New exact solution for the exterior gravitational field of a charged spinning mass

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An exact asymptotically flat solution of the Einstein-Maxwell equations describing the exterior gravitational field of a charged rotating axisymmetric mass possessing an arbitrary set of multipole moments is presented explicitly.

I. INTRODUCTION

In a recent paper by one of the authors [1] an exact solution of Einstein's equations describing the gravitational field of a spinning axisymmetric mass and possessing an arbitrary set of multipole moments has been constructed by application of the nonlinear superposition technique [2–4] to the stationary solution considered in Ref. [5]. The aim of the present article is to give an electrovacuum generalization of this solution which would already describe the field of a charged rotating arbitrary axisymmetric mass. To obtain such generalization, we first improve the solution of Ref. [1], avoiding the double summation in the expression for the function a, and then apply to the resulting formulas the Kramer-Neugebauer transformation [6], which allows us to derive the required electrovacuum solution and the corresponding metric functions in explicit form. Our solution has an event horizon with only a singularity at the pole ($y=-1$), and can be compared to the metric recently found by Quevedo and Mashhoon [7], which generalizes the Kerr-Newman spacetime [8] to the case of a mass with an arbitrary *quadrupole* moment and exhibits a singular event horizon.

II. STATIONARY VACUUM SOLUTION

As is well known [9], any stationary axisymmetric gravitational field is determined by the Papapetrou line element

$$ds^2=k^2f^{-1}\left[e^{2\gamma}(x^2-y^2)\left[\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right]+(x^2-1)(1-y^2)d\phi^2\right]-f(dt-\omega d\phi)^2\ , \tag{1}$$

where k is a real constant, and the three unknown metric functions f, γ, and ω, depending only on the prolate spheroidal coordinates (x,y), are defined through the complex Ernst potential ϵ, which satisfies the equation

$$(\epsilon+\epsilon^*)\Delta\epsilon=2(\nabla\epsilon)^2\ , \tag{2}$$

with

$$\begin{aligned}
&\epsilon=f+i\Phi,\quad \epsilon^*=f-i\Phi\ ,\\
&\Phi_x=k^{-1}(x^2-1)^{-1}f^2\omega_y,\quad \Phi_y=k^{-1}(y^2-1)^{-1}f^2\omega_x\ ,\\
&\gamma_x=\frac{1-y^2}{(x^2-y^2)(\epsilon+\epsilon^*)^2}[x(x^2-1)\epsilon_x\epsilon_x^*-x(1-y^2)\epsilon_y\epsilon_y^*-y(x^2-1)(\epsilon_x\epsilon_y^*+\epsilon_y\epsilon_x^*)]\ ,\\
&\gamma_y=\frac{x^2-1}{(x^2-y^2)(\epsilon+\epsilon^*)^2}[y(x^2-1)\epsilon_x\epsilon_x^*-y(1-y^2)\epsilon_y\epsilon_y^*+x(1-y^2)(\epsilon_x\epsilon_y^*+\epsilon_y\epsilon_x^*)]\ ,\\
&\Delta\equiv k^{-2}(x^2-y^2)^{-1}\{\partial_x[(x^2-1)\partial_x]+\partial_y[(1-y^2)\partial_y]\}\ ,\\
&\nabla\equiv k^{-1}(x^2-y^2)^{-1/2}[\mathbf{x}_0(x^2-1)^{1/2}\partial_x+\mathbf{y}_0(1-y^2)^{1/2}\partial_y]
\end{aligned} \tag{3}$$

($\mathbf{x}_0$ and $\mathbf{y}_0$ are unit vectors, while a subscript denotes a partial derivative with respect to x or y).

We shall use in the present work generating formulas that differ from those that were used to derive the metric obtained in Ref. [1] by the coordinate change $y \to -y$. Thus we will write now

$$
\begin{aligned}
&\epsilon = A_-/A_+, \quad f = 2p(x^2-1)e^{2\Psi}A/B\ ,\\
&e^{2\gamma} = e^{2\gamma'}A/(x+y)^8, \quad \omega = 2kq + kqp^{-1}(4y+\hat{p}-C/A)\ ,\\
&A_{\mp} = (p+1)(x\mp 1)[(x+y)^4 \mp iq(x\mp 1)(1\mp y)(x^2-1)e^{2a}]e^{\pm\Psi}\\
&\qquad + iq(x\pm 1)[(x+y)^4 \pm iq(x\pm 1)(1\pm y)(x^2-1)e^{2a}]e^{\mp\Psi}\ ,\\
&A = (x+y)^8 - q^2(1-y^2)(x^2-1)^3e^{4a}\ ,\\
&B = (p+1)(x+1)^2[(x+y)^8 + q^2(x+1)^2(1+y)^2(x^2-1)^2e^{4a}]\\
&\qquad + (p-1)(x-1)^2[(x+y)^8 + q^2(x-1)^2(1-y)^2(x^2-1)^2e^{4a}]e^{4\Psi} + 4q^2(x^2-1)^2(x+y)^5e^{2\Psi+2a}\ ,\\
&C = (x+y)^5[(p+1)(1+y)(x+1)^3e^{-2\Psi} + (p-1)(1-y)(x-1)^3e^{2\Psi}]e^{2a}\\
&\qquad + 2[q^2x(1-y^2)(x^2-1)^3e^{4a} + y(x+y)^8]\ ,
\end{aligned}
\tag{4}
$$

where Ψ is any solution of the static vacuum Weyl class satisfying Laplace's equation

$$\Delta\Psi = 0\ , \tag{5}$$

γ' is the γ function of the static Weyl solution corresponding to $\Psi' = \frac{1}{2}\ln[(x-1)/(x+1)] + \Psi$ (equations for γ can be found, e.g., in Ref. [10]); the functions a and $\hat{p}$ are, respectively, defined by the first-order differential equations

$$
\begin{aligned}
&(x+y)a_x = (xy+1)\Psi_x + (1-y^2)\Psi_y\ ,\\
&(x+y)a_y = -(x^2-1)\Psi_x + (xy+1)\Psi_y\ ,
\end{aligned}
\tag{6}
$$

and

$$\hat{p}_x = 2(y^2-1)\Psi_y, \quad \hat{p}_y = 2(x^2-1)\Psi_x\ , \tag{7}$$

while p and q are real constants subjected to the constraint

$$p^2 - q^2 = 1\ . \tag{8}$$

If one now chooses Ψ of the form

$$\Psi = \sum_{n=1}^{\infty} \frac{\alpha_n}{(x+y)^{n+1}} P_n\left[\frac{xy+1}{x+y}\right] \tag{9}$$

where the α_n's are real constants, and the P_n are the Legendre polynomials of argument $(xy+1)/(x+y)$, then one obtains, by integrating Eqs. (6), the following expression for the function a:

$$a = \sum_{n=1}^{\infty} \frac{\alpha_n}{(x+y)^{n+1}} P_{n+1}\ , \tag{10}$$

which turns out to be much simpler than the corresponding a in Ref. [1] because of the new generating formulas (4) that we are now using. On the other hand, the expression for γ' remains the same as in that solution [1], while $\hat{p}$ will differ from the respective expression in Ref. [1] only by its sign; i.e., we have

$$
\begin{aligned}
&\gamma' = \frac{1}{2}\ln\frac{x^2-1}{x^2-y^2} + \sum_{n=1}^{\infty}\alpha_n\left[\frac{P_{n+1}}{(x+y)^{n+1}} + \frac{1}{2^{n+1}}\sum_{l=0}^{n}\left[\frac{2^l(x-y)P_l}{(x+y)^{l+1}} - 1\right]\right]\\
&\qquad + \sum_{m,n=1}^{\infty}\frac{\alpha_m\alpha_n(m+1)(n+1)}{(m+n+2)(x+y)^{m+n+2}}(P_{m+1}P_{n+1} - P_mP_n)\ ,\\
&\hat{p} = 2\sum_{n=1}^{\infty}\frac{\alpha_n}{(x+y)^n}\left[P_{n-1} - \frac{xy+1}{x+y}P_n\right]\ .
\end{aligned}
\tag{11}
$$

Then Eqs. (4) and (9)–(11) will determine a stationary asymptotically flat metric that exhibits the same physical properties as the solution considered in Ref. [1]; i.e., it describes the exterior gravitational field of a stationary arbitrary axisymmetric mass; it possesses an event horizon (the hypersurface $x=1$) which, however, contains now only one singular point (the pole $y=-1$), while the area of the horizon again is given by Eq. (10) of Ref. [1]; in addition, it is defined by even more concise expressions than the mentioned solution [1]. The total mass M and angular momentum J of the metric (4), (9)–(11) are given by the relations

$$M = p, \quad J = q(p^2+2)/p\ , \tag{12}$$

while the higher relativistic mass-multipole moments [11,12] M_i, $i\geq 1$ contain the parameters α_n which describe the deformations of a mass and allow M_i to assume arbitrary values.

Now we shall proceed to derive a charged generalization of the above stationary vacuum solution, for which purpose we should consider the combined system of the Einstein-Maxwell equations.

III. STATIONARY ELECTROVACUUM SOLUTION

In this general case an axisymmetric Einstein-Maxwell field is determined, like in the previous case, by the Papapetrou line element, whereas the field equations assume Ernst's elegant form [13]

$$\begin{aligned}(\mathrm{Re}\epsilon+\Psi\Psi^*)\Delta\epsilon&=(\nabla\epsilon+2\Psi^*\nabla\Psi)\nabla\epsilon\ ,\\ (\mathrm{Re}\epsilon+\Psi\Psi^*)\Delta\Psi&=(\nabla\epsilon+2\Psi^*\nabla\Psi)\nabla\Psi\ ,\end{aligned}\tag{13}$$

with

$$\begin{aligned}&\epsilon=f-\Psi\Psi^*+i\Phi,\quad \Psi=A_4+iA_3'\ ,\\ &A'_{a,x}=k^{-1}(x^2-1)^{-1}fA'_{3,y},\\ &A'_{3,y}=k^{-1}(y^2-1)^{-1}fA'_{3,x}\ ,\end{aligned}\tag{14}$$

where A_3 and A_4, respectively, are the magnetic and electric components of the electromagnetic four-potential. The metric function ω is related to the potentials Φ and Ψ according to

$$\begin{aligned}\omega_x&=k(y^2-1)f^{-2}[\Phi_y+2\,\mathrm{Im}(\Psi^*\Psi_y)]\ ,\\ \omega_y&=k(x^2-1)f^{-2}[\Phi_x+2\,\mathrm{Im}(\Psi^*\Psi_x)]\ .\end{aligned}\tag{15}$$

Note, that the coefficient γ can be calculated once ϵ and Ψ are known [14].

As was shown by Kramer and Neugebauer [15], if one knows any solution ϵ_0 of the Ernst equation (2) one can construct potentials ϵ and Ψ satisfying the system (13) with the aid of the symmetric transformation

$$\epsilon=\frac{\epsilon_0-\beta^2}{1-\beta^2\epsilon_0},\quad \Psi=\frac{\beta(\epsilon_0-1)}{1-\beta^2\epsilon_0}\ ,\tag{16}$$

where β is a real constant subjected to the constraint $|\beta|\neq 1$ to avoid singularities in the metric (1) [see Eqs. (18)–(20) below]. Transformation (16) leads to an asymptotically flat solution of Eqs. (13) if the solution ϵ_0 is asymptotically flat; moreover, under (16), the function γ of the electrovacuum solution is equal to γ_0 of the seed vacuum metric.

Let us choose as ϵ_0 in (16) the solution from the previous section; then one arrives at ϵ and Ψ of the form

$$\epsilon=\frac{A_- -\beta^2A_+}{A_+-\beta^2A_-},\quad \Psi=\frac{\beta(A_- -A_+)}{A_+-\beta^2A_-}\ ,\tag{17}$$

which determine the following expression for the metric function f:

$$\begin{aligned}f&=2p(1-\beta^2)^2(x^2-1)e^{2\Psi}A/D\ ,\\ D&=(p+1)\{(x+y)^4[x+1-\beta^2(x-1)e^{2\Psi}]+(p-1)(x^2-1)[(x-1)^2(1-y)e^{2\Psi}+\beta^2(x+1)^2(1+y)]e^{2a}\}^2\\ &\quad+(p-1)\{(x+y)^4[(x-1)e^{2\Psi}-\beta^2(x+1)]+(p+1)(x^2-1)[(x+1)^2(1+y)+\beta^2(x-1)^2(1-y)e^{2\Psi}]e^{2a}\}^2\ .\end{aligned}\tag{18}$$

Since the potential γ of solution (17) is given by (4) and (11), it only remains to find the metric function ω to determine the new metric completely. Fortunately, it turns out possible to integrate Eqs. (15) for arbitrary ψ, the result being given by

$$\omega=[\omega_0(x,y,\psi)+\beta^4\omega_0(-x,-y,-\psi)]/(1-\beta^2)^2\ ,\tag{19}$$

where ω_0 is the function ω of the vacuum solution (4). By using (4) one finally has

$$\begin{aligned}\omega&=\frac{kq}{(1-\beta^2)^2}\{2(1+\beta^4)+p^{-1}[(1-\beta^4)(4y+\hat{p})-E/A]\}\ ;\\ E&=(x+y)^5\{[1-\beta^4+p(1+\beta^4)](1+y)(x+1)^3e^{-2\psi}+[p(1+\beta^4)-1+\beta^4](1-y)(x-1)^3e^{2\psi}\}e^{2a}\\ &\quad+2(1-\beta^4)[q^2x(1-y^2)(x^2-1)^3e^{4a}+y(x+y)^8]\ .\end{aligned}\tag{20}$$

Therefore we have derived all the necessary equations representing a charged generalization of the superposition formulas (4)–(8). With the choice of ψ, a, and $\hat{p}$ given by (9)–(11) the expressions obtained above for f, γ, and ω will describe the asymptotically flat gravitational field of a charged rotating arbitrary axisymmetric mass, whose deformations are characterized by the parameters α_n. The expressions for the total mass M, angular momentum J, total electric charge Q, and magnetic dipole moment μ of our solution can be found from (14), (17), and (18) with the aid of the coordinate transformation

$$x=(r-M)/k,\quad y=\cos\theta\ ,\tag{21}$$

and taking into account that the asymptotic behavior of the functions f, γ, ω, A_4, and A_3', as $r\to\infty$, is

$$f=1-2Mr^{-1}+O(r^{-2}),\quad e^{2\gamma}=1+O(r^{-2}),$$
$$\omega=2Jr^{-1}\sin^2\theta+O(r^{-2}),\quad A_4=Qr^{-1}+O(r^{-2}), \tag{22}$$
$$A'_3=\mu r^{-2}\cos\theta+O(r^{-9}),$$

then one gets

$$M=\frac{kp(1+\beta^2)}{1-\beta^2},\quad J=\frac{k^2q(1+\beta^2)(3+q^2)}{p(1-\beta^2)},$$
$$Q=-\frac{2\beta kp}{1-\beta^2},\quad \mu=\frac{2\beta k^2q(p^2+1)}{p(1-\beta^2)}. \tag{23}$$

Let us consider two particular cases of the solution obtained:

$$\text{(A)}\quad \alpha_n=0,\quad q\neq 0. \tag{24}$$

In this case one gets the three-parameter stationary electrovacuum solution which generalizes the solution of Ref. [5] and has the Schwarzschild metric as its static vacuum limit. Since it is the first solution of this kind different from the Kerr-Newman metric, we write explicitly its metric functions f, γ, and ω:

$$\begin{aligned}
&f=2p(1-\beta^2)^2(x^2-1)A'/B',\quad e^{2\gamma}=\frac{x^2-1}{x^2-y^2}\frac{A'}{(x+y)^8},\quad \omega=-\frac{2kq(1-y^2)C'}{p(1-\beta^2)A'},\\
&A'=(x+y)^8-q^2(x^2-1)^3(1-y^2),\\
&B'=(p+1)\{(x+y)^4[(1-\beta^2)x+1+\beta^2]+(p-1)(x^2-1)[(x-1)^2(1-y)+\beta^2(x+1)^2(1+y)]\}^2\\
&\qquad+(p-1)\{(x+y)^4[(1-\beta^2)x-1-\beta^2]+(p+1)(x^2-1)[(x+1)^2(1+y)+\beta^2(x-1)^2(1-y)]\}^2,\\
&C'=(x+y)^5[(1-\beta^4)(3x^2+3xy+y^2+1)+(1+\beta^4)(3px+py)]+q^2(x^2-1)^3[(1-\beta^4)(x+2y)+p(1+\beta^4)].
\end{aligned} \tag{25}$$

Note that in the absence of rotation ($q=0$) the above solution reduces, like the Kerr-Newman solution, to the Reissner-Nordström metric [16] describing the field of a spherically symmetric charged mass.

$$\text{(B)}\quad q=0,\quad \alpha_n\neq 0. \tag{26}$$

With this choice of the parameters one comes to the electrostatic solution which represents the exterior gravitational field of an arbitrary static axisymmetric mass possessing an electric charge. The metric function f and electric potential A_4 of this solution have the form

$$f=\frac{(1-\beta^2)^2(x^2-1)e^{2\psi}}{[x+1-\beta^2(x-1)e^{2\psi}]^2},$$
$$\psi=\sum_{n=1}^{\infty}\frac{\alpha_n}{(x+y)^{n+1}}P_n\left[\frac{xy+1}{x+y}\right], \tag{27}$$
$$A_4=-\frac{\beta[x+1-(x-1)e^{2\psi}]}{x+1-\beta^2(x-1)e^{2\psi}},$$

while γ is given by the right-hand side of the expression for γ' in Eq. (11). In the absence of electric charge ($\beta=0$) the metric (27) reduces to the static vacuum metric of Ref. [1]; the Reissner-Nordström solution is contained in (27) as the simplest electrostatic case corresponding to $\alpha_n=0$.

In the general case one gets the electrovacuum solution which has the Schwarzschild metric as its static pure vacuum limit and possesses an arbitrary set of multipole moments determined by the parameters α_n and q.

The investigation of the invariants of the Weyl tensor [17] shows that our solution degenerates to the Petrov type D only in the cases $\alpha_n=q=\beta=0$ (the Schwarzschild solution) and $\alpha_n=q=0, \beta\neq 0$ (the Reissner-Nordström solution), being algebraically general for all other values of the parameters. The solution has an event horizon defined by the hypersurface $x=1$, which can possess no more than a singular point located at the pole $y=-1$.

IV. CONCLUSION

An electrovacuum solution has been derived that corresponds to the exterior gravitational field of a charged rotating arbitrary axisymmetric mass, and, as such, its relevance to astrophysics is evident. A peculiar feature of our solution is that the whole metric is determined by very concise expressions which leads us to believe that a future more detailed analysis of its physical properties should be feasible.

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