Charged black holes with scalar hair

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We describe a family of static black-hole solutions arising in a theory with Einstein gravity coupled to electromagnetism and an axionlike scalar field. These solutions carry both electric and magnetic charge. In addition, the scalar field is spatially varying outside the horizon. Since this nontrivial scalar field is not required by any conservation law, these solutions shed further light on the black-hole no-hair conjecture.

In this article we describe a family of static black-hole solutions in a theory with Einstein gravity coupled to electromagnetism and an axionlike scalar field. The solutions have nonzero electric and magnetic charges. In addition, there is a spatially varying scalar field outside the horizon. Since this nontrivial scalar field is not required by any conservation law, these solutions shed further light on the black-hole no-hair conjecture [1].

Black-hole solutions with scalar hair have been found in a number of other theories. These include the case of an axion coupled to gravity through an $aR\tilde{R}$ coupling [2] and that of a dilaton coupled to Einstein-Maxwell theory [3]. A conformally coupled scalar field can have a static [4], but unstable [5], solution. In addition, there have been studies of black holes with quantum hair [6], which carries a vanishing energy-momentum tensor and can be detected only by quantum interference measurements; while very interesting in its own right, this is not directly related to the classical hair we investigate here.

Our theory is described by the action

$$S=\int d^4x\sqrt{g}\left[\frac{1}{16\pi G}R-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}\partial^\mu a\,\partial_\mu a+\frac{\lambda}{4}aF_{\mu\nu}\tilde{F}^{\mu\nu}\right], \tag{1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $\tilde{F}^{\mu\nu}=\frac{1}{2}g^{-1/2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The theory is characterized by a single dimensionless quantity λ^2/G. If $a(r)$ is identified with the standard axion, this quantity is proportional to the square of the ratio of the Planck mass to the Peccei-Quinn symmetry-breaking scale, and hence is large. We will later consider the effects of adding a potential for the axion field.

We concentrate on static spherically symmetric solutions, for which the metric may be written in the form

$$ds^2=-B(r)dt^2+C^{-1}(r)dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2\ , \tag{2}$$

with $C(\infty)=B(\infty)=1$. The only nonvanishing components of the electromagnetic field strength are $F_{rt}=E_r(r)$ and $F_{\theta\phi}=r^2\sin\theta\,B_r(r)$. These are determined by the Bianchi identity and Gauss's law (suitably modified to take into account the presence of the $aF\tilde{F}$ term [7]), which imply that

$$B_r=\frac{Q_M}{r^2} \tag{3}$$

and

$$E_r=\frac{1}{r^2}\left[\frac{B}{C}\right]^{1/2}(Q_E+\lambda aQ_M)\ , \tag{4}$$

with the magnetic and electric charges Q_M and Q_E being constants. The rr and tt components of the gravitational field equations can be combined to give

$$-R_{rr}-\frac{1}{BC}R_{tt}=\frac{1}{r}\frac{(B/C)'}{B/C}=8\pi Ga'^2\ , \tag{5}$$

where primes denote differentiation with respect to r. (In obtaining these equations, it is important to note that the $aF\tilde{F}$ term does not contribute to the energy-momentum tensor.) Integration of Eq. (5) gives

$$B(r)=C(r)\exp\left[-8\pi G\int_r^\infty dr\,ra'^2\right]\ . \tag{6}$$

Our problem is thus reduced to one involving two functions $C(r)$ and $a(r)$. These satisfy the differential equations

$$rC'+C-1=-4\pi G\left[r^2Ca'^2+\frac{1}{r^2}[Q_M^2+(Q_E+\lambda aQ_M)^2]\right] \tag{7}$$

and

$$(r^2Ca')'+4\pi GCr^3a'^3=\frac{\lambda Q_M}{r^2}(Q_E+\lambda aQ_M)\ . \tag{8}$$

For later use, we note that these can be combined to give

$$rC(ra')'=\frac{\lambda Q_M}{r^2}(Q_E+\lambda aQ_M)-ra'\left[1-\frac{4\pi G}{r^2}[Q_M^2+(Q_E+\lambda aQ_M)^2]\right]\ . \tag{9}$$

A solution of these equations is determined by fixing three constants. These may be chosen to be $a(\infty)$, together with a mass parameter M and an "axion charge" P, defined by the large distance expansion

$$C(r)=1-\frac{2MG}{r}+\frac{4\pi G(Q_M^2+Q_E^2+P^2)}{r^2}+\cdots\ , \tag{10}$$

$$a(r)=a(\infty)+\frac{P}{r}+\frac{2PMG+\lambda Q_E Q_M}{r^2}+\cdots\ . \tag{11}$$

(It should be stressed that this axion charge is not related to any conservation law.) Since Q_E and a enter the field equations only in the combination $Q_E+\lambda a Q_M$, a may be shifted by a constant, provided that Q_E is modified appropriately. We will use this freedom to set $a(\infty)=0$. This leaves a two-parameter family of solutions for each choice of Q_E and Q_M.

These solutions all develop singularities as the asymptotic form is integrated in toward smaller values of r. We will concentrate here on those solutions for which this singularity is hidden within a horizon (i.e., a zero of C) at some value $r=r_H$. As in other black-hole solutions, such a horizon can exist only if M is greater than some critical value $M_{\rm cr}$. However, this is not a sufficient condition. To see this, suppose that as the asymptotic solution is extended inward, C begins to approach zero. If the right-hand side of Eq. (9) does not simultaneously tend toward zero, a' will start to grow. This will cause the first term on the right-hand side of Eq. (7) to become large, thus changing the sign of C' before a horizon can be reached. This situation will be avoided only by special choices of the asymptotic parameters; these may be viewed as defining a critical value $P_{\rm cr}(M)$ for each $M>M_{\rm cr}$.

Even with M and P tuned so that the right-hand side of Eq. (9) vanishes at one zero of C, there is no reason to expect it to vanish at a second. We should therefore not expect to find an inner horizon, except possibly for exceptional values of M. This is consistent with the known instability of the Reissner-Nordström inner horizon.

We can obtain analytic results in the limits of very small and very large λ. In the former case, the effects of the axion field may be treated as perturbations about the Reissner-Nordström metric

$$C(r)=B(r)=1-\frac{2MG}{r}+\frac{4\pi G(Q_M^2+Q_E^2)}{r^2}\ . \tag{12}$$

This metric has horizons at

$$r_\pm=MG\pm[M^2G^2-4\pi G(Q_M^2+Q_E^2)]^{1/2} \tag{13}$$

as long as

$$M>M_{\rm cr}\equiv[4\pi(Q_M^2+Q_E^2)/G]^{1/2}\ . \tag{14}$$

To order λ, the axion field equation (8) reduces to

$$(r^2Ca')'=\frac{\lambda Q_M Q_E}{r^2} \tag{15}$$

with C given by Eq. (12). For $M>M_{\rm cr}$, this can be integrated to give

$$a(r)=\frac{1}{r_+-r_-}\left[\left[\frac{\lambda Q_M Q_E}{r_+}+P\right]\ln\left[\frac{r}{r-r_+}\right]-\left[\frac{\lambda Q_M Q_E}{r_-}+P\right]\ln\left[\frac{r}{r-r_-}\right]\right]+O(\lambda^2), \tag{16}$$

with the axion charge P entering as an arbitrary constant of integration. [A second integration constant has been adjusted to set $a(\infty)=0$.] The singularity at the outer horizon is eliminated by setting P equal to

$$P_{\rm cr}=-\frac{\lambda Q_M Q_E}{r_+}\ , \tag{17}$$

giving

$$a(r)=-\frac{\lambda Q_M Q_E}{r_+r_-}\ln\left[\frac{r}{r-r_-}\right]+O(\lambda^3)\ . \tag{18}$$

The $O(\lambda^2)$ corrections to the metric can be obtained by substituting this lowest-order solution for a into Eq. (7). With $P=P_{\rm cr}$, the result is

$$C(r)=1-\frac{2MG}{r}+\frac{4\pi G(Q_M^2+Q_E^2)}{r^2}+\frac{4\pi G\lambda^2Q_M^2Q_E^2}{r_+^2}\left[\left[\frac{r_++r_-}{r_-^2r}-\frac{2r_+}{r_-r^2}\right]\ln\left[\frac{r}{r-r_-}\right]-\frac{r_+}{r_-r^2}\right]+O(\lambda^4)\ . \tag{19}$$

Since the correction term is negative at $r=r_+$, the zero of C must occur at some $r_H>r_+$; thus, the effect of the axion field is to move the outer horizon outward. The inner horizon disappears, as indicated by the fact that the first-order corrections to C tend toward $-\infty$ as r approaches r_-.

These results can be taken over to the case $M<M_{\rm cr}$ by allowing r_+ and r_- to be complex. For all values of P the horizon is absent and there is a naked singularity at the origin.

For the case $M=M_{\rm cr}$, we find

$$a(r)=-\frac{\lambda Q_M Q_E}{r_H^2}\ln\left[\frac{r}{r-r_H}\right]+\frac{k}{r-r_H}+O(\lambda^3) \tag{20}$$

where $r_H\equiv r_+=r_-$. This is singular at the horizon, no matter what the value of the integration constant k.

The large-λ limit can be understood by considering the analogous flat-space problem. Thus, consider a static spherically symmetric configuration of electric, magnetic, and axion fields, with point electric and magnetic sources at the origin. The vanishing of $\nabla\cdot\mathbf{B}$ implies Eq. (3), while the modified Gauss law leads to Eq. (4), with $B(r)=C(r)=1$. With $C(r)$ set equal to unity and the term proportional to Newton's constant omitted, the axion field equation (8) becomes

$$(r^2a')'=\frac{\lambda^2Q_M^2}{r^2}\left[\frac{Q_E}{\lambda Q_M}+a\right]\ , \tag{21}$$

whose solution is

$$a=\frac{Q_E}{\lambda Q_M}\left[k\exp\left[-\frac{\lambda Q_M}{r}\right]+(1-k)\exp\left[\frac{\lambda Q_M}{r}\right]-1\right]. \tag{22}$$

[As before, one integration constant has been chosen so that $a(\infty)=0$.] The rising exponential must be rejected on energetic grounds. We therefore set $k=1$ and have

$$a=\frac{Q_E}{\lambda Q_M}\left[\exp\left[-\frac{\lambda Q_M}{r}\right]-1\right]. \tag{23}$$

For $r\gg\lambda Q_M$, this reduces to $a=-Q_E/r+\cdots$, and thus corresponds to an axion charge $P=-Q_E$. For $r\ll\lambda Q_M$, a is approximately constant, with precisely the value needed to make the electric field vanish [see Eq. (4)].

We now incorporate gravity. To do this, we first integrate Eq. (7) to obtain

$$C(r)=1-\frac{2MG}{r}+\frac{4\pi GQ_M^2}{r^2}+\frac{4\pi G}{r}\int_r^\infty dr\left[\frac{(\lambda aQ_M+Q_E)^2}{r^2}+r^2Ca'^2\right]. \tag{24}$$

For r much greater than both MG and $\sqrt{GQ_M^2}$, this will be close to unity, and a may be approximated by the flat space solution. Substitution of this into Eq. (24) gives

$$C(r)\approx 1-\frac{2MG}{r}+\frac{4\pi GQ_M^2}{r^2}+\frac{4\pi GQ_E^2}{\lambda Q_M r}\left[1-\exp\left[-\frac{2\lambda Q_M}{r}\right]\right]+\frac{4\pi GQ_E^2}{r}\int_r^\infty dr\frac{C-1}{r^2}\exp\left[-\frac{2\lambda Q_M}{r}\right]. \tag{25}$$

The contribution from the last term is small; it may be approximated by successive iterations of this equation. By doing so, we obtain the large-distance expansion

$$C(r)\approx 1-\frac{2MG}{r}+\frac{4\pi G(Q_M^2+2Q_E^2)}{r^2}+O(r^{-3}),\quad r\gg\lambda Q_M. \tag{26}$$

At somewhat smaller values of r, where a' and $a+(Q_E/\lambda Q_M)$ are both small (this region extends to within the horizon), we have

$$C(r)\approx 1-\frac{2MG}{r}+\frac{4\pi GQ_M^2}{r^2}+\frac{4\pi GQ_E^2}{\lambda Q_M r}+O(\lambda^{-2}),\quad r_H\lesssim r\ll\lambda Q_M. \tag{27}$$

In the large-λ limit, this reduces to the Reissner-Nordström metric corresponding to a purely magnetic charge Q_M. This has an outer horizon at $r_H=a_+$, where

$$a_\pm=MG\pm(M^2G^2-4\pi GQ_M^2)^{1/2}. \tag{28}$$

This result for the metric can now be used to extend the approximate axion solution inward. Substituting the Reissner-Nordström metric into Eq. (8) and dropping the term containing a'^3 gives a linear equation, valid when a is small, which can be recast in the form of a Legendre equation with singular points at $r=a_\pm$. Requiring that the solution be regular at a_+ leads to a singularity at the would-be inner horizon.

For intermediate values of λ, we must resort to numerical integration of the field equations to obtain solutions; typical results are shown in Fig. 1, where we show $a(r)$ and $C(r)$ for $\lambda=10.0G^{1/2}$, $Q_M=Q_E=(4\pi)^{-1/2}$, and $M=2.0G^{-1/2}$. For comparison, we have also plotted $C(r)$ for the corresponding Reissner-Nordström solution. The behavior of a and C as $r\to 0$ can be found by considering the large-C behavior of Eqs. (7) and (8). These imply that $a\sim\alpha G^{-1/2}\ln r$ and $C\sim-\beta r^{-4\pi G\alpha^2}$, where α and β are constants.

In Fig. 2, we plot the critical axion charge as a function of mass for various values of λ; analogous plots of r_H are given in Fig. 3.

We close with some brief remarks.

(1) The stability of these solutions can be studied by considering the effect of spherically symmetric perturbations; this requires that we take into account the time-derivative terms which were omitted in our original field equations. Doing so, we find that the magnetic and electric fields must still be of the form of Eqs. (3) and (4), while $B(r)$ is given by Eq. (6), provided that $B(r=\infty,t)=1$; this condition can be maintained by imposing an appropriate coordinate condition. Using Eq. (7) (which receives no corrections to first order in the perturbation), together with

$$R_{rt}=\frac{\dot C}{rC}=-8\pi Ga'\dot a, \tag{29}$$

one can show that the fluctuations $\delta C(r,t)$ and $\delta a(r,t)$ are related by

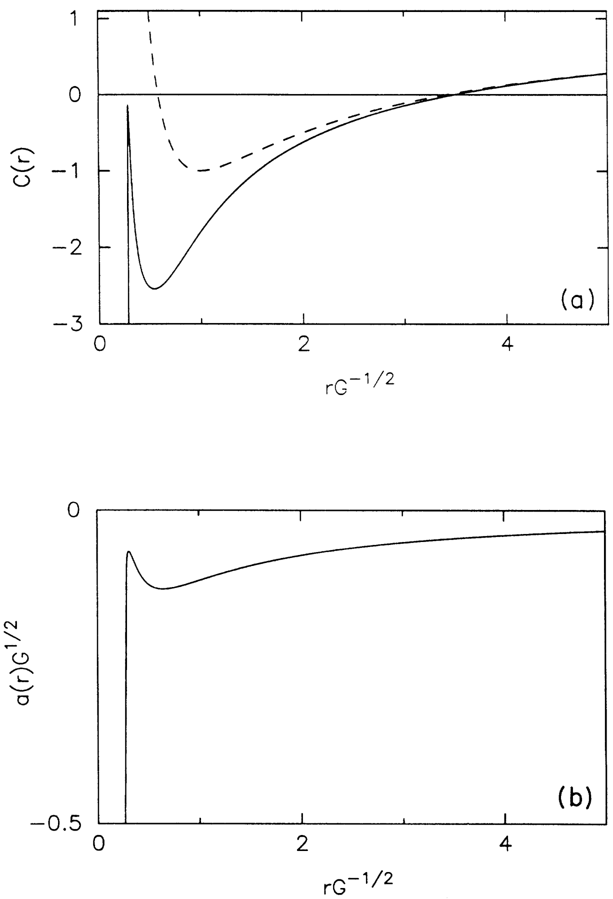

FIG. 1. Plots of (a) $C(r)$ and (b) $a(r)$ for $\lambda=10.0G^{1/2}$, $Q_M=Q_E=(4\pi)^{-1/2}$, and $M=2.0G^{-1/2}$. The corresponding Reissner-Nordström solution is indicated by the dashed line in (a).

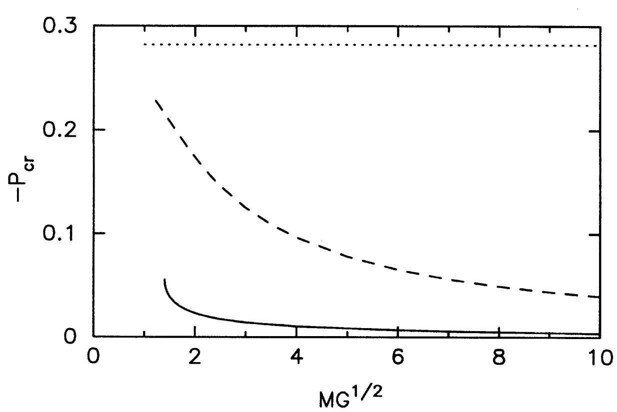

FIG. 2. The critical axion charge as a function of mass for $Q_M=Q_E=(4\pi)^{-1/2}$ and $\lambda=1.0G^{1/2}$ (solid line), $\lambda=10.0G^{1/2}$ (dashed line), and $\lambda=\infty$ (dotted line). For the first of these, the discrepancy between the small λ prediction of Eq. (17) and the results of numerical integration of the field equations is less than the width of the line.

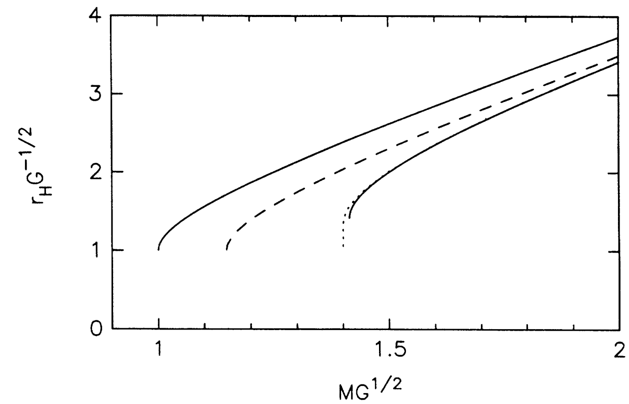

FIG. 3. The horizon distance as a function of mass for $Q_M=Q_E=(4\pi)^{-1/2}$ and $\lambda=\infty$ (leftmost solid line), $\lambda=10.0G^{1/2}$ (dashed line), $\lambda=1.0G^{1/2}$ (dotted line), and $\lambda=0.0$ (rightmost solid line).

$$\delta C=-8\pi G r C_0 a_0' \delta a - \frac{2G\delta M}{r}\exp\left[4\pi G\int_r^\infty dr\, r a_0'^2\right], \tag{30}$$

where subscript zeros denote the unperturbed static solution. The second term simply corresponds to a shift in the mass parameter by a constant δM, and may be omitted when looking for instabilities. With the aid of Eq. (30) the equation obeyed by δa can then be cast into the form

$$\left[-\frac{d^2}{dx^2}+V(x)\right]\phi=-\frac{d^2\phi}{dt^2} \tag{31}$$

where $\phi=r\delta a$,

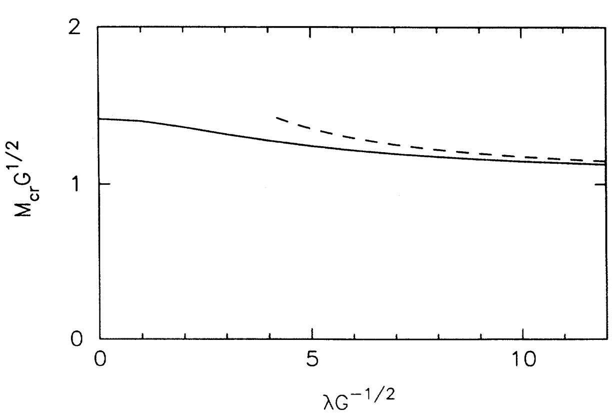

FIG. 4. The critical mass as a function of λ for $Q_M=Q_E=(4\pi)^{-1/2}$. The dashed line indicates the predictions which follow from Eq. (27).

$$V(r)=B_0(r)\left\{\frac{C_0'}{r^2}+4\pi GC_0(a_0')^2+\frac{(\lambda Q_M)^2}{r^4}\left[1+16\pi Gra_0'\left[a_0+\frac{Q_E}{\lambda Q_M}\right]\right]+8\pi G(a_0')^2[C_0+rC_0'+4\pi Gr^2C_0(a_0')^2]\right\} \tag{32}$$

and x is defined by $dx/dr=(B_0C_0)^{-1/2}$ (so that the region outside the horizon is mapped onto the range $-\infty<x<\infty$). The unperturbed static solution will be stable if the operator on the left-hand side of Eq. (31) is a positive operator on all functions which are finite at $x=\pm\infty$. A sufficient condition for this is that C and $(Q_E+\lambda aQ_M)^2$ be increasing functions of r outside the horizon. This is in fact the case for all of the analytical and numerical solutions we have found.

(2) Equation (30) shows that δC, when evaluated at the horizon of the unperturbed solution, is always of the opposite sign than δM. It follows that r_H is an increasing function of M. The critical mass $M_{\rm cr}$ should therefore correspond to the smallest possible value for r_H. Now note that at the horizon Eq. (7) reduces to

$$r_H^2=r_H^3C'(r_H)+4\pi G\{Q_M^2+[Q_E+\lambda a(r_H)Q_M]^2\} \ . \tag{33}$$

Since $C'(r_H)$ can never be negative, r_H will certainly be minimized if $C'(r_H)$ and $[Q_E+\lambda a(r_H)Q_M]$ both vanish; indeed, Eq. (8) shows that the vanishing of one of these implies the vanishing of the other, provided that $\lambda\neq 0$. Hence [8],

$$r_H^2=4\pi GQ_M^2, \quad M=M_{\rm cr}, \quad \lambda\neq 0 \ . \tag{34}$$

On the other hand, when $\lambda=0$ Eq. (13) gives

$$r_H^2=4\pi G(Q_M^2+Q_E^2), \quad M=M_{\rm cr}, \quad \lambda=0 \ . \tag{35}$$

This discontinuity at $\lambda=0$ is possible because $dr_H/dM_{\rm cr}$ diverges as $M\to M_{\rm cr}$, as can be seen from the results shown in Fig. 3. Our numerical results for $M_{\rm cr}$ as a function of λ are shown in Fig. 4. As $\lambda\to 0$, these approach the Reissner–Nordström value (14). For large λ, Eq. (27) leads to

$$M_{\rm cr}\approx\left[\frac{4\pi Q_M^2}{G}\right]^{1/2}+\frac{2\pi Q_E^2}{\lambda Q_M} \ . \tag{36}$$

This approximation is indicated by the dashed line in Fig. 4.

(3) We have considered the case of a scalar field endowed only with the axionic $aF\tilde{F}$ coupling. However, it is evident that static solutions with nontrivial scalar fields will continue to exist even if a potential $V(a)$ is introduced. (This may change the large distance behavior of the scalar field; a massive scalar will fall as $1/r^4$ as $r\to\infty$.) The persistence of solutions in the presence of a potential makes clear that the special properties of the axion, in particular the possibility of reformulating its dynamics in terms of an antisymmetric three-form field with an associated gauge symmetry, are not essential to the existence of the scalar hair.

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