Harmonic gauge in canonical gravity

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The Isham-Kuchař representation theory of the spacetime diffeomorphism group in canonical geometrodynamics is implemented in the context of harmonic coordinate conditions. The representation is carried by either an extended phase space, consisting of the cotangent bundle over the space of three-metrics, spacelike embeddings, and Lagrange multipliers which serve to enforce the harmonic gauge in the action, or by a reduced space in which the multipliers are eliminated. The approach used here is applicable to any generally covariant theory and to any coordinate conditions. The physical interpretation of the diffeomorphism Hamiltonians is discussed and compared with the analogous interpretation given by us elsewhere in terms of Gaussian coordinate conditions.

I. INTRODUCTION

The role of the spacetime diffeomorphism group is usually suppressed in the canonical approach to the dynamics of general relativity. As in gauge theories, the presence of this infinite-dimensional invariance group makes itself felt by the appearance of the Hamiltonian and momentum constraints, which serve both to limit the accessible portions of the gravitational phase space as well as to provide the dynamics of the theory by generating the appropriate canonical transformations. Unlike in gauge theories, however, the Poisson-brackets algebra provided by the constraints in a generally covariant theory does not replicate the gauge group Diff(M), or more precisely its Lie algebra diff(M), but rather takes the form of an unwieldy open algebra of hypersurface deformations.

In [1] Isham and Kuchař address this issue and find a way of explicitly isolating functions on an extended phase space for general relativity which provide a homomorphism from diff(M) to the Poisson algebra of functions on the phase space. The extension of the phase space is needed because in a generally covariant theory spacetime events are not distinguished *a priori* as there are no background structures which could be used to identify events. But an identification of spacetime events via canonical data is precisely what is needed to represent diff(M) on the phase space; after all, it is spacetime points which are acted upon by diffeomorphisms. Given this state of affairs, one can, and should, strive to isolate canonical variables on the geometrodynamical phase space which can serve as clocks and position markers, but no one has ever gotten very far with this enterprise. The approach of [1] is to adjoin embeddings, maps from the three-manifold on which the canonical formalism lives into spacetime, to the dynamical variables of geometrodynamics. The embedding variables select an instant of time and identify points of space. Spacetime diffeomorphisms act naturally on the embeddings and also on the gravitational variables once the embeddings are given geometrical content by linking them with the spacetime metric via coordinate conditions, which are imposed on the theory from the outset. The coordinate conditions chosen in [1] were the simplest possible: Gaussian coordinate conditions. The embedding variables did their job and led to the "diffeomorphism Hamiltonians," which generate dynamical evolution in the extended phase space via canonical transformations satisfying the Lie algebra of the spacetime diffeomorphism group.

The Isham-Kuchař construction of the diffeomorphism Hamiltonians relied heavily on special features of the Gaussian coordinates and it was not at all clear if the approach could be generalized to other coordinate conditions, e.g., the harmonic (or de Donder) conditions, which are technically superior to the Gaussian conditions in many applications [2]. In this paper we will show how to incorporate the harmonic gauge into the Isham-Kuchař representation theory of the Lie algebra of the spacetime diffeomorphism group in canonical gravity. The technique which will be used is quite general and is applicable to any generally covariant theory and any coordinate conditions whatsoever [3]. The key to the approach is to incorporate the coordinate conditions into the variational principle by adding the coordinate conditions to the Einstein-Hilbert action with Lagrange multipliers. The "gauge-fixed" action is of course no longer diffeomorphism invariant, but this is remedied by adjoining the preferred coordinates to the configuration space of the system. This is the "parametrization process" discussed in [1]. In the canonical formalism, the preferred coordinate functions become the embeddings, and their presence leads directly to the successful extraction of the diffeomorphism Hamiltonians. In [4] this approach was used for the diff(M) representation theory based on Gaussian coordinates and reproduced the results of [1] after the elimination of the nondynamical Lagrange multipliers. When applied to the harmonic coordinate conditions the formalism naturally leads to a doubly extended phase space because the Lagrange multipliers themselves become dynamical. If desired, the Lagrange multipliers can be eliminated and the phase space reduced to that of [1], but not without some cost in the way of technical complications.

Because the Isham-Kuchař approach to the diff(M) representation theory can be based on an action principle, it is possible to give a physical interpretation to the extended phase space; this was in fact the main emphasis of [4]. For the Gaussian coordinate conditions it was shown that the embeddings represent velocity potentials for a heat-conducting fluid. By using the fluid to identify spacetime events in the quantum canonical formalism, one can convert the hyperbolic "Wheeler-DeWitt equation" of vacuum geometrodynamics into a parabolic functional Schrödinger equation describing the evolution of geometry in the presence of the Gaussian fluid. The functional Schrödinger equation arises by requiring that the state vector of the system be annihilated by the operators corresponding to the diffeomorphism Hamiltonians, which are constrained to vanish in the classical theory. Thus the explicit incorporation of spacetime covariance into the theory leads, in principle, to considerable technical and conceptual simplifications. The scheme *does* have its disadvantages: the classical reference fluid, if it is to be physical, has to satisfy energy conditions which restrict the allowable regions of the classical geometrodynamical phase space. Because of this, the use of the fluid to define a privileged notion of *space* ultimately becomes physically untenable. In the quantum theory, a functional Schrödinger equation in the privileged Gaussian *time* can be retained, but the probability interpretation is hampered by the problem of constructing a suitable set of observables which would be compatible with the energy conditions.

From the point of view of [4], different coordinate conditions in the Isham-Kuchař representation theory correspond to the coupling of gravity to different types of reference systems. In the case of the harmonic gauge considered here, the reference system being used to define privileged notions of space and time is provided by massless scalar fields; thus the use of the harmonic gauge represents a step toward making the references systems more realistic and less phenomenological. As in [4], the use of scalar fields as clocks and position markers is not without its drawbacks, both mathematically and physically. We briefly discuss some of the advantages and disadvantages of the use of harmonic reference systems to interpret quantum gravity in the final section of the paper.

Our notation and conventions are as follows. Greek indices run from 0 to 3 and are used to label the harmonic coordinate functions. In Sec. II latin indices are spacetime indices and are spatial indices in Secs. III–V. The spacetime metric is denoted γ_{ab}, the spatial metric is g_{ab}, and the absolute values of their determinants are $|\gamma|$ and $|g|$, respectively. The Lie derivative along the vector field $\mathbf{N}$ is denoted $L_{\mathbf{N}}$.

II. LAGRANGIAN FORMULATION

We view the harmonic coordinate conditions as a set of locally imposed relations between the spacetime metric γ_{ab} and four externally prescribed, functionally independent scalar fields $X^\alpha : M \to R^4$, which are to serve as the harmonic coordinates. The relations we impose are

$$\gamma^{ab}\nabla_a\nabla_b X^\alpha = 0 \ , \tag{2.1}$$

where ∇_a is the torsion-free derivative operator compatible with γ_{ab}. Equation (2.1) is a coordinate-independent expression; we can of course evaluate it in the harmonic coordinates X^α themselves. This yields an alternative (possibly more familiar) condition:

$$\frac{1}{\sqrt{|\gamma|}}\partial_\beta\left[\sqrt{|\gamma|}\,\gamma^{\alpha\beta}\right] = 0 \ . \tag{2.2}$$

It is easy enough to find a variational principle that leads to both the Einstein equations and the gauge-fixing constraints (2.1). The action functional is given by

$$S[\gamma,\lambda] = S^E[\gamma] + S^F[\gamma,\lambda] \ , \tag{2.3}$$

where

$$S^E = \int_M |\gamma|^{1/2} R[\gamma] + \int_{\partial M} |g|^{1/2} K \ , \tag{2.4}$$

$$S^F = -\frac{1}{2}\int_M |\gamma|^{1/2}\gamma^{ab}\nabla_a\lambda_\alpha\nabla_b X^\alpha \ , \tag{2.5}$$

where K is the trace of the extrinsic curvature of the boundary. The action $S[\gamma,\lambda]$ is to be varied with respect to its arguments: the four-metric γ_{ab} and the four scalar fields λ_α, which are serving as Lagrange multipliers enforcing the harmonic conditions (2.1). For the moment we hold the harmonic coordinate functions X^α themselves fixed and do not vary them. When (2.3) is varied with respect to γ_{ab} we obtain the Einstein equations with sources

$$G^{ab} = \tfrac{1}{2} T^{ab} \ , \tag{2.6}$$

where

$$T^{ab} = 2|\gamma|^{-1/2}\frac{\delta S^F}{\delta\gamma_{ab}} = \nabla^{(a}\lambda_\alpha\nabla^{b)}X^\alpha - \tfrac{1}{2}\gamma^{ab}\nabla_c\lambda_\alpha\nabla^c X^\alpha \ . \tag{2.7}$$

Variation of the action (2.3) with respect to λ_α yields the harmonic coordinate conditions (2.1). We cannot obtain an equation of motion for λ_α directly from (2.3) because we are currently holding the fields X^α fixed; however, the contracted Bianchi identify used in conjunction with (2.6) yields

$$\gamma^{ab}\nabla_a\nabla_b\lambda_\alpha = 0 \ , \tag{2.8}$$

so that the multipliers λ_α evolve as a set of free massless scalar fields in the geometry defined by (2.6).

The content of (2.6), (2.8), and (2.1) is that of Einstein gravity, in the harmonic gauge, coupled to an energy-momentum tensor built from the Lagrange multipliers and harmonic coordinate functions. To recover the vacuum field equations we can eliminate the "sources" by simply restricting attention to the subspace of solutions to (2.1), (2.6), and (2.8) in which $\lambda_\alpha=0$. Alternatively, we can choose vanishing Cauchy data for λ_α. The field equations we have obtained form a hyperbolic set of partial differential equations [5] and so will generate a vacuum solution within the domain of dependence of this initial data set which, assuming that the spacetime (M,γ) is globally hyperbolic, is all of M.

While the action integral (2.3) is unchanged by (passive) coordinate transformations and hence is well-defined geometrically, it is not an invariant functional with respect to the (active) group of diffeomorphisms Diff(M). This is because we have fixed the scalar functions X^α and are not free to redefine them via the usual pull-back action of Diff(M). Following the strategy of [4] we can recover general covariance by "parametrizing" the theory associated with (2.3). This amounts to adjoining the harmonic coordinate functions themselves to the dynamical quantities to be varied. The consistency of this new variational principle is guaranteed by the fact that the equations coming from varying X^α are redundant:

$$\frac{\delta S}{\delta X^\alpha}=0 \leftrightarrow \nabla^a\nabla_a\lambda_\alpha=0\ , \tag{2.9}$$

which is (2.8), so that the extrema of the action, now viewed as a functional of $(\gamma_{ab},\lambda_\alpha,X^\alpha)$, are the same as before. Because the harmonic coordinate functions are no longer externally prescribed we recover the general covariance of the theory. In fact the redundancy of Eqs. (2.8) and (2.9) is the hallmark of diffeomorphism invariance. From the point of view of the parametrized theory, the action $S[g,X,\lambda]$ yields equations of motion interpretable as that of Einstein gravity coupled to eight free massless scalar fields. The action and energy-momentum tensor describing the "matter fields" are, at first sight, somewhat unconventional, and it will be useful in what follows to have available an alternative form for them.

Consider replacing the variables X^α, λ_α with

$$\phi^\alpha:=\tfrac{1}{2}(X^\alpha+\lambda^\alpha)\ , \tag{2.10}$$

$$\psi^\alpha:=\tfrac{1}{2}(X^\alpha-\lambda^\alpha)\ , \tag{2.11}$$

where the greek indices are raised and lowered with the Kronecker delta. The action (2.5) takes the form

$$S^F[\phi,\psi]=-\frac{1}{2}\int_M|\gamma|^{1/2}\gamma^{ab}(\nabla_a\phi_\alpha\nabla_b\phi^\alpha-\nabla_a\psi_\alpha\nabla_b\psi^\alpha)\ , \tag{2.12}$$

which shows that (2.5) amounts to coupling gravity to eight massless scalar fields, four of which have positive energy, four of which have negative energy:

$$T_{ab}=\nabla_a\phi_\alpha\nabla_b\phi^\alpha-\tfrac{1}{2}\gamma_{ab}\nabla_c\phi_\alpha\nabla^c\phi^\alpha -(\nabla_a\psi_\alpha\nabla_b\psi^\alpha-\tfrac{1}{2}\gamma_{ab}\nabla_c\psi_\alpha\nabla^c\psi^\alpha)\ . \tag{2.13}$$

The negative-energy scalar fields can be eliminated by observing that while the treatment of X^α as dynamical variables on the configuration space is vital for the establishment of diffeomorphism invariance, the multipliers are less crucial because they are, after all, true Lagrange multipliers (not associated with any gauge invariance). The multipliers can be eliminated by holding fixed the fields ϕ_α and extremizing (2.12) with respect to ψ^α, and then evaluating the action at the extremum. Varying (2.12) with respect to ψ^α yields

$$\nabla^a\nabla_a\psi^\alpha=0\ . \tag{2.14}$$

Substituting the general solution to this equation into (2.12) and integrating by parts amounts to dropping the second term in (2.12) and the addition of a surface term which we ignore. Substituting $\phi^\alpha=X^\alpha-\psi^\alpha$, where ψ^α satisfies (2.14), into this action yields, after additional partial integrations (again dropping surface terms), the reduced action

$$S^{F*}[\gamma,X]=-\frac{1}{2}\int_M|\gamma|^{1/2}\delta_{\alpha\beta}\gamma^{ab}\nabla_a X^\alpha\nabla_b X^\beta\ . \tag{2.15}$$

Thus, for *any* solution to the ψ^α equations of motion the reduced action (2.15) is the same (modulo surface terms [6]). While more elaborate choices are certainly possible, the simplest solution to (2.14) is $\psi^\alpha=0$, which, from (2.11), implies that the reduction can be accomplished by setting

$$\lambda^\alpha=X^\alpha\ . \tag{2.16}$$

The reduced formalism, because it derives from the manifestly invariant action (2.15), still retains its covariance under Diff(M). In fact the reduced formalism is equivalent to that of Einstein gravity minimally coupled to four free massless scalar fields X^α with positive energy.

III. HAMILTONIAN FORMULATION—EXTENDED PHASE SPACE

We will make the transition to the Hamiltonian formalism using the parametrized action (2.3) on the extended configuration space as the starting point. It is then relatively straightforward to obtain the phase-space version of the reduced formalism in which the multipliers and their conjugate momenta are eliminated. In each case we will explicitly isolate the "diffeomorphism Hamiltonians" which provide the homomorphism from the Lie algebra diff(M) into the Poisson algebra of functions on the appropriate phase space.

The Hamiltonian form of the action (2.3), viewed as a functional of $(\gamma_{ab},\lambda_\alpha,X^\alpha)$, follows the pattern of a collection of matter fields minimally coupled to gravity. Introduce a foliation

$$Y:R\times\Sigma\rightarrow M\ , \tag{3.1}$$

and denote the derivatives in the R direction (time derivatives) with an overdot. The spacetime metric, when pulled back by Y, decomposes into the lapse function N, shift vector N^a, and induced three-metric g_{ab}. The momenta p^{ab} conjugate to g_{ab} are defined exactly as in vacuum geometrodynamics. The phase-space form of S^E is also unchanged:

$$S^E[g,p;N,\mathbf{N}]=\int_{R\times\Sigma}(p^{ab}\dot{g}_{ab}-NH^E-N^aH^E_a)\ , \tag{3.2}$$

where

$$H^E=|g|^{-1/2}(p^{ab}p_{ab}-\tfrac{1}{2}p^2)-|g|^{1/2}R \ , \tag{3.3}$$

$$H_a^E=-2D_b p_a^b \ , \tag{3.4}$$

and D_a is the derivative operator on Σ which is compatible with g_{ab}. On $R\times\Sigma$ the gauge-fixing portion of the action takes the form

$$S^F=\frac{1}{2}\int_{R\times\Sigma}\sqrt{|g|}[N^{-1}(\dot{\lambda}_\alpha-L_{\mathbf{N}}\lambda_\alpha)(\dot{X}^\alpha-L_{\mathbf{N}}X^\alpha) -Ng^{ab}D_a\lambda_\alpha D_b X^\alpha] \ . \tag{3.5}$$

The momenta conjugate to the multipliers are given by

$$\mu^\alpha=\tfrac{1}{2}|g|^{1/2}N^{-1}(\dot{X}^\alpha-L_{\mathbf{N}}X^\alpha) \ , \tag{3.6}$$

while the momenta conjugate to the harmonic coordinate functions are given by

$$P_\alpha=\tfrac{1}{2}|g|^{1/2}N^{-1}(\dot{\lambda}_\alpha-L_{\mathbf{N}}\lambda_\alpha) \ . \tag{3.7}$$

The definitions of the momenta can be inverted to give the velocities $(\dot{X}^\alpha,\dot{\lambda}_\alpha)$ as functions of the conjugate momenta; the phase space action then takes the form

$$S^F[X,P;\lambda,\mu;g,N,\mathbf{N}] =\int_{R\times\Sigma}(P_\alpha\dot{X}^\alpha+\mu^\alpha\dot{\lambda}_\alpha-NH^F-N^aH_a^F) \ , \tag{3.8}$$

where

$$H^F=2|g|^{-1/2}P_\alpha\mu^\alpha+\tfrac{1}{2}|g|^{1/2}g^{ab}D_a\lambda_\alpha D_bX^\alpha \ , \tag{3.9}$$

$$H_a^F=\mu^\alpha D_a\lambda_\alpha+P_\alpha D_aX^\alpha \ . \tag{3.10}$$

The dynamical evolution of the combined system of gravitational field, harmonic coordinates, and Lagrange multipliers is governed by the Hamiltonian

$$H(N,\mathbf{N})=H(N)+H(\mathbf{N}) :=\int_\Sigma[NH+N^aH_a] \ , \tag{3.11}$$

where

$$H:=H^E+H^F \ , \tag{3.12}$$

$$H_a:=H_a^E+H_a^F \ . \tag{3.13}$$

Of course, as in any generally covariant theory, not all points of phase space are accessible; there are constraints which arise by variation of the lapse and shift:

$$H\approx 0 \ , \tag{3.14}$$

$$H_a\approx 0 \ . \tag{3.15}$$

Based on rather general arguments, one knows that the constraint functions satisfy the "Dirac algebra" of hypersurface deformations [7]:

$$[H(N),H(M)]=H(\mathbf{J}) \ ,$$
$$[H(\mathbf{N}),H(\mathbf{M})]=H(L_{\mathbf{N}}\mathbf{M}) \ , \tag{3.16}$$
$$[H(N),H(\mathbf{M})]=H(-L_{\mathbf{M}}N) \ ,$$

where

$$J^a=g^{ab}(N\partial_bM-M\partial_bN) \ .$$

Initial data satisfying (3.14) and (3.15), when evolved via the Hamiltonian (3.11), satisfy the field equations (2.1), (2.6), and (2.8), provided these equations are pulled back to $R\times\Sigma$ by the foliation Y, which in particular defines the lapse and shift. As discussed in Sec. II, we can restrict attention to vacuum spacetimes by choosing vanishing Cauchy data for the Lagrange multipliers λ_α. In the Hamiltonian formalism this is done by imposing the constraints

$$\lambda_\alpha\approx 0\approx P_\alpha \ . \tag{3.17}$$

One can check that the constraints (3.17) are preserved in time by the Hamiltonian (3.11) and that the constraints (3.14), (3.15), and (3.17) are "first class." As desired, the constraints (3.17) imply that the vacuum constraints are satisfied:

$$H^E\approx 0\approx H_a^E \ . \tag{3.18}$$

Similarly, the evolution equations for gravitational data that satisfy (3.14), (3.15), when evaluated on the subspace of phase space satisfying (3.17), are equivalent to the vacuum evolution equations.

Our goal is to extract diff(M), the Lie algebra of the *spacetime* diffeomorphism group, from the open algebra (3.16). As shown in [1,3,4], to do this we must rearrange the constraint functions such that the resulting constraint algebra is Abelian. This, in turn, is done by solving the constraints for the momenta conjugate to variables which represent embeddings of Σ into M. The embedding variables are identified with the harmonic coordinates themselves. To see this, note that the canonical variables $X^\alpha(x)$, $x\in\Sigma$, are, geometrically speaking, the composition (pull-back) of the harmonic coordinate functions with the embedding $Y_t(x)$, which is the foliation map evaluated at a fixed value of $t\in R$. Thus $X^\alpha(x)$ can be interpreted as the local (harmonic) coordinate expression of an embedding of Σ into M, i.e., an instant of time, and our task is to solve the constraints (3.14), (3.15) for the conjugate momenta P_α. These momenta enter linearly in the constraints, so one can simply solve for them by "brute force"; however, it is much more illuminating to solve for the embedding momenta by first isolating a "hypersurface basis" as a functional of the canonical data. The hypersurface basis is a quadruplet of four-vectors which, in a solution to the evolution equations, has one leg of the basis as the unit normal while the other three legs are everywhere tangent to the hypersurface specified by the embedding $X^\alpha(x)$. This basis can then be used to "unproject" the super-Hamiltonian and supermomenta functions; these unprojected functions will satisfy an Abelian Poisson-brackets algebra. We do this as follows.

Suppose we evolve the initial set of harmonic coordinate functions $X^\alpha(x)$ (the initial embedding) into a one-parameter family $X^\alpha(x,t)$—a (local) foliation of M. Then $X^\alpha(x,t)$ satisfies

$$\dot{X}^\alpha=[X^\alpha,H(N,\mathbf{N})]=2|g|^{-1/2}N\mu^\alpha+L_{\mathbf{N}}X^\alpha \ , \tag{3.19}$$

which is just a rearrangement of (3.6). This expression should be compared with the definition of the lapse and shift associated with a foliation $X^\alpha(x,t)$:

$$\dot{X}^\alpha = N n^\alpha + L_{\mathbf{N}} X^\alpha \ , \qquad (3.20)$$

where n^α is the unit normal to the embedding $X_t:\Sigma \to M$. Comparing (3.19) with (3.20) we see that the unit normal to the hypersurface is fixed by the momenta conjugate to the multipliers:

$$n^\alpha = 2|g|^{-1/2}\mu^\alpha \ . \qquad (3.21)$$

The hypersurface basis can then be taken to be $(n^\alpha, X^\alpha{}_{,a})$, where the triplet of four-vectors tangent to the hypersurface is given by $X^\alpha{}_{,a}$. In order to construct the unprojected constraint functions we will need to find the corresponding dual basis. To do this, introduce the alternating symbols $\eta_{\alpha\beta\gamma\delta}$ and η^{abc}. Now define

$$-n_\alpha = \frac{1}{3!} J^{-1} \eta_{\alpha\beta\gamma\delta} \eta^{bcd} X^\beta{}_{,b} X^\gamma{}_{,c} X^\delta{}_{,d} \ , \qquad (3.22)$$

$$X^a_\alpha = \frac{1}{2!} J^{-1} \eta_{\alpha\beta\gamma\delta} \eta^{abc} X^\beta{}_{,b} X^\gamma{}_{,c} (2|g|^{-1/2}\mu^\delta) \ , \qquad (3.23)$$

where

$$J = \frac{1}{3!}(2|g|^{-1/2}\mu^\alpha)\eta_{\alpha\beta\gamma\delta}\eta^{bcd} X^\beta{}_{,b} X^\gamma{}_{,c} X^\delta{}_{,d} \qquad (3.24)$$

is nonvanishing provided the canonical data are such that n^α and $X^\alpha{}_{,a}$ are four linearly independent vectors at each point. This requirement is equivalent to our initial assumption that the scalar fields are functionally independent in some open region of spacetime; so we can and will assume that the canonical data respect $J \neq 0$. It is easy to verify that (3.22)–(3.24) give a basis $(-n_\alpha, X^a_\alpha)$ which is dual to the basis $(n^\alpha, X^\alpha{}_{,a})$.

Because the hypersurface basis and its dual are known functions on the phase space, the quantities

$$\Pi_\alpha := -n_\alpha H + X^a_\alpha H_a \approx 0 \qquad (3.25)$$

are fixed, known functions on the phase space whose vanishing is equivalent to the vanishing of the original super-Hamiltonian and supermomentum constraint functions. The Π_α are the sought-after "unprojected" constraint functions as can be seen by explicitly evaluating (3.25):

$$\Pi_\alpha = P_\alpha + h_\alpha \ , \qquad (3.26)$$

where

$$h_\alpha = -n_\alpha (H^E + \tfrac{1}{2}|g|^{1/2} g^{ab} D_a \lambda_\beta D_b X^\beta) + X^a_\alpha (H^E_a + \mu^\beta D_a \lambda_\beta) \ . \qquad (3.27)$$

We can prove that the functions Π_α satisfy an Abelian Poisson-brackets algebra by bringing to bear the following argument. The constraints $\Pi_\alpha \approx 0$ are certainly first class because they arise via combinations (3.25) of the original first-class constraints (the super-Hamiltonian and supermomentum constraints). However, from the form of Π_α given in (3.26) it is clear that the bracket $[\Pi_\alpha(x), \Pi_\beta(y)]$ is independent of the momenta P_α because the embedding momenta only appear as shown in the first term of (3.26). Thus this bracket cannot be a combination of constraints—the constraints necessarily depend on P_α—and therefore the bracket must vanish strongly:

$$[\Pi_\alpha(x), \Pi_\beta(y)] = 0 \ . \qquad (3.28)$$

Having found a rearrangement of the constraints of the form (3.26) satisfying (3.28), the phase-space representatives of diff(M) are constructed using the techniques of [1]. Fix a set of coordinates X^α on M, then (locally) an infinitesimal diffeomorphism is represented by four functions $V^\alpha(\mathrm{X})$—the components of a spacetime vector field $\mathbf{V}$ in the X^α coordinate basis. Next we identify the canonical variables $\mathrm{X}^\alpha(x)$ as an embedding expressed parametrically in the coordinates X^α. The components of the vector field are pulled back by the embedding to become functions on Σ and functionals of the embedding: $V^\alpha(x,X] = V^\alpha(X(x))$. The dynamical variables

$$\Pi(\mathbf{V}) := \int_\Sigma V^\alpha(x,X]\Pi_\alpha(x) \qquad (3.29)$$

satisfy

$$[\Pi(\mathbf{V}), \Pi(\mathbf{W})] = \Pi(-[\mathbf{V},\mathbf{W}]) \ , \qquad (3.30)$$

where $[\mathbf{V},\mathbf{W}]$ is the commutator of vector fields. The functions (3.29) thus provide a homomorphism from diff(M) to the Poisson algebra of functions on the phase space.

IV. HAMILTONIAN FORMULATION: REDUCED PHASE SPACE

It is now possible to pass from the Hamiltonian formulation on the phase space which is extended by the addition of the Lagrange multipliers and embedding variables (along with conjugate momenta) to a reduced phase space in which the embeddings and their momenta only are present. The price to be paid, as we shall see, is a more complicated functional form for the diffeomorphism Hamiltonians, i.e., the reduced counterparts of (3.26), as well as a restriction on the available regions of the gravitational phase space.

The reduced Hamiltonian formulation can be obtained from the Hamiltonian formulation associated with the action (2.15), which is a standard computation. The reduction can also be performed in phase space by eliminating the multipliers and their conjugate momenta in favor of the embedding variables via the constraints

$$\lambda_\alpha - \delta_{\alpha\beta} X^\beta \approx 0 \ , \qquad (4.1)$$

$$\mu^\alpha - \delta^{\alpha\beta} P_\beta \approx 0 \ , \qquad (4.2)$$

which are the phase-space counterparts of (2.16). These constraints are preserved in time by the evolution generated by (3.11), and are second class, so they can be eliminated by using (4.1) and (4.2) everywhere λ_α and μ^α appear, and then using the induced symplectic structure on the submanifold of the phase space defined by (4.1), (4.2) to construct the appropriate set of induced Poisson brackets (the "Dirac brackets") for X^α and P_α:

$$[X^\alpha(x),P_\beta(y)]^*=\tfrac{1}{2}\delta^\alpha_\beta\delta(x,y)\ . \tag{4.3}$$

After a trivial rescaling

$$P_\alpha\to\tfrac{1}{2}P_\alpha\ ,$$

the Dirac brackets are canonical. Henceforth we will drop the asterick on the brackets and use the rescaled embedding momenta. This is equivalent to using the symplectic structure associated with (2.15).

The reduced Hamiltonian and momentum constraints, obtained either from the reduced action (2.15) or the constraints (4.1), (4.2), take the form

$$H^*=H^E+\tfrac{1}{2}(|g|^{-1/2}P_\alpha P^\alpha+|g|^{1/2}g^{ab}D_aX^\alpha D_bX_\alpha)\approx 0\ , \tag{4.4}$$

$$H_a^*=H_a^E+P_\alpha D_aX^\alpha\approx 0\ , \tag{4.5}$$

which are easily recognized to be the super-Hamiltonian and supermomenta associated with four free massless scalar fields coupled to gravity.

In contrast with (3.14), the constraint (4.4) is quadratic in the momenta conjugate to the harmonic coordinates (embeddings). Thus the simple "unprojection" used before will not be adequate to "Abelianize" the constraints. Still, the constraints are no worse than quadratic in the embedding momenta, so that it is possible to solve explicitly for these momenta. Let us briefly sketch this procedure.

By taking linear combinations of the X^α we can assume without loss of generality that they are in the form $X^\alpha=(T,X^i)$, where $X^i(x)$, $i=1,2,3$ are invertible functions of x. Similarly we can set $P_\alpha=(P,P_i)$, and the momentum constraints take the form

$$H_a^*=H_a^E+PD_aT+P_iD_aX^i\approx 0\ , \tag{4.6}$$

which can be solved for P_i:

$$P_i=-X_i^a(H_a^E+PD_aT)=:-h_i^*\ , \tag{4.7}$$

where

$$X_i^aX^j{}_{,a}=\delta_i^j\ . \tag{4.8}$$

Using (4.7), the Hamiltonian constraint becomes a quadratic equation for the momentum P:

$$\delta^{\alpha\beta}T_\alpha T_\beta P^2+\delta^{\alpha\beta}\mathcal{H}_\alpha T_\beta P+\delta^{\alpha\beta}\mathcal{H}_\alpha\mathcal{H}_\beta+2|g|^{1/2}H^E +|g|\delta_{\alpha\beta}g^{ab}D_aX^\alpha D_bX^\beta\approx 0\ , \tag{4.9}$$

where

$$T_\alpha=(-1,X_i^aD_aT)\ ,\qquad \mathcal{H}_\alpha=(0,H_a^EX^a{}_{,i})\ . \tag{4.10}$$

The constraint (4.9), being a quadratic equation for P, is easily solved:

$$P=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=:-h^*\ , \tag{4.11}$$

where

$$a=\delta^{\alpha\beta}T_\alpha T_\beta\ , \tag{4.12}$$

$$b=\delta^{\alpha\beta}\mathcal{H}_\alpha T_\beta\ , \tag{4.13}$$

$$c=\delta^{\alpha\beta}\mathcal{H}_\alpha\mathcal{H}_\beta+2|g|^{1/2}H^E+|g|\delta_{\alpha\beta}g^{ab}D_aX^\alpha D_bX^\beta\ , \tag{4.14}$$

although the explicit expression for P is unwieldy to say the least.

The difficulties with this way of solving the Hamiltonian constraint are associated with the question of existence and uniqueness of solutions. Existence of real solutions is not guaranteed because, given a set of gravitational data (g_{ab},p^{ab}), the discriminant b^2-4ac may be negative. And solutions will not be unique unless the discriminant is zero. Therefore, in order to treat the Hamiltonian constraint as a restriction on the momenta conjugate to the scalar field T we will have to restrict the gravitational phase space somewhat. A necessary and sufficient condition on the gravitational phase space which guarantees existence and uniqueness of real *positive* solutions for P is that

$$c\le 0\ . \tag{4.15}$$

To see this, note that a is manifestly positive and therefore (4.15) is sufficient to guarantee that the roots (4.11) cannot be complex. Furthermore, $c\le 0$ is a necessary condition ensuring that the roots of (4.9) cannot be both positive or both negative. So, given (4.15), the only possible result of solving (4.9) for P is a pair of distinct roots, one positive and one negative. We can consistently throw out the negative roots by requiring that the harmonic time coordinate T increase as we move from past to future; thus if we make a normal deformation of the hypersurface $T=T(x)$,

$$0\le\dot{T}=g^{-1/2}NP\ , \tag{4.16}$$

which implies $P\ge 0$. Another way of looking at this is that the non-negativity of P ensures that the lapse function is always positive. This requirement, along with (4.15), selects the plus sign in (4.11), and we see that the existence and uniqueness of solutions for P is guaranteed once we have agreed to accept only non-negative values for P and restrict the gravitational phase space by (4.15). Of course, given gravitational data satisfying (4.15), there is no reason why such data will continue to satisfy this restriction after a finite amount of time evolution. So one must conclude that the reduced phase-space approach we have discussed in this section is valid only locally in phase space and in spacetime; this latter locality being above and beyond that imposed by the fact that the harmonic coordinates themselves are locally defined.

Having emphasized the limitations of the present approach, the diff(M) representatives can now be constructed. Define

$$h_\alpha^*=(h^*,h_i^*)\ , \tag{4.17}$$

where it is understood that in h_i^* we substitute for P the solution of (4.9). We can now rewrite the constraints as

$$\Pi_\alpha^*:=P_\alpha+h_\alpha^*\approx 0\ , \tag{4.18}$$

and we can argue as before that

$$[\Pi^*_\alpha(x),\Pi^*_\beta(y)]=0 \ . \tag{4.19}$$

Therefore, when the functions $\Pi^*_\alpha(x)$ are smeared with spacetime vector fields $\mathbf{V}$ and $\mathbf{W}$ we recover the algebra diff(M):

$$[\Pi^*(\mathbf{V}),\Pi^*(\mathbf{W})]=\Pi^*(-[\mathbf{V},\mathbf{W}]) \ . \tag{4.20}$$

V. PHYSICAL INTERPRETATION

We have presented two versions of the Isham-Kuchař approach to phase-space representations of spacetime diffeomorphisms in the context of the harmonic gauge. The first version, while technically quite clean, needed a rather large phase space: in contrast with [1], we need not only the embeddings, but also the Lagrange multipliers λ_α and their conjugate momenta. The second approach, while managing to eliminate the multiplier part of the phase space, works locally at best in the gravitational phase space. For each of these two versions there is a corresponding physical interpretation, which is especially relevant for the quantum theory, analogous to that obtained for the diffeomorphism representations using Gaussian coordinate conditions [4].

Classically, in the extended phase-space approach we can physically interpret the resulting formalism as that of gravity coupled to eight massless scalar fields, four of which are being used to identify instants of time and points of space. The main defect in this interpretation is that it is hard to imagine setting up a physical system (even of the most idealized type) in which the fields λ_α and X^α (or ϕ^α and ψ^α) are coupled as in (2.5) [or (2.12)]. Indeed, the classical energy-momentum tensor (2.7) [or (2.13)] satisfies none of the usual energy conditions and is rather unconventional. It could only arise in a theory in which one postulates some sort of internal SO(4,4) symmetry for the matter fields, and such a symmetry is not present in any current physical models of matter.

Despite the last comment, let us briefly outline the quantized version of the extended phase-space approach for the sake of comparison with the results of [4]. The structure of the diffeomorphism Hamiltonians (3.26) suggests a quantum representation in which states Ψ depend on the three-metric g_{ab}, the multiplier momenta μ^α, and the embedding X^α (which represents time):

$$\Psi=\Psi[X,\mu,g] \ . \tag{5.1}$$

The quantum dynamical variables $(X^\alpha,\hat{\mu}^\alpha,\hat{g}_{ab})$ act on such wave functions via multiplication while the remaining variables $(P_\alpha,\hat{\lambda}_\alpha,\hat{p}^{ab})$ act by functional differentiation in the usual way. Then the physical state condition

$$\hat{\Pi}_\alpha\Psi=0 \ , \tag{5.2}$$

i.e.,

$$i\frac{\delta\Psi[X,\mu,g]}{\delta X^\alpha(x)}=\hat{h}_\alpha(x;X,\hat{\lambda},\hat{\mu},\hat{g},\hat{p})\Psi[X,\mu,g] \ , \tag{5.3}$$

is a functional Schrödinger equation, the solutions of which can be interpreted as probability amplitudes for localizing the values of the observables (g_{ab},μ^α) on the embedding specified by X^α. The formal structure of the quantum theory would be thus quite satisfying if not for the fact that the matter fields are not entirely realistic.

From the physical (as opposed to mathematical) point of view the reduced formalism is much more palatable as it is equivalent to gravity coupled to four positive-energy scalar fields which are being used to identify spacetime events. The formal quantization proceeds by using states which are functionals of the three-metric and the scalar fields (again playing the role of time):

$$\Psi=\Psi[X,g] \ , \tag{5.4}$$

with $\hat{g}_{ab}$ and X^α represented as multiplication operators and their conjugate momenta represented via functional differentiation. We again obtain a functional Schrödinger equation for the physical states:

$$\hat{\Pi}^*_\alpha\Psi=0 \ , \tag{5.5}$$

i.e.,

$$i\frac{\delta\Psi[X,g]}{\delta X^\alpha(x)}=\hat{h}^*_\alpha(x;X,\hat{g},\hat{p})\Psi[X,g] \ . \tag{5.6}$$

Such states represent the probability amplitude for localizing the value of the three-metric on the embedding $X^\alpha=X^\alpha(x)$. The difficulty here is to make sense of the classical restriction (4.15) in the quantum theory as well as to find a sensible operator representation of the square roots which appear in h^*_α. These difficulties are analogous to those found in [4] associated with energy conditions for the Gaussian reference fluid. As in [4], the present difficulties are mitigated somewhat if one is willing to decline the use of the scalar fields to define a privileged notion of space. This amounts to dropping the canonical pairs (X^i,P_i) from the classical phase space; thus we use a single scalar field to provide a (many-fingered) time. In this approach the Hamiltonian and momentum constraints are of the form

$$H^E+\tfrac{1}{2}(|g|^{-1/2}P^2+|g|^{1/2}g^{ab}D_aTD_bT)\approx 0 \ , \tag{5.7}$$

$$H^E_a+PD_aT\approx 0 \ . \tag{5.8}$$

The Hamiltonian constraint can be rewritten as

$$P+h'\approx 0 \ , \tag{5.9}$$

where

$$h'=-\sqrt{-2|g|^{1/2}H^E-|g|g^{ab}D_aTD_bT} \ . \tag{5.10}$$

On the constraint surface (5.7) the argument of the square root is non-negative, and will remain non-negative throughout the dynamical evolution.

In the quantum theory associated with (5.7) and (5.8) we have the states

$$\Psi=\Psi[T,g] \ , \tag{5.11}$$

which satisfy the functional Schrödinger equation

$$i\frac{\delta\Psi[T,g]}{\delta T(x)}=\hat{h}'(x;T,\hat{g},\hat{p})\Psi[T,g] \tag{5.12}$$

as well as a subsidiary constraint

$$\left[\hat{H}_a^E(x) - iD_a T \frac{\delta}{\delta T(x)}\right]\Psi[T,g]=0\ , \tag{5.13}$$

which means that the state functional is invariant under the action of diff(Σ) on its arguments. As in [4], however, the metric is no longer an observable and neither is the three-geometry, so the probability interpretation of the state functional is not directly available. What are needed to interpret the states are observables, which are built from the three-geometry, that preserve the subspace of physical states satisfying

$$(2|\hat{g}|^{1/2}\hat{H}^E + |\hat{g}|\hat{g}^{ab}D_a T D_b T)\Psi \le 0\ , \tag{5.14}$$

so that a meaningful square root can be extracted in (5.10).

So, as in [4], we see that the use of matter fields as a means of solving the old problems of time, observables, and interpretation in canonical quantum gravity is not without problems of its own. Whether these difficulties are preferable to those which arise in vacuum quantum gravity, where one is expected to find a purely geometrical solution to these persistent problems, remains to be seen.

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