Gravitational field of a hedgehog and the evolution of vacuum bubbles

E. I. Guendelman
Department of Nuclear Physics, Weizmann Institute of Science, Rehovot 76100, Israel

A. Rabinowitz
Department of Physics, Ben Gurion University of the Negev, Beer Sheva 84105, Israel
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The gravitational field produced by a spherically symmetric "hedgehog" configuration in scalar field theories with global SO(3) symmetry (or higher) is studied in the limit in which these models become nonlinear σ models. The same gravitational effect can be generated by a set of cosmic strings intersecting at a point, in the limit that one considers a continuous distribution of such intersecting strings in a spherically symmetric configuration (to be referred to as the "string hedgehog"). When the energy densities associated with the hedgehog are small, we obtain a static geometry, but for higher values, the resulting geometry is that of an anisotropic cosmology. The evolution of bubbles joining two phases, one of which contains a hedgehog (as defined above) is investigated. The role of such configurations in processes that lead to classical false-vacuum destabilization and in the evolution of inflationary bubbles is discussed. The generalization of our results to the gauged case, i.e., to magnetic-monopole hedgehogs, is discussed.

I. INTRODUCTION AND SUMMARY

In this paper, we want to discuss some new, self-consistent solutions of Einstein's equations of gravity coupled to scalar fields or cosmic strings. The characteristic feature of these solutions is that they contain a hedgehog configuration, that is, a spherically symmetric configuration with some feature pointing radially. For an isovector scalar $\boldsymbol{\phi}=(\phi_1,\phi_2,\phi_3)$, this means that $\boldsymbol{\phi}$ is parallel to $\hat{\mathbf{r}}$, the unit vector in the radial direction, while in a case of a configuration of cosmic strings, we mean a set of strings joined at a central region.

As we will see, such configurations give rise to a static geometry when the strength of the hedgehog is less than some critical value. For higher strengths, a cosmological solution is obtained. In the case of a hedgehog of isoscalar fields $\boldsymbol{\phi}$, exact solutions can be obtained in the limit for which the nonlinear σ model is appropriate.

In the nonlinear σ model the scalar-field part of the action is given by $S_m=\int\sqrt{-g}\,L_m d^4x$, with

$$L_m=\tfrac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi}g^{\mu\nu}+\lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi}-v^2)^2 \ . \qquad (1)$$

We study the limiting case $\lambda\to\infty$, so that (1) is equivalent to

$$L_m=\tfrac{1}{2}\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi}g^{\mu\nu} \qquad (2a)$$

with $\boldsymbol{\phi}$ constrained to

$$\boldsymbol{\phi}\cdot\boldsymbol{\phi}=v^2 \ . \qquad (2b)$$

It turns out then, that for values of $8\pi Gv^2<1$, the hedgehog configuration leads to a static solution, while for values of $8\pi Gv^2>1$, a cosmological solution is the relevant one.

The presence of this hedgehog—called a "defect"—can, as we will see, be responsible for the classical destabilization of a false vacuum. That is, a vacuum which is only quantum-mechanically unstable, can, due to the presence of a hedgehog, become unstable even at the classical level.

We study the evolution of bubbles separating two phases, one being the "false vacuum" and the other the "true vacuum." When the false vacuum is outside and the true vacuum is inside [1],[2], we have a problem of relevance to the vacuum stability mentioned above. A related problem is that of the evolution of inflationary bubbles where the false vacuum is inside and the true vacuum is in the outside region [3]–[8]. The presence of a hedgehog can be incorporated into such solutions.

As mentioned previously, we will show that the hedgehog solutions described above can be interpreted as due to a spherically symmetric ensemble of cosmic strings. A remarkable result that we will find is that the trajectory of a given geodesic in such a geometry is identical to the trajectory in the gravitational field of a single string (for references on the gravitational field of one string see Ref. [9]).

Finally, at the end of Sec. IV, we discuss how some of our results have applications to solutions with local gauge symmetry, i.e., when the hedgehog becomes a magnetic monopole.

II. THE GRAVITATIONAL FIELD OF A HEDGEHOG

A. The scalar-field hedgehog

We begin by studying the energy-momentum tensor produced by a scalar-field hedgehog, i.e., the $T_{\mu\nu}$ produced by (2a) and (2b) in the case of a hedgehog configuration:

$$\boldsymbol{\phi}=+v\hat{\mathbf{r}} \ \text{ or } \ \boldsymbol{\phi}=-v\hat{\mathbf{r}} \qquad (3a)$$

where

$$\hat{r}_1=\sin\theta\cos\varphi,\ \ \hat{r}_2=\sin\theta\sin\varphi,\ \ \hat{r}_3=\cos\theta\ . \tag{3b}$$

Both forms of ϕ in (3a) lead to the same stress-energy tensor $T_{\mu\nu}$.

It is easy to see that (3a) and (3b) are a solution of the scalar-field equations of motion when $g_{\mu\nu}$ is of a spherically symmetric form:

$$ds^2=-Adt^2+Bdr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\ . \tag{4}$$

The $T_{\mu\nu}$ produced by (2) is

$$T_{\mu\nu}=\partial_\mu\phi\cdot\partial_\nu\phi-g_{\mu\nu}[\tfrac{1}{2}(\partial_\rho\phi)\cdot(\partial_\gamma\phi)g^{\rho\gamma}] \tag{5}$$

which for ϕ given by (3) gives

$$T_0^0=T_r{}^r=-\frac{v^2}{r^2},\ \ T_\varphi{}^\varphi=T_\theta{}^\theta=0\ . \tag{6}$$

All other components of $T^\mu{}_\nu$ also vanish. When t in (4) is a timelike coordinate, (5) and (6) imply that $T_{00}\geq 0$.

The singularity of $T^\mu{}_\nu$ at the center of the geometry is a consequence of the constraint (2b). If instead λ is big but finite, this would allow $\phi=0$ at $r=0$, thus avoiding the singularity, while still having ϕ parallel to r everywhere. For very large λ (3a) will be very approximately satisfied, except for a very small region near $r=0$.

We now look for the geometry generated by the energy-momentum given by (6). Einstein's equations are

$$G^\mu_\nu\equiv R^\mu_\nu-\tfrac{1}{2}\delta^\mu_\nu R=8\pi G T^\mu_\nu\ .$$

For a metric of the form (4) with $A=M$ and $B=M^{-1}$, where M is a space-time constant, $8\pi G T^\mu{}_\nu=G^\mu{}_\nu$ gives

$$T^0{}_0=T^r{}_r=\frac{M-1}{8\pi G r^2},\ \ \text{all other } T^\mu{}_\nu=0\ . \tag{7}$$

That is, $M=1-8\pi G v^2$ for the hedgehog that produces the energy-momentum tensor (6). This agrees with the solution found by Vilenkin and Barriola [10], authors that have first studied the gravitational field of a global hedgehog.

Notice that for $8\pi G v^2<1$, the metric

$$ds^2=-(1-8\pi G v^2)dt^2+(1-8\pi G v^2)^{-1}dr^2+r^2d\Omega^2 \tag{8}$$

corresponds to a static geometry, but for $8\pi G v^2>1$, (8) becomes an anisotropic cosmology. This is because t becomes a spacelike coordinate while r becomes a timelike one, and the term $r^2d\Omega^2$ in (8) represents an expanding two-sphere with a big-bang singularity at $r=0$. The other dimension, associated with the coordinate t, does not suffer any expansion or contraction whatsoever.

Notice that positivity of energy (T_{00} for $M>0$ or T_{rr} for $M<0$) is assured provided $M<1$.

B. The gravitational field of a hedgehog string configuration

We now show that the geometry discussed in Sec. II A can be produced by an ensemble of strings in a "hedgehog" configuration even though the cosmic strings are quite different physically from the source in the nonlinear σ model.

For a single cosmic string located along the z axis, we have an energy-momentum of the form

$$T^0{}_0=T^z{}_z=-\sigma\delta(x)\delta(y)\ , \tag{9}$$

with all other components vanishing. $\sigma>0$ ensures positivity of energy, i.e., $T_{00}>0$ when t is the timelike coordinate.

Now, let us suppose that instead of a single string, we have an ensemble of many strings, all of them intersecting at a central point. In the limit that the strength of each string is taken to be very small, but the number of strings is taken to be very big, the resulting $T^\mu{}_\nu$ has approximate spherical symmetry (which can become exact as we let the number of strings go to infinity, i.e., in a continuum limit).

In such a case (which we call the "string hedgehog"), we expect (9) to be replaced by

$$T^0{}_0=T^r{}_r\ , \tag{10}$$

with all other components vanishing.

In addition, we expect that for a sphere centered on the intersection point of the ensemble of strings, the energy content inside will be linear in the size of the sphere, with the size of the sphere defined as the proper distance from the center of the geometry to the surface of the sphere.

All these conditions are satisfied, assuming that Einstein's equations hold, for the metric

$$ds^2=-Mdt^2+M^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \tag{11}$$

with M a space-time constant. From Einstein's equations we find $T^0{}_0=T^r{}_r=(M-1)/r^2$ all other $T^\mu{}_\nu=0$. The energy contained inside a sphere $r=r_0$ is defined by $E(r_0)\equiv\int_0^{r_0}\sqrt{-g}\,T_{\mu 0}\xi^\mu d^3x$, ξ^μ being the unit timelike Killing vector, $\xi^\mu=(1/\sqrt{M}\,)(1,0,00)$, in the case $M>0$, i.e., if t is the timelike coordinate. Then, $E(r_0)=(1-M)M4\pi D$, where $D=r_0/\sqrt{M}$ is the proper distance from the sphere to the center of the geometry. The energy inside the sphere goes linearly with D as we expect for an ensemble of cosmic strings.

Again we find that as the energy density is increased for a given fixed r, M becomes negative and, as in the discussion of the scalar hedgehog, we find a cosmological rather than a static solution. The point $r=0$, where the strings meet is now interpreted as a big-bang singularity.

It is interesting to study the motion of geodesics of the metric with M positive. In fact, because of the spherical symmetry of the problem, there is a conserved angular momentum, and geodesics are therefore contained in a plane. We can therefore set $\sin\theta=1$ (i.e., restrict the motion to the x-y plane). Then

$$ds^2=-Mdt^2+M^{-1}dr^2+r^2d\varphi^2\ .$$

Defining $\bar{t}=\sqrt{M}t$, $\bar{r}=\sqrt{M^{-1}}r$, and $\bar{\varphi}=M\varphi$, we get

$$ds^2=-d\bar{t}^{\,2}+d\bar{r}^{\,2}+\bar{r}^{\,2}d\bar{\varphi}^{\,2}\ , \tag{12a}$$

$$0\leq\bar{\varphi}\leq 2\pi M\ , \tag{12b}$$

so that the net effect of the ensemble of strings on the motion of the particle is the same as that of a single string [9] which is oriented perpendicularly to the plane in which the motion takes place.

C. Generalizations

It is very simple to consider the effects of a cosmological constant or superpose to our solution a Schwarzschild metric or a Reissner-Nordström one. This is because for the form $ds^2=-Adt^2+A^{-1}dr^2+r^2d\Omega^2$, the Einstein tensor becomes

$$G^0{}_0=G^r{}_r=\frac{A-1}{r^2}+\frac{A'}{r} \tag{13a}$$

$$G^\theta{}_\theta=G^\varphi{}_\varphi=\frac{A''}{2}+\frac{A'}{r},\quad \text{other } G^\mu{}_\nu=0 \tag{13b}$$

so that Einstein's equations are linear [11] in $A-1$. Also in the cases mentioned above, $T^\mu{}_\nu$ is a function of r alone (not of A).

This results from the fact that the metric in the presence of only a cosmological constant is of the form (4) with $B=A^{-1}$ and similarly for the Schwarzschild solution, as well as for the pure hedgehog. The combination of all these situations is a linear problem, as explained above.

Thus, in particular, in the presence of a cosmological constant or, what is the same, a false-vacuum state with uniform energy density ρ_0 and of a central mass, we have, for a spherically symmetric scalar field or string hedgehog, the metric

$$A=M-\frac{2Gm}{r}-\chi^2r^2,\quad B=A^{-1} \tag{14}$$

where $M=\text{const}<1$, $m=\text{const}\geq 0$, and χ^2 is given in terms of ρ_0 through the relation $\chi^2=(8\pi G/3)\rho_0$. χ in the case $v=m=0$, i.e, when we have a pure de Sitter space, corresponds to the Hubble expansion parameter of that space.

This can be generalized even further to include the presence of an electric charge for example, since the Reissner-Nordström solution is also of the form (4) with $B=A^{-1}$.

Notice that for M positive (14) is a space-time such that $A=0$ at a certain value of r. For $M<0$ however, r is always a timelike variable and (14) is to be interpreted as an anisotropic cosmological solution.

III. CLASSICAL FALSE-VACUUM DESTABILIZATION

A. Matching a medium containing a hedgehog configuration to a vacuum region

It is interesting to match a vacuum region with a space-time where there is a hedgehog configuration, producing a metric of the form (14) (in particular, in this section for the choice $m=0$). In this way we can study the growth of a true-vacuum region, where the growth is helped by the presence of a hedgehog.

If the false-vacuum region contains, for example, a string hedgehog configuration, these strings can actually pull the surface of a true-vacuum bubble outwards, further into the false-vacuum region, so that at some value of the strength of this "pulling," an arbitrarily small region of true vacuum can expand and overcome the whole false-vacuum region.

This is an entirely classical process, since the origin of the bubble can be interpreted as a very small classical perturbation of the original system (in the presence of the hedgehog). What results then is the phenomenon of classical destabilization of the false vacuum: that is we do not have to rely on the quantum-mechanical bubble nucleation processes to destabilize the false vacuum—it occurs even at the classical level.

From a mathematical point of view, it does not matter whether the hedgehog we consider is made of a scalar field or of strings, the equations of motion of the false-vacuum destabilization are the same for both cases.

Technically, the matching of the two regions can be implemented using Israel's method [12]. Assume that the interface of the two regions consists of a domain wall with an energy-momentum tensor of the form [9]

$$T_{\mu\nu}=-\sigma(g_{\mu\nu}-\xi_\mu\xi_\nu)\delta(\eta)\ , \tag{15}$$

where ξ_μ=normal to the surface of the wall and where Gaussian normal coordinates have been used in (15). Defining geodesics normal to the $(2+1)$-dimensional surface of the wall, $|\eta|$ is defined as the length along one such geodesic, starting from the surface to a given point outside the surface. η is taken to be positive in the false-vacuum region and negative in the true-vacuum region. η=const represents surfaces normal to such geodesics and $\eta=0$ is of course the position of the wall.

At least in a neighborhood of the wall, any point will be intersected by one and only one such geodesic, at a proper distance $|\eta|$ from the surface (the sign specifying on which side of the wall the point is). The coordinates in the $(2+1)$-dimensional surface where the geodesics originates gives the rest of the coordinates needed. The coordinates so defined are called Gaussian normal coordinates.

A convenient choice for coordinates in the wall can be: τ=proper time as measured by an observer at rest with respect to the wall; for a spherical wall it is convenient also to choose standard angular variables (θ,φ). Therefore we can define a Gaussian normal coordinates system using the coordinates $(\tau,\theta,\varphi,\eta)$.

Also, it is convenient to extend the definition of ξ_μ to all space: We define ξ_μ to be, in general, the normal to an η=const surface. In the system defined above, it is easy to see that (using i,j,k,e,m to denote τ,θ,φ)

$$g^{\eta\eta}=g_{\eta\eta}=1,\quad g^{\eta i}=g_{\eta i}=0$$

and $\xi^\mu=\xi_\mu=(0,0,0,1)$, i.e., the normal is taken to point from the inside towards the outside region.

We then define the extrinsic curvature

$$K_{ij}=\xi_{i;j}=\frac{\partial\xi^i}{\partial x^j}-\Gamma^\mu_{ij}\xi_\mu=-\Gamma^\eta_{ij}=\tfrac{1}{2}\partial_\eta g_{ij}\ . \tag{16}$$

In terms of these variables, Einstein's equations take the form (as usual $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$)

$$G^{\eta}_{\eta}=-\tfrac{1}{2}{}^{(3)}R+\tfrac{1}{2}[(K^2)^i_i-K^i_iK^j_j]=8\pi GT^{\eta}_{\ \eta}\ ,$$
$$G^{\eta}_i=K^j_{i|j}-K^j_{j|i}=8\pi GT^{\eta}_{\ i}\ ,$$
$$G^i_j={}^{(3)}G^i_{\ j}+K_e{}^eK_j{}^i-\tfrac{1}{2}\delta_j{}^i[(K^2)_e{}^e+K^e_eK^m_m]$$
$$+\partial_\eta(K_j{}^i-\delta^i_jK^e_e)=8\pi GT^i_{\ j}\ , \tag{17}$$

where a vertical bar means covariant derivative in a three-dimensional sense [in the (2+1)-dimensional space of the coordinates (τ,θ,ϕ) labeled variously by latin indices i,j,k,e,m]. Also quantities denoted with a superscript three in parentheses, such as, for example, ${}^{(3)}R$, ${}^{(3)}G^i_{\ j}$, are to be evaluated as if they concerned a purely three-dimensional metric g_{ij}, ignoring information on the embedding in the higher (3+1)-dimensional space.

The discontinuity of $K^i_{\ j}$ can be obtained by integrating Eq. (17). Since three-dimensional quantities must be continuous and $K^i_{\ j}$ may have a discontinuity, but is otherwise finite, we have [using (15) for $T_{\mu\nu}$, which implies $T^i_{\ j}=-\sigma\delta^i_{\ j}\delta(\eta)$], integrating from $\eta=-\varepsilon$ to $\eta=+\varepsilon$, $\varepsilon>0$, that $\gamma^i_{\ j}\equiv K^i_{\ j}$ (outside)$-K^i_{\ j}$ (inside) is given by

$$\gamma^i_{\ j}-\delta^i_{\ j}\gamma^e_{\ e}=8\pi G\sigma\delta^i_{\ j}\ ,$$

or

$$\gamma^i_{\ j}=-4\pi G\sigma\delta^i_{\ j}\ . \tag{18}$$

In order to extract information from (18), we must evaluate $K^i_{\ j}$. We start by studying $K_{\theta\theta}$. From (16), we have that

$$K_{\theta\theta}=\tfrac{1}{2}\partial_\eta g_{\theta\theta}\equiv\tfrac{1}{2}\partial_\eta r^2$$

where by definition $g_{\theta\theta}=r^2$, r being the proper circumferential radius, which has an invariant meaning. Working in an arbitrary coordinate system, we have

$$K_{\theta\theta}=\tfrac{1}{2}\xi^\mu\partial_\mu r^2$$

which can be easily evaluated in a coordinate system where the metric is of the form $ds^2=-Adt^2+A^{-1}dr^2+r^2d\Omega^2$ ($d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2$). There, the (2+1)-dimensional membrane is characterized by a velocity $u^\mu=(\dot t,\dot r,0,0)$ and a unit normal to this velocity $\xi^\mu=(A^{-1}\dot r,A\dot t,0,0)$, so that

$$K_{\theta\theta}=\tfrac{1}{2}\partial_\eta r^2=\tfrac{1}{2}\xi^\mu\partial_\mu r^2=r\xi^r=rA\dot t=r\sqrt{A+\dot r^2}\equiv r\beta\ . \tag{19}$$

We see therefore that taking the θ-θ component of Eq. (18) gives us the discontinuity of $K_{\theta\theta}$, and using (19), we get

$$\beta_- -\beta_+=4\pi G\sigma r\ , \tag{20}$$

where $\beta_\pm=(A_\pm+\dot r^2)^{1/2}$, with plus and minus referring to the outside and inside, respectively.

We demand that the induced metric in the wall be well defined. As a result the proper circumferential radius r of the inside and outside metrics $ds^2_\pm=-A_\pm dt^2_\pm+A^{-1}_\pm dr^2+r^2d\Omega^2$, must be continuous at the boundary.

Integration of other components of Einstein's equations are consistent with (20), i.e., $(d/d\tau)(\sigma r^2)=\sigma(d/d\tau)(r^2)$ [for a domain wall with $T_{\mu\nu}$ given by Eq. (15)], that is, $\sigma=$const.

B. Classical false-vacuum destabilization

We now turn our attention to processes relevant to the decay of a false vacuum, that is, to the growth of a bubble of true vacuum in the midst of a false-vacuum region. In particular, we want to analyze the effect of introducing a hedgehog in the false-vacuum region.

We will choose the true-vacuum region to have zero energy density and therefore be just flat space. Therefore, we have $A_-=1$ for the inside of the bubble.

For the outside of the bubble, we specify a region with higher energy density. In the standard treatment of false-vacuum decay [2], this outside region is taken to be a de Sitter space, i.e., $A_+=1-\chi^2r^2$, but now we want to consider the presence of the hedgehog in the false vacuum. According to our discussion in Sec. II C, this is simply achieved by replacing the one in $1-\chi^2r^2$ by M, where M is a constant. We further require $M<1$, for positivity of the energy density. Therefore, $A_+=M-\chi^2r^2$.

Introducing the above expressions for A_- and A_+ into Eq. (20), then solving for β_- in terms of β_+ and squaring, and then, from the resulting equation solving for β_+ and squaring once again, we get that

$$\dot r^2-\frac{(M-1)^2}{4K^2r^2}-\frac{\chi_+^4r^2}{4K^2}=-\left[\frac{M\chi_+^2-\chi_-^2}{2K^2}\right] \tag{21}$$

where $K=4\pi\sigma G$, $\chi_+^2=\chi^2+K^2$, $\chi_-^2=\chi^2-K^2$.

It is very useful to notice that (21) is just like the equation of a particle (of "mass"=2) in a potential U, given by

$$U=-\frac{(M-1)^2}{4K^2r^2}-\frac{\chi_+^4r^2}{4K^2} \tag{22}$$

provided the energy of the "particle" is given by

$$E_{\rm part}=-\left[\frac{M\chi_+^2-\chi_-^2}{2K^2}\right]\ . \tag{23}$$

For the case $M=1$, there is no hedgehog: in the scalar-field case $M=1-8\pi Gv^2$, so $v^2=0$. Likewise, $M=1$ corresponds to zero strength for the set of strings in the string-hedgehog configuration. This case is the usual Coleman–De Luccia bounce solution [2], where we have a classical solution that contracts, reaches a minimum radius $r_m=2K/\chi_+$ and then reexpands. From a semiclassical point of view, there is a Euclidean tunneling solution that interpolates $r=0$ with $r=r_m$, the minimum radius of the classical solution, i.e., the radius at which the bubbles of true vacuum nucleate in the midst of the false vacuum.

In the absence of a hedgehog, i.e., for $M=1$, the formation and initial growth of the true-vacuum bubbles in the midst of a false-vacuum region necessitates quantum-mechanical tunneling because in the classical regime, the bubbles must be of radius $r=r_m$ or bigger. We now show, however, that in the presence of a hedgehog, the formation of a true-vacuum bubble in the midst of a false-vacuum environment can take place purely classically, i.e, the bubble can start from zero size and eventually "absorb" the whole false-vacuum region. In this way we achieve destabilization of the false vacuum even at the classical level.

To see this, notice that in order to have a solution in which a bubble can start from $r=0$ and then go all the way to $r=\infty$, it is necessary that $E_{\rm part}$ as given by (23) is always equal to or greater than U as given by (22). It is clear that this can be achieved by making the hedgehog sufficiently strong or M sufficiently small. (For example, in the scalar-field case where $M=1-8\pi G v^2$, increasing v^2, i.e., the strength of the hedgehog, reduces M. Likewise in the string-hedgehog case, making the hedgehog stronger also reduces the value of M.)

The condition that $E_{\rm part}>U$ for all r is of course equivalent to $E_{\rm part}>U_{\rm max}$, where $U_{\rm max}$ is the value of U at its maximum, where the maximum is determined by setting $\partial U/\partial r=0$. This gives us the condition $M<\chi^2/\chi_+^2$. For all these values of M (all of them physically allowed since $\chi^2/\chi_+^2<1$), we have a situation where by introducing a hedgehog, there is a bubble solution that starts arbitrarily small (from $r=0$) and then expands to infinity. The above condition is equivalent to demanding that the scale of symmetry breaking is big enough: $v>(8\pi G)^{-1/2}\{1-\chi^2[\chi^2+(4\pi G\sigma)^2]^{-1}\}^{1/2}$.

Notice that for the solutions (in the presence of a hedgehog) which start from $r=0$ and then expand to infinity, one has, due to the behavior of U near $r=0$, that $dr/d\tau=\infty$ for $r=0$. This explosive-type behavior is directly correlated to the singularity of $T^{\mu}{}_{\nu}$ for a hedgehog at the point $r=0$. Replacing the physics of a hedgehog in the nonlinear σ model limit [i.e., taking $\lambda\to\infty$ in (1)] by considering instead λ big but finite should remove these singularities, since then we can have $\phi=0$ at the center of the geometry. The problem is then however, that simple analytic expressions are no longer available.

Notice furthermore, that only in the case $\lambda=\infty$ are we allowed to consider an arbitrarily thin wall, as we have done here. This is because the wall matches a point where $|\phi|=v$ and where in a neighborhood of this point $U(|\phi|)\sim\lambda(|\phi|^2-v^2)^2+\text{const}$ (the constant is introduced in order to have a positive energy density for the false vacuum), to a point where $|\phi|=0$ and where $U'(0)=U(0)=0$ (in order to obtain at this point the true vacuum with zero cosmological constant). $\lambda=\infty$ implies that the local minimum of U is infinitely steep in the false-vacuum region, so that (3a) is an exact result there. An arbitrarily steep potential (both near $|\phi|=0$ and near $|\phi|=v$) is also needed in order to ensure that the transition from $|\phi|=0$ to $|\phi|=v$ takes place in an arbitrarily thin domain wall. If λ is big but not infinite, that wall will have a nonzero thickness and this in turn puts a lower bound on the minimum size of the bubbles we are considering. What happens at very small distances cannot be described by the thin-wall approximation, and it is possible that a small amount of quantum tunneling (arbitrarily small as we let λ become bigger and bigger) may be required to form the initial bubble. This point merits further research.

Finally we end this section by giving the explicit solutions of the equations of motion in the cases discussed above. For $M<\chi^2/\chi^2{}_+$, the trajectory is described by

$$r^2=\frac{1}{\chi_+^2}\left[F^{1/2}\sinh\left[\frac{\chi_+^2}{K}\tau\right]-\frac{2K^2}{\chi_+^2}E_{\rm part}\right] \tag{24}$$

where

$$F=(M-1)^2-\frac{4K^4E_{\rm part}^2}{\chi_+^2}\ . \tag{25}$$

For $M<\chi^2/\chi_+^2$, $F>0$. Since $\sinh(\chi_+^2\tau/K)$ takes all possible real values, there is a $\tau=\tau_0$ such that $r^2=0$, for $\tau>\tau_0$ $r^2>0$, and as $\tau\to\infty$ $r^2\to\infty$.

At the point $M=\chi^2/\chi_+^2$, $F=0$ and $E_{\rm part}=-\frac{1}{2}$, then r^2 is given by

$$r^2=\frac{1}{\chi_+^2}\left[\frac{K^2}{\chi_+^2}\pm\exp\left[\frac{\chi_+^2\tau}{K}\right]\right]\ ; \tag{26}$$

of course $\tau\to-\tau$ gives us solutions also. (26) and its time-reversed versions are solutions that start at either $r=0$ or $r=\infty$ and take an infinite time to get to $r^2=K^2/\chi_+^4$, but never cross that point.

For $M>\chi^2/\chi_+^2$, $F<0$, $F=-|F|$, $E_{\rm part}<0$ in this region and r^2 is given by

$$r^2=\frac{1}{\chi_+^2}\left[-\frac{2K^2E_{\rm part}\chi_+^2}{\chi_+^2}\pm|F|^{1/2}\cosh\left[\frac{\chi_+^2}{K}\tau\right]\right]\ . \tag{27}$$

Again, in this case we have solutions that start and finish at $r=0$ or $r=\infty$ but never cross the point $r^2=(1/\chi_+^2)[-(2K^2E_{\rm part}/\chi_+^2)]$. The solutions (27) bounce back at this point, producing therefore time-reversal-invariant solutions.

C. Hedgehogs and inflationary bubbles

It is of interest to study the problem of a bubble of false vacuum in the midst of a true vacuum of zero energy density. Such a false vacuum will have an associated constant positive energy density and constant negative pressure, which therefore leads to an exponentially expanding (or inflationary) phase. In this way, a local version of the inflationary scenario [13] can be studied. In such a local scenario, inflation takes place not over all space, but in a bubble that could expand to become a very big region.

When a false-vacuum region and a true-vacuum region are matched through a domain wall, such that the false vacuum is inside, Eq. (20) holds, with the negative subscript now denoting the true vacuum. One finds [3]–[8] that when inflation takes place the false-vacuum region does not displace the true-vacuum region (which has a higher pressure), but it expands forming a space that disconnects from the exterior and builds a wormhole. One therefore matches a false vacuum with $A_-=1-\chi^2r^2$ to a true vacuum with $A_+=1-(2Gm/r)$. Equation (20) holds with $\beta_+=\pm\sqrt{A_++\dot r^2}$ and

$\beta_- = \pm\sqrt{A_- + \dot{r}^2}$. The signs of β_- are determined by carefully demanding [7], in a legitimate (i.e., one-to-one) set of coordinates (Kruskal for the Schwarzschild region and Gibbons-Hawking coordinates for the de Sitter region), that the normal which points from inside to outside always points from the de Sitter region towards the Schwarzschild region.

The analysis of that matching reveals however that inflation can take place, but in some cases it does not. In fact, some bubbles that start from arbitrarily small size (i.e., from $r=0$), recollapse instead of becoming arbitrarily large ones. We now show that introducing a hedgehog of sufficient strength in the false-vacuum region makes sure that an arbitrarily small bubble of false vacuum will always expand to an arbitrarily large size.

In the case of a false vacuum with a hedgehog, we have to replace $A_- = 1-\chi^2 r^2$ by $A_- = M - \chi^2 r^2$, $M<1$. For a sufficiently strong hedgehog, $M<0$ (for example, if $8\pi G v^2 > 1$ in the scalar case). In such case $A_- < 0$ for all possible r. From the identity $\dot{r}^2 + A_- = (A_- dt/d\tau)^2$, which tell us that $\dot{r}^2 + A_- \geq 0$, we have that $\dot{r}$ can never be zero (since $-A_-$ is bounded from below by $-M>0$ in the case at hand). As a result, recollapse (which requires a point with $\dot{r}=0$) cannot take place. Therefore a bubble of false vacuum with a sufficiently strong hedgehog is guaranteed to expand from $r=0$ to $r=\infty$, and this only because the bubble has to have a kinematically allowed motion from the point of view of the internal region (the region denoted with a negative subscript).

IV. DISCUSSION OF THE RESULTS

In this paper, we have solved and analyzed the gravitational field of a hedgehog. In the case of a scalar field with a global SO(3) symmetry, this can be achieved in the limit $\lambda \to \infty$ in (1), which gives rise to a nonlinear σ model. Of course the global symmetry can be bigger than SO(3): for any model that contains SO(3) as a subgroup of the global symmetry, the hedgehog solutions discussed here holds. In the case of strings, the solution holds for a set of radially pointing strings joined at $r=0$.

For the strength of the hedgehog bigger than some critical value ($8\pi G v^2 > 1$ in the scalar field case), $M<0$ and r becomes a timelike variable. In this way we go from a static solution to an anisotropic cosmological solution. This transformation of r into a timelike variable is similar to that which occurs when crossing the Schwarzschild horizon of a black hole—crossing from outside to inside. In fact the phenomenon of making $M<0$ in the case of the hedgehog can be interpreted as making the energy density so large that the entire manifold is inside the Schwarzschild horizon.

We also studied what happens when we match the hedgehog to another vacuum solution: a false-vacuum region containing a hedgehog becomes unstable with regard to the formation of a true-vacuum bubble in its midst, even at the classical level, provided the hedgehog is strong enough. Such a hedgehog in the false vacuum can be interpreted as the introduction of some impurity or "defect" into the vacuum.

Notice also that, in the theory we consider, the introduction of a single hedgehog requires the expenditure of infinite energy, because of the long-distance behavior of the hedgehog. This is however not a problem, because the same effect can be achieved by the introduction of a hedgehog-antihedgehog pair, a configuration which has a finite total energy, where the hedgehog can be separated from the antihedgehog by a large distance (the antihedgehog to be defined in the scalar field theory as $\boldsymbol{\phi} = -v\hat{\mathbf{r}}$ rather than $\boldsymbol{\phi} = +v\hat{\mathbf{r}}$ in the case of the hedgehog).

If the hedgehog-antihedgehog separation is large, when considering the field configuration very close to that of the center of the hedgehog, the field configuration is almost the same as that produced by a pure single hedgehog. Therefore the evolution of bubbles originating from this point is almost the same, when the bubble is quite small, as is the case with the evolution in the presence of just one hedgehog. Once the bubble gets to a rather bigger size, it will continue to expand regardless of the existence or nonexistence of the hedgehog. The effect of the vacuum destabilization relies only on the short-distance behavior near one hedgehog and that is not very much affected by the existence of an antihedgehog very far away.

We also discussed the effect of a hedgehog on the evolution of false-vacuum bubbles. We find that introducing a strong enough hedgehog in a false-vacuum bubble guarantees that this bubble will expand to arbitrarily large sizes.

Concerning the origin of these hedgehogs, in a cosmological context it appears quite natural to consider them, since there is no reason to have the order parameter ϕ perfectly aligned all over space: for regions separated by more than one horizon length, we expect no correlations concerning the orientations of ϕ, so a rough estimate gives us a hedgehog in a volume of (one horizon)^3, at the time of the creation of the phase of the vacuum supporting the hedgehog.

Finally, it is interesting to investigate what happens in the gauged case. In such a situation, the SO(3) hedgehog becomes a finite-energy solution, i.e., the 't Hooft–Polyakov magnetic monopole [14].

Naively, one could expect to get back the global solution in the limit when the gauge coupling constant e goes to zero. This is actually true if we do not look to asymptotically large distances, since it is easy to see that the limits $e \to 0$ and $r \to \infty$ do not commute. For the region near the core of the monopole, the limit $e \to 0$ gives configurations physically equivalent to those of the global case. Likewise, for small values of the coupling constant, the local and global case do not differ very much as long as we choose to look at the field configurations near the core of the monopole.

In this regime of small gauge field coupling constant, looking also at the core of the monopole and also taking λ, the Higgs-field self-coupling [as in Eq. (1)], very large, we have that the solutions found in Sec. II of this paper hold.

For the hedgehogs discussed here, where the strength of the hedgehog is large enough, r becomes a timelike variable; i.e., we are inside the Schwarzschild radius. This implies the following: for the case of a magnetic monopole, for sufficiently strong λ and sufficiently small gauge field coupling constant, near the center of the monopole, we are inside the Schwarzschild radius; i.e., there is no static solution.

In such a case we can have gravitational collapse of the monopole, after which magnetic charge survives as "hair," but where all the scalar field structure of the 't Hooft–Polyakov monopole collapses.

It is quite likely that systems of gravity coupled to scalar fields in the context of cosmology, as the ones studied here, will continue to provide an interesting framework for developing theoretical ideas in relation to many diverse problems. (For example, aspects of nonlinear scalar fields with global gauge symmetry coupled to gravity, different from those studied here, have recently been studied in Ref. [15]).

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