{a}_2}(n_2)=0.5; \quad (34)$$

thus

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})\geq\frac{1}{s^2}(0.25)(0.5)\ . \quad (35)$$

Note that the above result holds for both integer and half-integer spins.

2. $\alpha\neq\beta$

We now consider the most general case when $\alpha\neq\beta$ and $x_{\min}$ and $y_{\min}$ are irrational real numbers. First we assume that $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ is less than or equal to δ (where we assume that δ vanishes faster than $1/s^2$); then by rotating the axes so that $(\alpha\to\alpha,\beta\to-\beta)$, we shall show that the magnitude of violation for such a configuration vanishes approximately as $1/s^2$. The strategy of such an argument is as follows.

The function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})$ is assumed to satisfy

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})=\frac{1}{s^2}\sum_{m_1m_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)\times\left[\frac{\sin\beta}{\sin\alpha}m_1-\frac{\sin(\alpha+\beta)}{\sin\alpha}m_2+m_3\right]^2\leq\delta\ . \quad (36)$$

Let n_2 take values in a subset of $\{-s,\ldots,s\}$ (for example n_2 may take any value in the set $\{-s,\ldots,1\}$, or $\{0,2,s-1\}$, or $\{s\}$, etc.). Because the sum in (36) is term by term non-negative, it is bounded by any partial sum

$$\frac{1}{s^2}\sum_{m_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,n_2,m_3)\times(x_{\min}m_1+y_{\min}n_2+m_3)^2\le\delta\ . \quad (37)$$

Given any n_2, we divide the values that m_1 take into two sets: those denoted by n_1, which satisfy

$$0.6<\{x_{\min}n_1+y_{\min}n_2\}<0.4\ , \quad (38)$$

and those denoted by q_1, which satisfy

$$0.4\le\{x_{\min}q_1+y_{\min}n_2\}\le 0.6\ , \quad (39)$$

where $\{\mu\}$ is the fractional part of μ. Our first goal is to obtain a lower bound on $\sum_{n_1n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)$. Because the sum in (36) is term by term non-negative,

$$\frac{1}{s^2}\sum_{q_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(q_1,n_2,m_3)(x_{\min}q_1+y_{\min}n_2+m_3)^2\le\delta\ . \quad (40)$$

Because the smallest term in the sum is $(0.4)^2$,

$$\frac{1}{s^2}0.16\sum_{q_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(q_1,n_2,m_3)\le\delta\ , \quad (41)$$

summing over m_3, we obtain

$$\sum_{q_1n_2} p_{\hat{a}_1\hat{a}_2}(q_1,n_2)\le\frac{\delta s^2}{0.16}\ . \quad (42)$$

Our goal, however, is to obtain a lower bound on $\sum_{n_1n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)$ rather than on $\sum_{q_1n_2} p_{\hat{a}_1\hat{a}_2}(q_1,n_2)$. We use the identity

$$\sum_{q_1n_2} p_{\hat{a}_1\hat{a}_2}(q_1,n_2)=\sum_{n_2} p_{\hat{a}_2}(n_2)-\sum_{n_1n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)\ . \quad (43)$$

Substituting (43) in (42), we finally obtain

$$\sum_{n_1n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)\ge\sum_{n_2} p_{\hat{a}_2}(n_2)-\frac{\delta s^2}{0.16}\ . \quad (44)$$

Now rotate the axes so that $(\alpha\to\alpha,\beta\to-\beta)$ or $(x_{\min}\to -x_{\min},y_{\min}\to z_{\min})$ (note that it is absolutely crucial that $|x_{\min}|$ does not change when we rotate the axes). For the new set of axes, we shall show that the magnitude of violation vanishes approximately as $1/s^2$ in the classical limit. The function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})$ is defined as

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})=\frac{1}{s^2}\sum_{m_1m_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,m_2,m_3)\times\left[\frac{-\sin\beta}{\sin\alpha}m_1-\frac{\sin(\alpha-\beta)}{\sin\alpha}m_2+m_3\right]^2. \quad (45)$$

Since $z_{\min}+y_{\min}$ is assumed to be irrational, according to Kronecker's theorem, $(z_{\min}+y_{\min})m_2$ is uniformly distributed in the interval (0,1) as $s\to\infty$. Therefore, 50% of m_2, which we denote them by n_2, satisfy the relation

$$0.25\le\{(z_{\min}+y_{\min})n_2\}\le 0.75\ . \quad (46)$$

That is

$$\sum_{n_2} p_{\hat{a}_2}(n_2)=\tfrac{1}{2}\ . \quad (47)$$

Now we obtain a lower bound on the function $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})$. We choose only those m_2 which satisfy (46) and denote them by n_2. Since each term of the sum in Eq. (45) is non-negative,

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})\ge\frac{1}{s^2}\sum_{m_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,n_2,m_3)(-x_{\min}m_1+z_{\min}n_2+m_3)^2\ . \quad (48)$$

Adding and subtracting $y_{\min}n_2$, we obtain

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})\ge\frac{1}{s^2}\sum_{m_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(m_1,n_2,m_3)[-x_{\min}m_1-y_{\min}n_2+(z_{\min}+y_{\min})n_2+m_3]^2\ . \quad (49)$$

Now we choose only those m_1 which satisfy (38) and denote them by n_1; thus,

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})\ge\frac{1}{s^2}\sum_{n_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(n_1,n_2,m_3)[-x_{\min}n_1-y_{\min}n_2+(z_{\min}+y_{\min})n_2+m_3]^2\ . \quad (50)$$

Using the inequalities (46) and (38), one can easily show that the smallest term on the right-hand side of (50) is $(0.4-0.25)^2$, thus,

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})\ge\frac{1}{s^2}0.15\sum_{n_1n_2m_3} p_{\hat{a}_1\hat{a}_2\hat{a}_3}(n_1,n_2,m_3)\ . \quad (51)$$

Summing over m_3, we obtain

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})\ge\frac{1}{s^2}0.15\sum_{n_1n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)\ . \quad (52)$$

But according to (44) and (47),

$$\sum_{n_1n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)\ge\frac{1}{2}-\frac{\delta s^2}{0.16}\ . \quad (53)$$

Since we have assumed that δ vanishes faster than $1/s^2$,

$$\lim_{s\to\infty}\frac{\delta s^2}{0.15}\to 0\ , \tag{54}$$

or

$$\sum_{n_1 n_2} p_{\hat{a}_1\hat{a}_2}(n_1,n_2)\geq\tfrac{1}{2}\ . \tag{55}$$

Thus, we finally obtain

$$f_{\hat{a}_1\hat{a}_2\hat{a}_3}(-x_{\min},z_{\min})\geq\frac{1}{s^2}(0.15)\tfrac{1}{2}\ . \tag{56}$$

By symmetry and using exactly the same argument, one can easily show that if $f_{\hat{a}_1\hat{a}_2\hat{a}_3}(x_{\min},y_{\min})\leq\delta$ for the geometry (α,β), then for the choices of axes $(\alpha\to\alpha,\beta\to\pi-\beta)$, $(\alpha\to-\alpha,\beta\to\beta)$, $(\alpha\to-\alpha,\beta\to\pi+\beta)$, $(\alpha\to\pi-\alpha,\beta\to\beta)$, $(\alpha\to\pi+\alpha,\beta\to-\beta)$, the lower bound on the magnitude of violation is given by the right-hand side of (56). Thus the magnitude of violation of Bell inequalities vanishes approximately as $1/s^2$ for at least $\frac{6}{7}\approx 85.7\%$ of the volume of the three-axis (coplanar) configuration space.

IV. CONCLUSION

To summarize, we have derived a set of generalized spin-s Bell inequalities with a magnitude of violation that vanishes approximately as $1/s^2$, considerably more slowly than the inequalities of Garg and Mermin. From the results derived in this paper, we conclude that the emergence of local realism in the classical limit is signaled by the vanishing (approximately as $1/s^2$) of the magnitude of violation of Bell inequalities. We finally wish to point out that it may be possible to derive Bell inequalities that vanish even more slowly, approximately as $1/s$, for a nonvanishing range of angles in the classical limit.

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