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Towards a unified treatment of Yang-Mills and Higgs fields

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Starting from a noncommutative algebra $\mathcal{A}$ of the form $\mathcal{C}\otimes\mathcal{M}$, where $\mathcal{C}$ is the algebra of smooth functions on space-time and $\mathcal{M}$ is the algebra of $n\times n$ Hermitian matrices, we construct an exterior algebra of differential forms over $\mathcal{A}$. We use the one-forms of this algebra to describe Yang-Mills and Higgs fields on a similar footing and construct a Lagrangian from its two-forms. We show how, in the resulting geometrical description, a Higgs potential that leads to spontaneous symmetry breaking arises naturally. We discuss the application of this formalism to the bosonic sectors of the standard electroweak theory and a grand-unified model based on SU(5)⊗U(1).

I. INTRODUCTION

Of all the shortcomings of the standard model, the most serious one is perhaps the symmetry-breaking mechanism using fundamental scalar fields known as Higgs fields. Unlike the gauge sector involving the fermions and vector bosons, the form, the content, and the couplings in the Higgs sector are not determined by gauge principles alone. As a consequence, additional *ad hoc* assumptions seem to be indispensable in model building. There are, of course, in addition the well-known fine tuning and hierarchy problems coupled with the fact that the Higgs particle masses are not predictable. It is not surprising, therefore, that there are several attempts in the literature to eliminate some of these difficulties.

In the early 1980s, for instance, several authors [1] proposed Kaluza-Klein ideas to unify gauge and Higgs fields. In space-times with extended space degrees of freedom, the additional components of fields other than four were identified as the Higgs fields: $A_M(x^0,x^1,\dots,x^{d-1})$ give gauge fields for $M=0,\dots,3$ and Higgs fields for $M=4,\dots,d-1$. By imposing suitable conditions on the dependence of the fields on the additional degrees of freedom, such theories could be reduced to four-dimensional Yang-Mills theories with built-in spontaneous symmetry breaking. Such models had the virtue of being able to predict, in the case of the Glashow-Salam-Weinberg model of electroweak interactions, the Weinberg angle, and the mass of the Higgs particle. However, the results found in specific models were not in accord with experiments. Hence, in spite of their aesthetic appeal, they were not pursued further.

More recently, some new ideas have made their appearance under the general category characterized by noncommutative geometry [2]. In one such approach advocated by Connes and applied to the standard model by Lott and Connes [3], there are more than one identical copies of space-times characterized by a discrete index $p=1,2,\dots$. While in a given space-time the customary gauge fields provide the connections through the one-forms $A_\mu dx^\mu$, the connections between discrete space-times are identified as Higgs fields. The mixed differential forms are assembled together in the form of a matrix with graded multiplication properties [4]. An exterior derivative defined on such matrices enables one to calculate two-forms and construct a gauge-invariant Lagrangian that has some interesting properties as well as predictive power. However, from our point of view, the starting point of such models is rather arbitrary. If one demands, at the beginning, appropriate transformation properties under allowed arbitrary gauge transformations, one finds that a number of additional fields are necessary.

In this paper, we follow an alternate approach initiated by Dubois-Violette, Kerner, and Madore [5]. The central idea here, as in Kaluza-Klein theory, is to extend the space-time degrees of freedom; however, the extension is not by continuous space degrees of freedom, but by finite matrices, which are supposed to describe the internal structure of a particle. The starting point is then an associative but noncommutative algebra $\mathcal{A}$ of the form $\mathcal{C}\otimes\mathcal{M}$, where $\mathcal{C}$ is the algebra of smooth functions of space-time coordinates and $\mathcal{M}$ is the algebra of $n\times n$ matrices. By defining an exterior algebra of forms over $\mathcal{A}$, we construct a Lagrangian from the generalized two forms.

In this approach, the Higgs fields originate as $\mathcal{C}$-valued coefficients when one takes the exterior derivatives of the generators of $\mathcal{M}$. In the resulting geometrical description, Yang-Mills and Higgs fields appear on the same footing. The content of the Higgs fields is determined by the choice of the exterior derivative. Further, apart from the overall gauge coupling constant, the only free parameters in the Lagrangian are a limited number of scale parameters. They too depend on the manner in which the exterior derivative is defined. The resulting model is therefore quite restrictive. A Higgs potential with spontaneously broken symmetry appears naturally in the Lagrangian. It consists of a sum of terms, each of which is a perfect square, its absolute minimum given by the vanishing of each term. The model also implies certain relations among the coupling constants at the classical level. However, these relations, whose origin is not clear at the moment, appear to exist at only one set of mass scales in the Lagrangian. Quantum corrections induce departures from these relations [6].

In the next section, we describe the general mathematical framework. It is followed by a discussion of the most straightforward choice of the exterior derivative resulting in the models proposed by Dubois-Violette, Kerner, and Madore [5]. In Sec. III, we review our previously considered SU(2)⊗U(1) model [6] and discuss briefly another model based on embedding SU(2)⊗U(1) generators into SU(2|1) algebra. In Sec. IV, we discuss its generalization to SU(5)⊗U(1), grand unified, and explore some of its physical consequences. The final section is devoted to a summary and conclusions.

II. GENERAL FRAMEWORK

As stated in the preceding section, the noncommutative algebra $\mathcal{A}$ is made of two factors. One of them is the commutative algebra $\mathcal{C}$ of all smooth functions on the Minkowski space-time M^4. The second factor is taken to be the algebra $\mathcal{M}_n$ of complex $n\times n$ matrices. Thus

$$\mathcal{A}=\mathcal{C}\otimes\mathcal{M}_n \ . \tag{2.1}$$

One may choose the generalized Gell-Mann matrices λ_a, $a=1,\ldots,n^2-1$, along with the identity 1 (or $\lambda_0\equiv\sqrt{2/n}$) as a Hermitian basis for the algebra $\mathcal{M}_n$. Then a generic element of $\mathcal{A}$ can be written as

$$F=f_0(x)\lambda_0+f_a(x)\lambda_a \ . \tag{2.2}$$

Here x's are the coordinates of M^4. Summation over a repeated index (here a) is assumed. Two elements of $\mathcal{A}$ can be multiplied, as usual, knowing the products $\lambda_a\lambda_b$ in terms of λ's and the identity.

In the ordinary case of M^4, it is found convenient to extend the algebra $\mathcal{C}$ to the so-called exterior algebra $\mathcal{C}^*$. $\mathcal{C}^*$ can be written as a direct sum

$$\mathcal{C}^*=\mathcal{C}^0\oplus\mathcal{C}^1\oplus\cdots\oplus\mathcal{C}^4 \ . \tag{2.3}$$

$\mathcal{C}^0$ is simply $\mathcal{C}$. To define others, one introduces a set of anticommuting objects dx^μ's. Then a generic element of $\mathcal{C}^1$ is the form $f_\mu(x)dx^\mu$, and this space is called the space of one-forms. And, more generally, $\mathcal{C}^p$ is made of p-forms of the kind

$$f_{\mu_1\cdots\mu_p}(x)dx^{\mu_1}\cdots dx^{\mu_p} \ . \tag{2.4}$$

There exists a differential operator on $\mathcal{C}^*$, the so-called exterior derivative. It is defined to be $d=dx^\mu\partial_\mu$. This operator satisfied $d^2=0$ and the Leibniz rule

$$d(fg)=d(f)g+(-1)^p f d(g) \ , \tag{2.5}$$

where f and g are respectively p- and q-forms. Its action on a p-form f is given by

$$df=\partial_\mu f_{\mu_1\cdots\mu_p}(x)dx^\mu dx^{\mu_1}\cdots dx^{\mu_p} \ . \tag{2.6}$$

We wish to extend the above notions to $\mathcal{M}_n$ and hence to $\mathcal{A}$. To extend the concept of forms to the algebra $\mathcal{M}_n$, we introduce anticommuting objects $\theta_a, a=1,\ldots,n^2-1$. These are analogous to dx^μ's. For simplicity, we denote them by θ's instead of the more obviously consistent notation $d\theta$'s. Define $\mathcal{M}_n^*$ to be the direct sum

$$\mathcal{M}_n^*=\mathcal{M}_n^0\oplus\mathcal{M}_n^1\oplus\cdots\oplus\mathcal{M}_n^{n^2-1} \ . \tag{2.7}$$

$\mathcal{M}_n^0$ is simply $\mathcal{M}_n$. $\mathcal{M}_n^1$ is made of objects like $F_a\theta_a$, where F_a is an element of $\mathcal{M}_n, F_a=f_{a0}\lambda_0+f_{ab}\lambda_b$. In general, $\mathcal{M}_n^p$ is made of objects like

$$F_{a_1\cdots a_p}\theta_{a_1}\cdots\theta_{a_p} =(f_{a_1\cdots a_p 0}\lambda_0+f_{a_1\cdots a_p a}\lambda_a)\theta_{a_1}\cdots\theta_{a_p} \ . \tag{2.8}$$

The exterior algebra associated with $\mathcal{A}$ can be taken to be the tensor product

$$\mathcal{A}^*=\mathcal{C}^*\otimes\mathcal{M}_n^*=\mathcal{A}^0\oplus\mathcal{A}^1\oplus\cdots\oplus\mathcal{A}^{n^2+3} \ . \tag{2.9}$$

In particular, $\mathcal{A}^p$, the space of generalized p-forms, is

$$\mathcal{A}^p=\sum_{k=0}^{p}\mathcal{C}^k\otimes\mathcal{M}_n^{p-k} \ . \tag{2.10}$$

It is assumed in the above that $\mathcal{C}^k$ and $\mathcal{M}_n^l$ vanish respectively for $k>4$ and $l>n^2-1$. Since forms can be added and multiplied, $\mathcal{A}^*$ represents an algebra, called the exterior algebra associated with $\mathcal{A}$. $\mathcal{A}^p$ consists of matrices each element of which is like a p-form in dx^μ's and θ's. In this way we seek a generalization of the usual non-Abelian gauge theories in which the differential forms are forms in $\mathcal{C}$ only. The coefficients of dx^μ's are the gauge potential matrices. By allowing space-time scalars as coefficients of θ's, in addition to space-time vectors as coefficients of dx^μ's, we can treat Higgs fields and gauge fields on a similar footing. It should be noted that, unlike in Refs. [3,4], every matrix element in $\mathcal{A}^p$ will be of a definite order as a generalized form. This way, we preserve the homogeneity of each term under multiplication. Forms of different order in diagonal and off-diagonal terms will lead to an inhomogeneous sum of terms of different orders. Gauge invariance will then require introducing additional fields or making *ad hoc* restrictions on the allowed gauge transformations.

Next we want to define an exterior derivative associated with the algebra $\mathcal{A}^*$. We have already noted that such an operator for $\mathcal{C}^*$ is $d=dx^\mu\partial_\mu$. This is constructed from the generators of the group of translations that act as automorphisms of $\mathcal{C}$. In other words,

$$e^{ia\cdot P}f(x)e^{-ia\cdot P}=f(x)+i[a\cdot P,f(x)]+\cdots =f(x)+a^\mu\partial_\mu f(x)+\cdots \ , \tag{2.11}$$

where P_μ's are the generators of translations, gives the infinitesimal change in $f(x)$ as $df(x)=a^\mu\partial_\mu f(x)$. Replacing a^μ's by anticommuting objects dx^μ's, we obtain the expression for the exterior derivative: $d=dx^\mu\partial_\mu$. In a similar fashion, if we can find an automorphism of the algebra $\mathcal{M}_n$, we can construct a differential operator associated with it and hence an exterior derivative for $\mathcal{M}_n^*$. This will let us generalize these notions to the full algebra $\mathcal{A}$. There is a natural choice for this automorphism, the so-called inner automorphism, given by

$$e^{i\alpha_b\lambda_b}\lambda_a e^{-i\alpha_c\lambda_c}=\lambda_a+i[\alpha_b\lambda_b,\lambda_a]+\cdots$$
$$\equiv\lambda_a+d\lambda_a+\cdots \ . \tag{2.12}$$

This forms the group SU(n) or SU(n)$\otimes$U(1), where the U(1) generator λ_0 drops out of the commutators. The infinitesimal change in λ_a is, thus,

$$d\lambda_a=i[\alpha_b\lambda_b,\lambda_a] \ . \tag{2.13}$$

This gives us a derivation E_a that acts on the elements (say, F) of the algebra $\mathcal{M}_n$ as

$$E_a(F)=\frac{m}{2}[\lambda_a,F] \ , \tag{2.14}$$

where m, having dimension of mass, introduces a scale into the theory (this is in analogy to ∂_μ that has dimension of mass in natural units). Being derived from an automorphism, this operator automatically obeys the Leibniz rule

$$E_a(FG)=E_a(F)G+FE_a(G) \ , \tag{2.15}$$

where F and G belong to $\mathcal{M}_n$. There exist other choices for the automorphisms as well, and hence for the derivations, that will be discussed in the next section. Unlike ∂_μ's, the above operators do not commute among themselves. Instead they satisfy

$$[E_a,E_b]=imf_{abc}E_c \ , \tag{2.16}$$

where f_{abc}'s are the structure constants of SU(n):

$$[\lambda_a,\lambda_b]=2if_{abc}\lambda_c \ . \tag{2.17}$$

We thus have a set of differential operators (∂_μ,E_a) acting on the full algebra $\mathcal{A}$. As for the exterior derivative, we follow Bowick and Gürsey [7] and construct the Becchi-Roueti-Stora-Tyutin- (BRST-) like operator Q as

$$Q=\theta_a(L_a+\tfrac{1}{2}S_a) \ , \tag{2.18}$$

where

$$L_a=E_a, \quad S_a=-imf_{abc}\theta_b\frac{\partial}{\partial\theta_c} \ . \tag{2.19}$$

L_a and S_a commute with each other, and both satisfy the commutation relations (2.16). Thus the resulting operator

$$Q=\theta_aE_a-i\frac{m}{2}f_{abc}\theta_a\theta_b\frac{\partial}{\partial\theta_c} \tag{2.20}$$

obeys $Q^2=0$. This will be the exterior derivative associated with $\mathcal{M}_n$. Now

$$D=d+Q \tag{2.21}$$

provides a natural exterior derivative for $\mathcal{A}^*$, since it follows from the automorphisms of $\mathcal{A}$, the direct product of the group of translations and that of the inner automorphisms of $\mathcal{M}_n$. θ's are taken to be anti-Hermitian to be consistent with the reality property of d. It is assumed that the objects dx^μ's and θ's anticommute among themselves so that d anticommutes with Q. This and the fact that $d^2=Q^2=0$ lead to the desired property $D^2=0$.

The natural object to be investigated next is the connection one-form in $\mathcal{A}$. It has the following structure:

$$\omega=A+\Phi \ . \tag{2.22}$$

A is an element of $\mathcal{C}^1$ taking values in $\mathcal{M}_n$:

$$A=-igA_\mu dx^\mu$$
$$=-ig\tfrac{1}{2}[A_{\mu0}(x)\lambda_0+A_{\mu a}(x)\lambda_a]dx^\mu \ , \tag{2.23}$$

where g is the coupling constant of the theory. These will turn out to be our usual gauge fields. The Higgs field is going to be Φ. It is an element of $\mathcal{M}^1$ taking values in $\mathcal{C}$:

$$\Phi=g\Phi_a\theta_a=g\tfrac{1}{2}[\phi_{a0}(x)\lambda_0+\phi_{ab}(x)\lambda_b]\theta_a \ . \tag{2.24}$$

A connection one-form comes with a gauge transformation. In this case it has the infinitesimal form

$$\delta\omega=D\epsilon+\omega\epsilon-\epsilon\omega \ , \tag{2.25}$$

where ϵ is an element of $\mathcal{A}$ and hence has the decomposition

$$\epsilon=-ig\tfrac{1}{2}[\epsilon_0(x)\lambda_0+\epsilon_a(x)\lambda_a] \ . \tag{2.26}$$

In terms of A and Φ the gauge transformation becomes

$$\delta A_{\mu0}=\partial_\mu\epsilon_0, \quad \delta A_{\mu a}=\partial_\mu\epsilon_a+gf_{abc}A_{\mu b}\epsilon_c \ ,$$
$$\delta\phi_{a0}=0, \quad \delta\phi_{ab}=-mf_{abc}\epsilon_c+gf_{bcd}\phi_{ac}\epsilon_d \ . \tag{2.27}$$

We can get rid of the nonhomogeneous part in the transformation of ϕ_{ab} by shifting ϕ_{ab} to $\phi_{ab}-(m/g)\delta_{ab}$ [that is Φ to $\Phi-(m/2)\lambda_a\theta_a$]. We assume in the following that this has been done. Now it is clear from these transformations that we have SU(n)$\otimes$U(1) gauge invariance. The Higgs sector, in this approach, consists of (n^2-1) singlets and (n^2-1) adjoint representations of SU(n) all having zero U(1) charge.

The field strength or the curvature two-form is the next object to be studied. Constructing it as usual we have

$$\Omega=D\omega+\omega^2 \ . \tag{2.28}$$

Its components are given by

$$\Omega=\tfrac{1}{2}\Omega_{\mu\nu}dx^\mu dx^\nu+\Omega_{\mu a}dx^\mu\theta_a+\tfrac{1}{2}\Omega_{ab}\theta_a\theta_b \ , \tag{2.29}$$

where

$$\Omega_{\mu\nu}=-igF_{\mu\nu}=-ig(\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu]) \ ,$$
$$\Omega_{\mu a}=g\mathcal{D}_\mu(\Phi_a)=g(\partial_\mu\Phi_a-ig[A_\mu,\Phi_a]) \ , \tag{2.30}$$
$$\Omega_{ab}=-ig(mf_{abc}\Phi_c+ig[\Phi_a,\Phi_b]) \ .$$

$F_{\mu\nu}$ gives the field strength of the SU(n)$\otimes$U(1) gauge fields. $\mathcal{D}_\mu(\phi_a)$ is the covariant derivative of the Higgs fields. As we shall see, the term Ω_{ab} is responsible for the Higgs potential.

Given the curvature two-form, one can write down a gauge-invariant Lagrangian. Consider

$$L=\frac{1}{2g^2}\,\mathrm{Tr}(\Omega_{ij}\Omega^{ij}) \;, \tag{2.31}$$

where the indices i and j are assumed to be summed over the full range: the space-time and the internal (noncommuting) directions. "Tr" stands for trace in the fundamental representation of SU(n). We thus have $\mathrm{Tr}(\lambda_a\lambda_b)=2\delta_{ab}$, where a,b run from 0 to n^2-1. In terms of the components of Ω, L becomes

$$L=-\tfrac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})+\mathrm{Tr}[\mathcal{D}_\mu(\Phi_a)\mathcal{D}^\mu(\Phi_a)]-V \;, \tag{2.32}$$

where the Higgs potential V is given by

$$V=\tfrac{1}{2}\,\mathrm{Tr}\{(mf_{abc}\Phi_c+ig[\Phi_a,\Phi_b])(mf_{abd}\Phi_d+ig[\Phi_a,\Phi_b])\} \;. \tag{2.33}$$

One of the features of the above Higgs potential is that it has a minimum that corresponds to spontaneous symmetry breaking. Being a sum of squares, V is (absolutely) minimized whenever $\Omega_{ab}=0$ or

$$mf_{abc}\Phi_c+ig[\Phi_a,\Phi_b]=0 \;. \tag{2.34}$$

This equation has two solutions. One of them is the trivial one, $\Phi_a=0$, that does not lead to any spontaneous symmetry breaking. The other one

$$\Phi_a=\frac{m}{2g}\lambda_a, \quad \text{or } \phi_{a0}=0, \quad \phi_{ab}=\frac{m}{g}\delta_{ab} \;, \tag{2.35}$$

breaks the gauge symmetry SU(n)⊗U(1) to U(1). This corresponds to SU(n) being completely broken and U(1) being untouched.

The approach outlined above is essentially that of Ref. [5]. There are some reasons why, with $n=2$, this cannot give rise to a realistic picture of the electroweak theory. First of all, we get only adjoint representations of Higgs fields. For the electroweak theory this is phenomenologically untenable, since the ρ-parameter constraint rules out the possibility of spontaneous symmetry breaking by triplet Higgs fields. Secondly, the unbroken U(1) is the same U(1) that appears explicitly in the gauge group SU(2)⊗U(1). In other words, there is no mixing of this U(1) with the one from SU(2). This will lead to zero value of the Weinberg angle. In Ref. [6] an approach is presented, which, though similar to the one presented above, cures these drawbacks and leads to a more realistic picture. In the next section we first give a brief review of this and then present a more general scheme that leads to a somewhat realistic grand-unified framework.

III. APPLICATION TO ELECTROWEAK THEORY

In the previous section, we obtained a Higgs potential that leads to spontaneous symmetry breaking. In the case of $n=2$, this corresponds to SU(2)⊗U(1) being broken down to U(1). As we have already noticed, the unbroken U(1) is that which appears explicitly in SU(2)⊗U(1). In the basis of Pauli matrices, τ_i's and $\tau_0\equiv 1$, this means that the generator τ_0 remains untouched. We wish to modify this scenario to bring the model closer to the standard model. One may trace the origin of this problem to the definition of the exterior derivative. The minimum of the Higgs potential corresponds to a solution of $\omega=0$. This equation is invariant under the gauge transformation (2.25) whenever the gauge parameter ϵ is annihilated by D. Because the operator D happened to satisfy $D(\tau_0)=0$, a U(1) gauge symmetry generated by τ_0 survives at the end. It thus follows that those generators annihilated by D will remain unbroken in the present approach. In the following we make use of this observation and define the exterior derivative operator suitably to keep the desired generators unbroken. First we will review this approach for the case of the standard model [8], and then we will generalize it to a grand-unified framework.

We want the electric charge to survive symmetry breaking. The Gell-Mann–Nishijima formula tells us that this generator is given by $(\tau_3+Y)/2$, the hypercharge Y being essentially τ_0. Thus we assume the existence of an exterior derivative D that satisfies

$$D(\tau_0+\tau_3)=0 \;. \tag{3.1}$$

We find it convenient to work with the generators of the algebra, q and $\bar{q}$, where

$$q=\tfrac{1}{2}(\tau_1+i\tau_2), \quad \bar{q}=\tfrac{1}{2}(\tau_1-i\tau_2) \;. \tag{3.2}$$

In terms of these, we may write (3.1) as $D(q\bar{q})=0$. This definition of the exterior derivative implies some identities satisfied by $D\tau$'s . For example, qDq and Dqq (and their conjugates) vanish, since

$$\begin{aligned} qDq&=qD(q\bar{q}q)=qD(q\bar{q})q+q(q\bar{q})Dq=0 \;,\\ Dqq&=Dq(q\bar{q}q)=D(qq\bar{q})q-qD(q\bar{q})q=0 \;, \end{aligned} \tag{3.3}$$

using $q^2=0$ and $D(q\bar{q})=0$. Other identities are

$$\begin{aligned} &\tau_0Dq=\tau_3Dq=Dq, \quad Dq\tau_0=-Dq\tau_3 \;,\\ &D\tau_0=-D\tau_3, \quad \tau_0D\tau_3=-\tau_3D\tau_3, \quad D\tau_3q=0 \;, \end{aligned} \tag{3.4}$$

and their conjugate relations. As a consequence of these relations, we will be able to work with a minimum number of Higgs fields and incorporate the Higgs doublet of the electroweak theory. The final Higgs potential will lead to a pattern of symmetry breaking that is closer to the desired one.

It is very useful to have an explicit expression for the exterior derivative. As we have noted earlier, this means looking for an automorphism of the algebra $\mathcal{M}_2$. Here we will make use of an outer automorphism that happens to be an inner automorphism of $\mathcal{M}_3$. To this end, we first embed the algebra of Pauli matrices $\mathcal{M}_2$ into that of Gell-Mann λ matrices $\mathcal{M}_3$. We identify τ_a's with the generators of I-spin λ_a's ($a=1,2,3$) and τ_0 with $Y+\frac{2}{3}$, where Y is the hypercharge $\lambda_8/\sqrt{3}$. Other elements of $\mathcal{M}_3$ are the so-called U and V spins given by

$$U_\pm=\tfrac{1}{2}(\lambda_6\pm i\lambda_7),\quad U_3=\tfrac{1}{2}[U_+,U_-]\;,\qquad V_\pm=\tfrac{1}{2}(\lambda_4\pm i\lambda_5),\quad V_3=\tfrac{1}{2}[V_+,V_-]\;. \tag{3.5}$$

The inner automorphism we will be concerned with reads

$$e^{i\alpha_b U_b}\tau_a e^{-i\alpha_c U_c}=\tau_a+i[\alpha_b U_b,\tau_a]+\cdots \equiv\tau_a+d\tau_a+\cdots\;. \tag{3.6}$$

This gives the infinitesimal variations $d\tau_a=i[\alpha_b U_b,\tau_a]$. This allows us to define the derivations, E's, as

$$E_\pm(F)=\frac{m}{\sqrt{2}}[U_\pm,F],\quad E_3(F)=\frac{m^2}{M}[U_3,F]\;, \tag{3.7}$$

where F is an element of $\mathcal{M}_2$, obeying the following commutation relations:

$$[E_+,E_-]=ME_3,\quad [E_3,E_\pm]=\pm\frac{m^2}{M}E_\pm\;. \tag{3.8}$$

m and M are two mass scales that need to be introduced into the theory. We have chosen to include them in Eq. (3.7) for later convenience [8]. The full exterior derivative is, as before, $D=d+Q$, where

$$Q=\theta_+E_-+\theta_-E_++\theta_3E_3 +M\theta_+\theta_-\frac{\partial}{\partial\theta_3}+\frac{m^2}{M}\theta_3\theta_+\frac{\partial}{\partial\theta_+}+\frac{m^2}{M}\theta_-\theta_3\frac{\partial}{\partial\theta_-}\;. \tag{3.9}$$

As before, we take θ_3 anti-Hermitian and $\theta_+^\dagger=-\theta_-$. One may easily check that the operator D, so defined, annihilates $\tau_0+\tau_3$ as a consequence of the fact that E's do so. Further, all the identities presented in Eqs. (3.3) and (3.4) follow, as they should.

Now we are ready to define the connection one-form or the generalized gauge potential. It has the following structure:

$$\omega=A+\frac{g}{\sqrt{2}}H\theta_-+\frac{g}{\sqrt{2}}H^\dagger\theta_++g\Delta\theta_3\;, \tag{3.10}$$

where

$$A=-igA_\mu dx^\mu=-ig\tfrac{1}{2}(B_\mu\tau_0+W_{\mu a}\tau_a)dx^\mu\;,\qquad H=H_+V_++H_0U_+,\quad \Delta=\tfrac{1}{2}(\Delta_0\tau_0+\Delta_a\tau_a)\;, \tag{3.11}$$

and g is the coupling constant of the theory. We may note at this stage that B and W's are going to be the gauge bosons, while H's and Δ's are the scalars. This gauge potential should not be viewed as obtained from a one-form based on the algebra $\mathcal{M}_3$ by dropping certain fields. It may be worth emphasizing here that our model is based on the noncommuting algebra $\mathcal{M}_2$ and not $\mathcal{M}_3$. In writing ω, we used the basis involving U's and V's only for convenience. It is possible to rewrite the scalar sector entirely in terms of $D\tau$'s as

$$-\frac{M}{2}(\Delta_0-\Delta_3)\bar{q}Dq+\frac{M}{2}(\Delta_0+\Delta_3)qD\bar{q} +\left[\frac{M}{2}(\Delta_0-\Delta_3)-mH_0\right]\tau_3D\tau_3 -M(\Delta_1+i\Delta_2)\tau_3D\bar{q}+mH_+qD\tau_3-\text{H.c.}\;, \tag{3.12}$$

multiplied by g/m^2. This is an expansion in a complete basis of differentials of $\mathcal{M}_2$. Other differentials ($Dq,D\bar{q},D\tau_3$, and $D\tau_0$) can be expressed in this basis:

$$Dq=-D(q\tau_3)=-Dq\tau_3-qD\tau_3\;,\qquad D\tau_3=-D\tau_0=-D(\tau_3^2)=-D\tau_3\tau_3-\tau_3D\tau_3\;. \tag{3.13}$$

An expansion of the form (3.12) is not possible for the most general gauge potential based on $\mathcal{M}_3$.

Under a generalized gauge transformation, the gauge potential ω transforms as $\delta\omega=D\epsilon+\omega\epsilon-\epsilon\omega$, where the gauge parameter ϵ is a zero-form: $\epsilon=-ig(\epsilon_0\tau_0+\epsilon_a\tau_a)/2$. Writing this down explicitly, one finds that the theory has SU(2)⊗U(1) gauge invariance. W and B transform, respectively, as SU(2) and U(1) gauge bosons. After performing the following shifts

$$H_0\to H_0-\frac{m}{g},\quad \Delta_0\to\Delta_0+\frac{m^2}{2gM},\quad \Delta_3\to\Delta_3+\frac{m^2}{2gM}\;, \tag{3.14}$$

we find that the scalar sector transforms as the usual doublet Higgs field H with hypercharge one as well as a singlet and a triplet Higgs field, Δ_0 and Δ_i's respectively, with zero hypercharge. It is assumed in the following that these shifts have been performed.

The next step is to construct the field strength Ω. As before, we consider $\Omega=D\omega+\omega^2$, which transforms as $\delta\Omega=\Omega\epsilon-\epsilon\Omega$. Its components can be inferred from the expansion

$$\Omega=\tfrac{1}{2}\Omega_{\mu\nu}dx^\mu dx^\nu+\Omega_{\mu+}dx^\mu\theta_-+\Omega_{\mu-}dx^\mu\theta_++\Omega_{\mu3}dx^\mu\theta_3 +\Omega_{+-}\theta_-\theta_++\Omega_{+3}\theta_-\theta_3+\Omega_{3-}\theta_3\theta_+\;. \tag{3.15}$$

$\Omega_{\mu\nu}=-igF_{\mu\nu}$ gives the field strength associated with the gauge fields. The next three terms in the expansion yield covariant derivatives of the scalar fields,

$$\Omega_{\mu+}=\frac{g}{\sqrt{2}}\mathcal{D}_\mu H,\quad \Omega_{\mu-}=\Omega_{\mu+}^\dagger,\quad \Omega_{\mu3}=g\mathcal{D}_\mu\Delta\;, \tag{3.16}$$

where $\mathcal{D}_\mu=\partial_\mu-igA_\mu$ and $A_\mu\Delta$ stands for $[A_\mu,\Delta]$. The remaining three terms can be computed to be

$$\Omega_{+-}=\frac{g^2}{2}[H,H^\dagger]-gM\Delta-m^2\tau_0+\frac{m^2}{2}\;,\qquad \Omega_{+3}=-\frac{g^2}{\sqrt{2}}\Delta H,\quad \Omega_{3-}=\Omega_{+3}^\dagger\;. \tag{3.17}$$

Just as in the preceding section, we next write down a Lagrangian that is of the form (2.31). The trace is now taken over 3×3 matrices. The indices i and j are summed over the full range that includes the θ directions (+, −, and 3). After some algebra, one obtains the usual covariant kinetic terms for the various fields,

$$-\tfrac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})+\mathcal{D}_\mu H^\dagger \mathcal{D}^\mu H+\mathrm{Tr}(\mathcal{D}_\mu \Delta \mathcal{D}^\mu \Delta)\ , \tag{3.18}$$

and a Higgs potential $V(H,\Delta)$ given by

$$\begin{aligned}\frac{1}{g^2}V(H,\Delta)=&\frac{1}{4}\left[H^\dagger H-\frac{m^2}{g^2}\right]^2\\&+\frac{1}{2}\left[\frac{1}{2}H^\dagger H-\frac{M}{g}\Delta_0-\frac{m^2}{g^2}\right]^2\\&+\frac{1}{2}\left[\frac{1}{2}H^\dagger\tau_a H-\frac{M}{g}\Delta_a\right]^2\\&+\frac{1}{4}H^\dagger(\Delta_0+\Delta_a\tau_a)^2H\ .\end{aligned} \tag{3.19}$$

In the above expressions H is written as a two-component column vector with entries H_+ and H_0. The potential, being a sum of squares, can be minimized easily. It leads to a symmetry-broken vacuum given by

$$H_0=\frac{m}{g},\quad H_\pm=0,\quad \Delta_0=\Delta_3=-\frac{m^2}{2gM},\quad \Delta_{1,2}=0\ , \tag{3.20}$$

where only the electromagnetism survives symmetry breaking.

Coming to the mass spectrum of the model, one finds that the masses of Z and W bosons are, respectively, m and $m(1+m^2/2M^2)^{1/2}/\sqrt{2}$. Further, the ρ parameter turns out to be $1+m^2/2M^2$. Since experimentally ρ is very close to one, we require $M\gg m$. The Higgs sector gives two heavy scalars with masses M and $M(1+2m^2/M^2)^{1/2}$ and a light scalar with mass m. There are also three massless scalars, the would-be Goldstone bosons, that are eaten by the gauge bosons.

In this model we get certain relations among the coupling constants. For example, the two coupling constants g and g' that correspond to SU(2) and U(1), respectively, happen to be equal. This leads to a prediction for the Weinberg angle to be 45° from the relation $\tan\theta_W=g'/g=1$ or $\sin^2\theta_W=0.5$. We assume that such relations hold at the scale M that is, perhaps, given by the Planck scale. Because the couplings evolve, these relations get modified as we approach the electroweak scale. Assuming three generations of fermions, this brings down the value of $\sin^2\theta_W$ from 0.5 to 0.26 as we evolve from the scale M to the scale m. This is to be compared with the experimental value of 0.23. Similar evolution takes place for the Higgs self-coupling. Neglecting Yukawa couplings, this predicts the light Higgs mass to be about 80 GeV.

Before ending this section, we wish to point out that there is an alternate way of constructing the exterior derivative. Instead of embedding our algebra $\mathcal{M}_2$, loosely speaking SU(2), into SU(3) as we have done above, we now do the embedding into SU(2|1). SU(2|1) generators are essentially given by those of SU(3). However, now there is a grading involved; $U_\pm$ and $V_\pm$ are to be taken "fermionic," while all the others are to be taken "bosonic." This means replacing the commutators in Eq. (3.5) by anticommutators, thus redefining U_3 and V_3. This will change the "statistics" of $E_\pm$ as well; thus the first commutator in Eq. (3.8) will become an anticommutator. To have a fermionic BRST operator Q, we take $\theta_\pm$ to be bosonic, while θ_3 remains fermionic. We no longer need the last two terms, those proportional to $\partial/\partial\theta_\pm$, in the expression for Q to ensure $Q^2=0$:

$$Q=\theta_+E_-+\theta_-E_++\theta_3E_3-M\theta_+\theta_-\frac{\partial}{\partial\theta_3}\ . \tag{3.21}$$

With these modifications, we proceed exactly as before. The field strength Ω, corresponding to (3.15), turns out to be almost the same, Ω_{+-} being

$$\Omega_{+-}=\frac{g^2}{2}\{H,H^\dagger\}-gM\Delta-\frac{m^2}{2} \tag{3.22}$$

and the other components differing at most by a sign. In writing down the Lagrangian, it is natural to use the supertrace rather than the usual trace. This involves subtracting the third entry in the diagonal from the sum of the other two. However, the resulting Lagrangian does not involve any kinetic term for the Higgs doublet. We may thus consider taking the usual trace, though this is not natural from the point of view of SU(2|1) symmetry. This yields a Lagrangian that is identical to the one obtained before, with the same Higgs potential. All the results that follow are therefore the same, in particular, the Weinberg angle is 45° before the quantum corrections. Thus our results are markedly different from the SU(2|1) models of Ref. [9].

IV. GRAND-UNIFIED VERSION

Here we attempt to generalize the approach of the preceding section. This will lead us to a framework that is, though not completely realistic, closer to the bosonic sector of a grand-unified theory based on SU(5). As in the preceding section, we consider the noncommuting algebra $\mathcal{A}$ along with the set (∂_μ,E_a) acting as derivatives on $\mathcal{A}$. We define operators E_a by embedding the algebra of Hermitian $n\times n$ matrices ($\mathcal{M}_n$) into that of Hermitian $(n+1)\times(n+1)$ matrices ($\mathcal{M}_{n+1}$). For the sake of concreteness, we shall concentrate on $n=5$. The generalization to arbitrary n is straightforward.

In what follows, we find it convenient to utilize the non-Hermitian basis (corresponding to q and $\bar{q}$ of the preceding section) consisting of matrices T_a^b, where

$$(T_a^b)_{cd}=\delta_{ac}\delta_{bd},\quad a,b,c,d=1,\ldots,6\ . \tag{4.1}$$

These matrices form a basis for $\mathcal{M}_6$. They are normalized to satisfy $\mathrm{Tr}(T_a^bT_c^d)=\delta_{ad}\delta_{bc}$. The identity element in this basis is $\sum T_a^a$. If we restrict the indices to run from 1 to 5 only, these matrices can be regarded as forming a basis for $\mathcal{M}_5$ as well. This corresponds to an embedding of $\mathcal{M}_5$ into $\mathcal{M}_6$ that we referred to earlier. The commutation relations satisfied by these matrices are

$$[T_a^b,T_c^d]=\delta_c^bT_a^d-\delta_a^dT_c^b\ , \tag{4.2}$$

which gives the Lie algebra of U(6) [or U(5) when the indices run from 1 to 5 only].

We construct the derivatives acting on $\mathcal{M}_5$ as follows. We want SU(3) color and U(1) electromagnetism to survive spontaneous symmetry breaking, or, in other words, their generators to be annihilated by D. They are assumed to be embedded into the above algebra in the usual way: SU(3) being generated by traceless combinations of T_a^b's for $a,b=1,2,3$ and U(1) being diag$(-1/3,-1/3,-1/3,1,0,0)$. We thus choose the automorphism group associated with the subalgebra SU(2)⊗U(1) that is generated by $T_5^6, T_6^5, (T_5^5-T_6^6)/2$ and S, where

$$S=\text{diag}(1,1,1,-3,0,0) , \tag{4.3}$$

and define the derivatives, E_a's, in terms of these generators:

$$E_+(F)=\frac{m}{\sqrt{2}}[T_5^6,F],\quad E_-(F)=\frac{m}{\sqrt{2}}[T_6^5,F] ,$$
$$E_3(F)=\frac{m^2}{2M}[T_5^5-T_6^6,F],\quad E_4(F)=\Lambda[S,F] , \tag{4.4}$$

where F is an element of $\mathcal{M}_5$ and m, M, and Λ are three mass scales that need to be introduced into the theory. $E_\pm$ and E_3 satisfy the commutation relations (3.8), while E_4 commutes with them.

Next, it is straightforward to introduce anticommuting objects dx^μ's and θ's, and define the exterior algebra $\mathcal{A}^*$. The exterior derivative is, as before, $D=d+Q$, where Q is given by our earlier expression (3.9) except for an additional term, $\theta_4 E_4$. The connection one-form has the structure

$$\omega=A+\frac{g}{\sqrt{2}}H\theta_-+\frac{g}{\sqrt{2}}H^\dagger\theta_++g\Delta\theta_3+g\Sigma\theta_4 , \tag{4.5}$$

with

$$A=-igA_\mu dx^\mu,\quad A_\mu=A_{\mu b}^a T_a^b,$$
$$H=H^a T_a^6,\quad \Delta=\Delta_b^a T_a^b,\quad \Sigma=\Sigma_b^a T_a^b , \tag{4.6}$$

where g is again the coupling constant of the theory. A's turn out to be the gauge bosons, while H's, Δ's, and Σ's are the scalars.

As before, under a generalized gauge transformation, the gauge potential ω transforms as $\delta\omega=D\epsilon+\omega\epsilon-\epsilon\omega$, where the gauge parameter ϵ has the decomposition $\epsilon=-ig\epsilon_b^a T_a^b$. There are inhomogeneous terms in the transformations of the scalar sector. Hence we shift the scalars as

$$H\to H-\frac{m}{g}T_5^6,\quad \Delta\to\Delta-\frac{m^2}{2gM}(T_5^5+T_6^6-1) ,$$
$$\Sigma\to\Sigma-\frac{\Lambda}{g}S . \tag{4.7}$$

We will assume in the following that these shifts have been performed. Writing down the transformations more explicitly, we find that the model has SU(5)⊗U(1) gauge invariance. H transforms in the fundamental representation "5" of SU(5) with a U(1) charge equal to one, while Δ and Σ (apart from singlets) transform in the adjoint representation "24" with zero U(1) charge.

The field strength, $\Omega=D\omega+\omega^2$, transforms under the gauge transformation as $\delta\Omega=\Omega\epsilon-\epsilon\Omega$. To find its components it is convenient to first decompose it as

$$\Omega=\tfrac{1}{2}\Omega_{\mu\nu}dx^\mu dx^\nu+\Omega_{\mu+}dx^\mu\theta_-+\Omega_{\mu-}dx^\mu\theta_+$$
$$+\Omega_{\mu3}dx^\mu\theta_3+\Omega_{\mu4}dx^\mu\theta_4+\Omega_{+-}\theta_-\theta_+$$
$$+\Omega_{+3}\theta_-\theta_3+\Omega_{3-}\theta_3\theta_++\Omega_{+4}\theta_-\theta_4$$
$$+\Omega_{4-}\theta_4\theta_++\Omega_{34}\theta_3\theta_4 , \tag{4.8}$$

where again $\Omega_{\mu\nu}=igF_{\mu\nu}$ gives the field strength associated with the gauge fields. The next four terms yield covariant derivatives of the scalar fields just as in Eq. (3.16). The remaining terms in the expansion turn out to be

$$\Omega_{+-}=\frac{g^2}{2}[H,H^\dagger]-gM\Delta+m^2T_6^6-\frac{m^2}{2} ,$$
$$\Omega_{+3}=-\frac{g^2}{\sqrt{2}}\Delta H ,$$
$$\Omega_{3-}=\Omega_{+3}^\dagger ,$$
$$\Omega_{+4}=-\frac{g^2}{\sqrt{2}}\Sigma H,\quad \Omega_{4-}=\Omega_{+4}^\dagger,\quad \Omega_{34}=g^2[\Delta,\Sigma] . \tag{4.9}$$

Continuing this approach, we next write down a Lagrangian using this field strength as in Eq. (2.31). The trace is now taken over 6×6 matrices. The indices i and j are summed over the full range: the space-time and the internal directions (+, −, 3, and 4). Along with the usual covariant kinetic terms,

$$-\tfrac{1}{2}\text{Tr}(F_{\mu\nu}F^{\mu\nu})+\text{Tr}(\mathcal{D}_\mu H^\dagger\mathcal{D}^\mu H)$$
$$+\text{Tr}(\mathcal{D}_\mu\Delta\mathcal{D}^\mu\Delta)+\text{Tr}(\mathcal{D}_\mu\Sigma\mathcal{D}^\mu\Sigma) , \tag{4.10}$$

where one obtains a Higgs potential $V(H,\Delta,\Sigma)$ given by

$$\frac{1}{g^2}V(H,\Delta,\Sigma)=\frac{1}{4}\left[H^\dagger H-\frac{m^2}{g^2}\right]^2$$
$$+\text{Tr}\left[\frac{1}{2}HH^\dagger-\frac{M}{g}\Delta-\frac{m^2}{2g^2}\right]^2$$
$$+H^\dagger\Delta^2H+H^\dagger\Sigma^2H-\text{Tr}[\Delta,\Sigma]^2 , \tag{4.11}$$

where H is written as a five-component column vector.

The Higgs potential, again being a sum of squares, can be minimized quite easily. The vacuum expectation values of H and Δ can be taken to be

$$H^\dagger=(0,0,0,0,m/g),\quad \Delta=\frac{m^2}{2gM}\text{diag}(1,1,1,1,0) . \tag{4.12}$$

The expectation value of Σ is not determined uniquely, except for the fact that its fifth row and fifth column vanish. Radiative corrections to the Higgs potential will settle this issue, most probably making this expectation value zero. This corresponds to SU(5)⊗U(1) being broken down to SU(4)⊗U(1).

This model will be more realistic if Σ gets an expectation value of, say,

$$\Sigma=\kappa\frac{\Lambda}{g}\mathrm{diag}(1,1,1,0,0) \tag{4.13}$$

(for some constant κ) breaking $SU(5)\otimes U(1)$ to $SU(3)\otimes SU(2)\otimes U(1)\otimes U(1)$ so that further breaking to $SU(3)\otimes U(1)\otimes U(1)$ is achieved by the expectation value of H. To obtain the resulting gauge boson masses, we define various gauge fields by

$$A_\mu=\frac{1}{\sqrt{2}}\left[\mathcal{A}_\mu+\frac{1}{\sqrt{5}}C_\mu\right],\quad \mathcal{A}_\mu=\left[\begin{array}{c|cc} G-\frac{2}{\sqrt{30}}B & X & Y \\ \hline \bar{X} & \frac{1}{\sqrt{2}}W_3+\frac{3}{\sqrt{30}}B & W_+ \\ \bar{Y} & W_- & -\frac{1}{\sqrt{2}}W_3+\frac{3}{\sqrt{30}}B \end{array}\right]_\mu, \tag{4.14}$$

where X and Y stand for the leptoquark gauge bosons, while G collectively represents gluons. $W_\pm$, W_3, and B are the usual fields of the electroweak theory. C is an extra gauge boson that comes from the U(1) of our $SU(5)\otimes U(1)$ model. The mass terms for these fields can be evaluated to be

$$(\kappa\Lambda)^2\bar{X}^\mu X_\mu+\left[(\kappa\Lambda)^2+\frac{1}{2}m^2\left[1+\frac{m^2}{2M^2}\right]\right]\bar{Y}^\mu Y_\mu +\frac{1}{2}m^2\left[1+\frac{m^2}{2M^2}\right]W_+^\mu W_{-\mu} +\frac{1}{2}m^2\left[-\frac{1}{\sqrt{2}}W_3+\frac{3}{\sqrt{30}}B+\frac{1}{\sqrt{5}}C\right]_\mu^2 . \tag{4.15}$$

This shows that, in the neutral sector, there is only one massive state, "Z," given by the combination of W_3, B and C that appears above. It couples to diag(0,0,0,0,1). Of the two massless states orthogonal to Z, one should correspond to the photon. It is not clear at this stage, which this combination is. In the context of a flipped $SU(5)\otimes U(1)$ model [10], electric charge is embedded as diag(2/3,2/3,2/3,1,0). In this case, the photon corresponds to the combination

$$\left[\frac{3}{7}\right]^{1/2}\left[\frac{1}{\sqrt{2}}W_3-\frac{1}{\sqrt{30}}B+\frac{3}{\sqrt{5}}C\right]_\mu ,$$

which is orthogonal to Z and hence massless. The second massless state,

$$\frac{2}{\sqrt{7}}\left[\frac{1}{\sqrt{2}}W_3+\frac{\sqrt{30}}{5}B-\frac{1}{2\sqrt{5}}C\right]_\mu ,$$

couples to diag($-1/2,-1/2,-1/2,1,0$) and hence couples to the bosonic sector of the electroweak theory like the photon. An alternate possibility is to embed the electric charge as diag(0,0,0,1,0,) in which case the second massless state couples to diag(1,1,1,0,0) and hence decouples from the bosonic sector of the electroweak theory. Because we have not as yet incorporated fermions into the theory, it is not clear what the full implications of these results are.

V. SUMMARY AND CONCLUSIONS

The geometric formulation of the bosonic sector of nonabelian gauge theory has clearly several attractive features. Geometry dictates the origin of both the gauge fields and Higgs fields. In the extended algebra, we find the gauge fields and the Higgs fields unified and on the same footing. While the $\mathcal{C}$-valued coefficients of the anticommuting differentials dx^μ's describe the gauge fields, the $\mathcal{C}$-valued coefficients of the corresponding anticommuting θ's are the Higgs fields. These Higgs fields are determined by the subalgebra used to construct the derivation operator Q, which itself is dictated by which symmetries one wants unbroken at the end. Thus, in the case of $SU(5)\otimes U(1)$ studied in this paper, we chose a specific subalgebra $SU(2)\otimes U(1)$, where the surviving symmetry contains $SU(3)\otimes U(1)$ so that we can identify SU(3) as the SU(3) color and U(1) as the electric charge. The most interesting feature of the Higgs potential that emerges is that it consists of a sum of terms each of which is a perfect square. It defines, barring the trivial solution in which all the fields vanish, and hence none of the symmetries are broken, an absolutely minimized symmetry-broken vacuum given by the vanishing of each term in the potential.

It should be noted, however, that the Lagrangian we obtain is not the most general one allowed by gauge invariance and renormalizability. In fact, there are no quartic terms in the Higgs fields belonging to the adjoint representations. There are only the coupling terms implying the vanishing of a number of arbitrary parameters, which generally are present when one constructs a Higgs potential starting with a given set of fields. It should also be noted that the quartic coupling is equal to the only dimensionless coupling in the model, namely, the gauge coupling. These facts seem to suggest some underlying symmetries whose origin is not apparent in the present formulation. Whether these symmetries prevail and the restrictions on the couplings remain valid when one considers higher-order corrections to the potential is an extremely interesting and important question that needs investigation.

Likewise, the scale parameters we introduce are arbitrary. Making some plausible assumptions, we have derived physical results concerning the Weinberg angle and the masses of the surviving Higgs particles. These results appear to be reasonable. However, the model is far from a realistic one without fermions. Adding the fermion sector in an *ad hoc* way to the bosonic Lagrangian discussed in this paper is not the solution. A natural way, which is also geometric in origin, has to be found to include the fermions if this approach is to succeed in providing a realistic model. A possible way that suggests itself is the extension of this work to incorporate supersymmetry. Indeed, the form of the Higgs potential (a sum of squares) and the relations in the coupling constants at prescribed mass scales strongly suggests a supersymmetric framework. The nonrenormalization of certain interactions in such theories will enable us to give a meaningful interpretation of the results discussed in this paper.

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