Gravitational breather and topological properties of gravisolitons

V. Belinsky
National Institute for Nuclear Physics (INFN) and International Center for Relativistic Astrophysics (ICRA), Rome, Italy
(Received 31 May 1991)

It is shown that for a wide class of cosmological solutions to the vacuum Einstein equations the notions of gravisolitons and antigravisolitons with respect to some topological charge can be introduced. The presence of attractive forces between two gravitational solitons with charges of opposite signs and repulsive forces between solitons of the same charges is shown. The construction of the gravitational breather as a bound state of the gravisoliton and antigravisoliton is described.

I. INTRODUCTION

In Ref. [1], the application of the inverse scattering method to general relativity and the procedure for calculating the exact solitonic solutions to the Einstein equations was described. It was established that, from the mathematical point of view, these solutions have the full status of solitons, although they possess a number of unusual features. Perhaps because of the latter reason, the theory of gravisolitons has received minor attention so far. In particular, up to now it was unclear whether the gravisoliton can represent a topological object and, if so, whether topological charge can be associated with it. If the gravitational topological charge exists, can one introduce the notions of the gravisoliton (S) and antigravisoliton (A) with respect to this charge and, if so, can this charge give rise to a repulsive force between gravisolitons of the same charge (SS or AA system) and an attractive force between a gravisoliton and antigravisoliton (SA system)? If this were the case, would it permit the existence of a time-oscillating gravisoliton-antigravisoliton bound state, i.e., a solution to the Einstein equations which would be the gravitational analogue of the breather?

I am going to give arguments in favor of positive answers to all these questions for a wide class of gravitational solitonic solutions. This class constitutes the solutions of a cosmological type which (i) describes inhomogeneous cosmological models containing "normal" solitonic waves, i.e., waves propagating with subluminal velocity [2], and (ii) are solutions to the essentially nonlinear field equations, i.e., for the essentially nondiagonal metric.

The nonstationary integrable metrics are

$$-ds^2=f(t,z)(-dt^2+dz^2)+g_{ab}(t,z)dx^a dx^b \ , \tag{1}$$

where $a,b=1,2$. The function f is positive, the signature of g_{ab} is $(++)$, and the square of the norm of the timelike vector is negative. The Einstein equations for (1) in a vacuum give, for the matrix g (with components g_{ab}),

$$(\alpha g_{,\zeta}g^{-1})_{,\eta}+(\alpha g_{,\eta}g^{-1})_{,\zeta}=0 \ , \tag{2}$$

$$\zeta=\tfrac{1}{2}(z+t), \quad \eta=\tfrac{1}{2}(z-t), \quad \alpha=(\det g)^{1/2} \ . \tag{3}$$

Here and below a comma denotes the usual partial derivative and power exponent $\frac{1}{2}$ indicates only positive values of the square root. We are forced to consider only half of the α axis because, in general, the points $\alpha=0$ correspond to the physical singularity through which the metric cannot be extended. After we solve problem (2), the calculation of the metric component f does not represent any principal difficulties [1,3].

The trace of (2) gives

$$\alpha_{,\zeta\eta}=0 \ . \tag{4}$$

In addition to α, we need a second independent solution β of the wave equation (4). These two solutions are

$$\alpha=a(\zeta)+b(\eta), \quad \beta=a(\zeta)-b(\eta) \ , \tag{5}$$

where a and b are arbitrary functions. Because of the freedom of coordinate transformation $\zeta=\zeta(\zeta')$, $\eta=\eta(\eta')$, which is still permitted in (1), one can specialize a and b to any form consistent with the adopted structure of the spacetime.

In order to have cosmological-type solutions, α should be timelike everywhere in spacetime ($\alpha_{,\zeta}\alpha_{,\eta}<0$). In this case, the curve $\alpha=0$ corresponds to the cosmological singularity. The variable β will be automatically spacelike everywhere. We also postulate that α,β form the single patch of the natural coordinates which cover the whole related two-dimensional section of the maximally extended physical spacetime and each pair of real numbers α,β from the ranges $\alpha>0$, $-\infty<\beta<+\infty$ represent one and only one point of this spacetime and vice versa. The variables t,z we consider as a second equivalent coordinate system in the same spacetime, which means that the map between t,z and α,β is smooth everywhere and one to one in both directions. As a consequence we have the right to use only those functions $\alpha(t,z)$ and $\beta(t,z)$ for which the Jacobian

$$\alpha_{,t}\beta_{,z}-\alpha_{,z}\beta_{,t}=\tfrac{1}{2}(\alpha_{,\zeta}\beta_{,\eta}-\alpha_{,\eta}\beta_{,\zeta})=-\alpha_{,\zeta}\alpha_{,\eta}$$

has nowhere zeros or infinities.

II. GRAVISOLITON'S TOPOLOGICAL CHARGE

To construct a one-soliton solution for (1), we need to "dress" some background metric [1]. Let us take this seed solution in diagonal form:

$$-ds^2=f_0(-dt^2+dz^2)+\alpha e^{u_0}(dx^1)^2+\alpha e^{-u_0}(dx^2)^2 , \tag{6}$$

where $\alpha(t,z)$ is a solution of (4) and $u_0(t,z)$, as follows from (2), should satisfy

$$(\alpha u_{0,\xi})_{,\eta}+(\alpha u_{0,\eta})_{,\xi}=0 . \tag{7}$$

The one-solitonic metric on this background can be written in the form

$$g=\begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22}\end{bmatrix}=\frac{1}{\mu\cosh\rho}\begin{bmatrix}(\mu^2e^{\rho}+\alpha^2e^{-\rho})e^{u_0} & \alpha^2-\mu^2 \\ \alpha^2-\mu^2 & (\alpha^2e^{\rho}+\mu^2e^{-\rho})e^{-u_0}\end{bmatrix} , \tag{8}$$

$$f=f_0\alpha^{1/2}\mu\cosh\rho(\alpha^2-\mu^2)^{-1} . \tag{9}$$

It is necessary to keep in mind that, for any solution g of Eq. (2), the matrix $-g$ is also a solution and the same symmetry is valid for the metric component f. Because of this freedom, one can always choose the correct signs in front of g and f in (8) and (9) in order to ensure positivity for f and a $(++)$ signature for g.

The function $\mu(t,z)$ is a solution to the quadratic equation

$$\mu^2+2(\beta-w)\mu+\alpha^2=0 \tag{10}$$

in which w is an arbitrary real constant. This function has to be substituted into (8) and (9) only in those parts of the α,β plane where it is real. In the region $(\beta-w)^2-\alpha^2<0$, the quantity μ becomes complex and the real physical continuation of the solution (8) and (9) into this region coincides with the unperturbed seed metric (6). On the borders $(\beta-w)^2-\alpha^2=0$, the one-soliton metric (8) is continuous but its first derivatives are not (see the details in Ref. [1]).

The function $\rho(t,z)$ can be found by quadratures from the following differential equations [4]:

$$\rho_{,\xi}=(\alpha+\mu)(\alpha-\mu)^{-1}u_{0,\xi} ,$$
$$\rho_{,\eta}=(\alpha-\mu)(\alpha+\mu)^{-1}u_{0,\eta} . \tag{11}$$

To ensure subluminal soliton speed, it is necessary to choose the background metric coefficient $u_0(t,z)$ to have spacelike character ($u_{0,\xi}u_{0,\eta}>0$). In this case, as follows from (11), the variable $\rho(t,z)$ also has spacelike character and the curve $\rho=0$ is timelike. This curve should be considered qualitatively as the soliton world line. Indeed, it can be seen from (8) that the field of the soliton perturbation is concentrated at the points where $\rho=0$ (which is especially clear in the approximation where α and μ can be considered as slowly varying functions with respect to ρ and u_0, see below) [5].

So, we shall use in what follows only the background solutions having spacelike u_0 and also only those functions $u_0(t,z)$ for which the pair α,u_0 can serve as acceptable time and space coordinates, i.e., for which the Jacobian $\alpha_{,\xi}u_{0,\eta}-\alpha_{,\eta}u_{0,\xi}$ vanishes or diverges nowhere.

There are two solutions to Eq. (10) for μ: if the first one is μ, the second is $\alpha^2\mu^{-1}$. Consequently, if the first solution, which can be denoted by $\mu=\mu_{\rm in}$, belongs to the interval $[-\alpha,\alpha]$, the second one $\mu=\mu_{\rm out}$ is out of this interval. By definition these solutions are

$$\mu_{\rm in}=(w-\beta)\{1-[1-\alpha^2(\beta-w)^{-2}]^{1/2}\} ,$$
$$\mu_{\rm out}=(w-\beta)\{1+[1-\alpha^2(\beta-w)^{-2}]^{1/2}\} . \tag{12}$$

It can be seen that, under the earlier adopted spacetime topological restrictions and due to general continuity requirements, the property of the pole trajectory $\mu(t,z)$ to be $\mu_{\rm in}$ or $\mu_{\rm out}$ is global: if, at some spacetime point, $\mu=\mu_{\rm in}$, then μ will remain $\mu_{\rm in}$ everywhere (and the same for $\mu_{\rm out}$) [6]. It follows from (12) that both solutions have positive values in the "left" causal domain $\beta-w<-\alpha$ and they are negative in the "right" region $\beta-w>\alpha$. This should be taken into account in choosing the correct signs for g and f in (8) and (9) for each causal domain.

So, there exist two one-solitonic solutions: the metric (8) and (9) for $\mu=\mu_{\rm in}$ and $\mu=\mu_{\rm out}$, respectively. It can be shown now that there is enough evidence to consider these two solutions as belonging to different topological sectors and, consequently, having different topological indices. In other words, it can be assumed that there is no homotopy between these two solutions. If such homotopy were to exist, it should also manifest itself (at least in some way) in the approximation where the function α (together with β and μ) is slowly varying with respect to the functions ρ and u_0. However, in this extremal case, the theory based on Eqs. (2)–(4) tends to coincide with sine-Gordon theory for which solutions with $\mu=\mu_{\rm in}$ and $\mu=\mu_{\rm out}$ emerge as well-known topologically different solutions associated with topological charges plus one and minus one. This correspondence is of help in finding, for the exact gravitational case, those one-dimensional manifolds between which one-soliton maps act, which is necessary for the homotopy notion to be sensible.

To verify these assertions, we need to pass from the metric matrix g to more appropriate field variables which do not depend on arbitrary linear transformations of the dummy coordinates x^1 and x^2. It is possible to construct such invariants only from the matrices $g_{,\xi}g^{-1}$ and $g_{,\eta}g^{-1}$. The first three nontrivial quantities of this kind are $\mathrm{Tr}[(g_{,\xi}g^{-1})^2]$, $\mathrm{Tr}[(g_{,\eta}g^{-1})^2]$, and $\mathrm{Tr}(g_{,\xi}g^{-1}g_{,\eta}g^{-1})$. [The simplest invariants $\mathrm{Tr}(g_{,\xi}g^{-1})=2\alpha_{,\xi}\alpha^{-1}$ and $\mathrm{Tr}(g_{,\eta}g^{-1})=2\alpha_{,\eta}\alpha^{-1}$ are trivial, because they do not carry any information about the soliton's behavior.] With the notation

$$\begin{aligned} &[\ln(g_{11}\alpha^{-1})]_{,\zeta}=R_1\cos(\gamma/2+\omega/2)\ , \\ &(g_{12}g_{11}^{-1})_{,\zeta}g_{11}\alpha^{-1}=R_1\sin(\gamma/2+\omega/2)\ , \\ &[\ln(g_{11}\alpha^{-1})]_{,\eta}=R_2\cos(\gamma/2-\omega/2)\ , \\ &(g_{12}g_{11}^{-1})_{,\eta}g_{11}\alpha^{-1}=R_2\sin(\gamma/2-\omega/2)\ , \end{aligned} \tag{13}$$

we get

$$\begin{aligned} &\mathrm{Tr}[(g_{,\zeta}g^{-1})^2]=2R_1^2+2(\alpha_{,\zeta})^2\alpha^{-2}\ , \\ &\mathrm{Tr}[(g_{,\eta}g^{-1})^2]=2R_2^2+2(\alpha_{,\eta})^2\alpha^{-2}\ , \\ &\mathrm{Tr}(g_{,\zeta}g^{-1}g_{,\eta}g^{-1})=2R_1R_2\cos\omega+2\alpha_{,\zeta}\alpha_{,\eta}\alpha^{-2}\ . \end{aligned} \tag{14}$$

Thus, the invariants we need are R_1, R_2, and ω. It is remarkable that Eq. (2) [together with self-consistency conditions for (13)] splits into the system containing invariants only:

$$\begin{aligned} &\omega_{,\zeta\eta}+\left[\frac{R_{1,\eta}}{R_1}+\frac{\alpha_{,\eta}}{2\alpha}\right]\omega_{,\zeta}+\left[\frac{R_{2,\zeta}}{R_2}+\frac{\alpha_{,\zeta}}{2\alpha}\right]\omega_{,\eta} \\ &\quad=\left[\left[\frac{\alpha_{,\zeta}R_2}{2\alpha R_1}\right]_{,\zeta}+\left[\frac{\alpha_{,\eta}R_1}{2\alpha R_2}\right]_{,\eta}+R_1R_2\right]\sin\omega\ , \end{aligned} \tag{15}$$

$$\begin{aligned} &2\alpha R_{1,\eta}+\alpha_{,\eta}R_1+\alpha_{,\zeta}R_2\cos\omega=0\ , \\ &2\alpha R_{2,\zeta}+\alpha_{,\zeta}R_2+\alpha_{,\eta}R_1\cos\omega=0\ , \end{aligned} \tag{16}$$

and the system from which the function γ can be found by quadratures involving the given invariants

$$\begin{aligned} (\gamma/2+\omega/2)_{,\eta}&=-R_2\sin(\gamma/2-\omega/2) \\ &\quad+\alpha_{,\zeta}R_2(2\alpha R_1)^{-1}\sin\omega\ , \\ (\gamma/2-\omega/2)_{,\zeta}&=-R_1\sin(\gamma/2+\omega/2) \\ &\quad-\alpha_{,\eta}R_1(2\alpha R_2)^{-1}\sin\omega \end{aligned} \tag{17}$$

[the self-consistency of (17) is ensured by (15) and (16)].

It is reasonable to consider the field $\omega(\mathrm{mod}2\pi)$ as the main gravisolitonic characteristic. Just this field has qualitative features we usually associate with solitons. Indeed, one can define the solitonic vacuum states as those exact solutions of the system (15)–(17) which correspond to the discrete set of the constant values of the ω field: $\omega=2\pi n$, where n is an integer. With these values of ω, Eqs. (15)–(17) can be solved exactly. For the functions R_1 and R_2 we get $R_1=\phi_{,\zeta}$, $R_2=\phi_{,\eta}$, where ϕ should be a solution to the equation $(\alpha\phi_{,\zeta})_{,\eta}+(\alpha\phi_{,\eta})_{,\zeta}=0$. Equations (17) are now equivalent to the Backlund transformation and gives, for γ, the solitonic-type solution

$$\gamma=4\arctan\exp[-(-1)^n\phi-C]\ ,$$

where $C=\mathrm{const}$. However, it turns out that this is a fictitious or pure gauge soliton because it can be removed by a linear transformation (with constant coefficients) of the dummy coordinates x^1 and x^2 (i.e., because γ is not invariant with respect to this transformation). Indeed, after calculating the g matrix from (13), it is easy to see that, in this way, the transformed matrix g can be made diagonal

$$g=\mathrm{diag}\{\alpha\exp[(-1)^n\phi+C],\alpha\exp[-(-1)^n\phi-C]\}\ ,$$

which means that $\gamma_{\mathrm{new}}=2\pi m$ (m is an integer). Any diagonal solution for g has this form so we conclude that any diagonal matrix g represents one of the vacuum states with respect to the invariant solitonic field ω. This picture conforms to our intuitive desire to consider the solutions to Eq. (2) with diagonal g as containing no real solitons because the Einstein field equations for g in this case are linear.

Because the function $\mu(t,z)$ satisfies the differential equations $\mu_{,\zeta}=2\alpha_{,\zeta}\mu(\alpha-\mu)^{-1}$ and $\mu_{,\eta}=2\alpha_{,\eta}\mu(\alpha+\mu)^{-1}$ (see Ref. [1] or just differentiate relation (10)), it is easy to show that, for timelike α, the variable $\mu\alpha^{-1}$ is also timelike. Then it can be seen that, for solution (8), the variables $\mu\alpha^{-1}$ and ρ also form a pair of acceptable time and space coordinates. The analysis shows that, for each fixed value of the new time $\mu\alpha^{-1}$ [i.e., on the straight lines $(\beta-w)\alpha^{-1}=\mathrm{const}$ in the α,β plane], the function $\omega(\rho)$ acts as a regular map between the one-dimensional ρ space and the ω circle: the map is one to one in both directions and the angle ω covers exactly one time segment $[0,2\pi]$ when ρ runs between its natural boundaries (from $\rho_{\alpha=0}$ to $\rho_{\alpha=\infty}$). It turns out that, for this map, the integer quantity $\mathrm{sgn}(\alpha^2-\mu^2)$ corresponds to the Brouwer degree. This quantity is equal to plus one for $\mu=\mu_{\mathrm{in}}$ and to minus one for $\mu=\mu_{\mathrm{out}}$. The above arguments show that two related one-solitonic solutions are associated with different topological indices and act like bridges between neighboring vacuums of the field ω, which is in agreement with general properties of topological solitons.

These assertions can be supported by the qualitative picture for the extreme case where α [or a and b in (5)] can be considered as a slowly varying variable with respect to ρ and u_0. In this case, the variables β and μ are also slowly varying because they are expressed in terms of a and b algebraically. In the first approximation, (16) gives $R_{1,\eta}=0$, $R_{2,\zeta}=0$, and without essential loss of generality one can take $R_1=\mathrm{const}$, $R_2=\mathrm{const}$. From (15) it follows $\omega_{,\zeta\eta}=R_1R_2\sin\omega$ and the Backlund transformation (17) for γ. Solution (8) has its own natural place in this approximation. Equation (7) gives $u_{0,\zeta\eta}=0$, and for the spacelike u_0 one can choose the coordinates in such a way that $u_0=mz$, where $m>0$ is an arbitrary constant. Integrating (11), which is trivial in the first-order approximation (α and μ are constants), and substituting (8) into (13) (at this step we differentiate only the rapidly varying functions ρ and u_0), we get $R_1=R_2=m$ and

$$\begin{aligned} &\omega=4\arctan e^{\rho}\ , \\ &\rho=m\frac{\alpha^2+\mu^2}{\alpha^2-\mu^2}\left[z+\frac{2\alpha\mu}{\alpha^2+\mu^2}t+z_0\right]\ , \end{aligned} \tag{18}$$

where z_0 depends only on the slowly varying quantities α,

β, μ, i.e., is a constant in the first approximation. Because of the obvious identity

$$\left[\frac{\alpha^2+\mu^2}{\alpha^2-\mu^2}\right]^2 \equiv \left[1-\left[\frac{2\alpha\mu}{\alpha^2+\mu^2}\right]^2\right]^{-1}, \quad (19)$$

expression (18) describes the sine-Gordon soliton with mass m and local velocity $v=-2\alpha\mu(\alpha^2+\mu^2)^{-1}$. It is now clear that the "α-slow" approximation is valid when this soliton is heavy enough, i.e., when its mass m is much bigger than the spacetime derivatives of α (and we are not too close to the singularity $\alpha=0$). We see also that a topological charge $\text{sgn}(\alpha^2-\mu^2)$ can be associated with this soliton, which coincides with the result of our previous exact analysis.

III. INTERACTION BETWEEN TWO GRAVISOLITONS

Following the previous line of thinking, we may call the solutions (8) and (9) for the case $\mu=\mu_{\text{in}}$ a gravisoliton (S) and for the case $\mu=\mu_{\text{out}}$ an antigravisoliton (A). However, the real physical manifestation of the topological charge can be seen only in the collision process of two such objects. If the notion of topological charge was introduced in the correct way, the attractive forces in the system SA and the repulsion for the combinations SS or AA should be expected to appear. It is somewhat problematic to see this in a direct way, but we can use the same trick as in sine-Gordon theory. First of all we need to show that there exist three types of two-soliton solutions: the first one describing the SS-scattering state, a second for the SA-scattering process, and a third describing the oscillating-in-time bound state of two gravisolitons. The latter, if it exists, can be called the *gravibreather*, a term which precisely expresses the fact that this solution is the direct gravitational analogue of the sine-Gordon breather. If it turns out that the gravibreather can represent the SA-bound state only, i.e., if, for the combinations SS and AA, it would be impossible to have a solution of this kind, this would be a proof of the presence of an attraction between gravisolitons and antigravisolitons and a repulsion between gravisolitons of the same charge. If so, the real metric of correct signature for the gravibreather should follow from the SA-scattering solution by its analytic continuation to purely imaginary values of the relative collision velocity of the colliding gravisolitons. At the same time, the analogous analytic continuation of the SS- and AA-type solutions should lead to an unphysical (complex) metric tensor. Let us show that all this is really the case.

The two-solitonic solution for the metric (1) on the background (6) can be written as

$$g_{11}=\{1+D^{-1}[\cosh 2\tau+(\mu_1+\mu_2)(\mu_1-\mu_2)^{-1}\sinh 2\tau+\cosh 2\sigma-(\alpha^2+\mu_1\mu_2)(\alpha^2-\mu_1\mu_2)^{-1}\sinh 2\sigma]\}\alpha e^{u_0},$$

$$g_{22}=\{1+D^{-1}[\cosh 2\tau-(\mu_1+\mu_2)(\mu_1-\mu_2)^{-1}\sinh 2\tau+\cosh 2\sigma+(\alpha^2+\mu_1\mu_2)(\alpha^2-\mu_1\mu_2)^{-1}\sinh 2\sigma]\}\alpha e^{-u_0}, \quad (20)$$

$$g_{12}=2\alpha D^{-1}[(\mu_1+\mu_2)(\mu_1-\mu_2)^{-1}\sinh\sigma\sinh\tau+(\alpha^2+\mu_1\mu_2)(\alpha^2-\mu_1\mu_2)^{-1}\cosh\sigma\cosh\tau],$$

$$f=f_0\mu_1\mu_2 D(\alpha^2-\mu_1^2)^{-1}(\alpha^2-\mu_2^2)^{-1}, \quad (21)$$

where

$$D=4\mu_1\mu_2[\alpha^2(\alpha^2-\mu_1\mu_2)^{-2}\cosh^2\sigma+(\mu_1-\mu_2)^{-2}\sinh^2\tau]. \quad (22)$$

The pole trajectories μ_1,μ_2 follow from the quadratic equations

$$\mu_1^2+2(\beta-w_1)\mu_1+\alpha^2=0,$$
$$\mu_2^2+2(\beta-w_2)\mu_2+\alpha^2=0, \quad (23)$$

where w_1 and w_2 are arbitrary constants. The functions σ,τ can be found by quadrature from the following differential equations [7]:

$$\sigma_{,\zeta}=(\alpha^2-\mu_1\mu_2)(\alpha-\mu_1)^{-1}(\alpha-\mu_2)^{-1}u_{0,\zeta},$$
$$\sigma_{,\eta}=(\alpha^2-\mu_1\mu_2)(\alpha+\mu_1)^{-1}(\alpha+\mu_2)^{-1}u_{0,\eta}, \quad (24)$$

$$\tau_{,\zeta}=\alpha(\mu_1-\mu_2)(\alpha-\mu_1)^{-1}(\alpha-\mu_2)^{-1}u_{0,\zeta},$$
$$\tau_{,\eta}=-\alpha(\mu_1-\mu_2)(\alpha+\mu_1)^{-1}(\alpha+\mu_2)^{-1}u_{0,\eta}. \quad (25)$$

In Ref. [1] it was shown that μ_1 and μ_2 can be either both real or complex conjugate to each other. Because β can be replaced by $\beta+\text{const}$ without any physical consequences, we can set $w_1=w$, $w_2=-w$ in (23), which now gives

$$\mu_1^2+2(\beta-w)\mu_1+\alpha^2=0, \quad \mu_2^2+2(\beta+w)\mu_2+\alpha^2=0 \quad (26)$$

and for the real μ_1,μ_2 we should take real w, while the complex-conjugate μ pair corresponds to the purely imaginary values of w.

Let w be real and positive and the real solutions to (26) be chosen to be $\mu_1=(\mu_1)_{\text{in}}$ and $\mu_2=(\mu_2)_{\text{in}}$. Consider on the α,β plane the interior region of the triangle T with vertices ($\alpha=w$, $\beta=0$), ($\alpha=0$, $\beta=w$), and ($\alpha=0$, $\beta=-w$). In this region, as follows from (12) and (26), we have $\mu_1>0$ and $\mu_2<0$. Just inside this triangle solution (20) describes the head-on collision of the two gravisolitons. Indeed, because u_0 is spacelike from (24) and (25), it follows that σ is spacelike and τ is timelike. Let us look at the asymptotic form of the g matrix (20) in the T region for large absolute values of σ and τ in the case of slowly varying functions α, β, μ_1, and μ_2. It is not difficult to prove that, at an early "time" $\tau \ll -1$ and in the space region far enough to the "left" $\sigma \ll -1$ (but inside T), the asymptotic form of solution (20) coincides exactly with the functional form of the one-soliton solution (8) for the seed metric

$$g_0 = \mathrm{diag}[(\alpha\mu_1)^{-1}\alpha e^{u_0}, (\mu_1\alpha^{-1})\alpha e^{-u_0}]$$

and pole trajectory $\mu = \mu_2$. At $\tau \ll -1$, but far to the "right" $\sigma \gg 1$, (20) gives the one-soliton solution (8) which corresponds to the seed metric

$$g_0 = \mathrm{diag}[(\mu_2\alpha^{-1})\alpha e^{\mu_0}, (\alpha\mu_2^{-1})\alpha e^{-u_0}]$$

and pole $\mu = \alpha^2\mu_1^{-1}$. When τ increases, these two solitons are going to collide (the σ distance between them is going to decrease) and at $\tau \gg 1$ the state (20) decays into two free solitons again: in the region $\sigma \ll -1$ (20) gives (8) for the seed

$$g_0 = \mathrm{diag}[(\alpha\mu_2^{-1})\alpha e^{u_0}, (\mu_2\alpha^{-1})\alpha e^{-u_0}]$$

and pole $\mu = \alpha^2\mu_1^{-1}$ and in the region $\sigma \gg 1$, (20) coincides with (8) for

$$g_0 = \mathrm{diag}[(\mu_1\alpha^{-1})\alpha e^{u_0}, (\alpha\mu_1^{-1})\alpha e^{-u_0}]$$

and $\mu = \mu_2$. So, if α is sufficiently slowly varying, the picture is very clear: the two-soliton solution (20) before the collision and after it describes the pair of free gravisolitons on the background

$$g_0 = \mathrm{diag}[\alpha \exp(u_0 + \text{slow terms}), \alpha \exp(-u_0 - \text{slow terms})]$$

and "free poles" associated with them are $\alpha^2\mu_1^{-1}$ and μ_2 instead of μ_1 and μ_2. This means that the two-soliton solution (20) in the case $\mu_1 = (\mu_1)_{\text{in}}$ and $\mu_2 = (\mu_2)_{\text{in}}$ describes the collision process between a gravisoliton with pole $\mu_{\text{in}} = \mu_2$ and an antigravisoliton with pole $\mu_{\text{out}} = \alpha^2\mu_1^{-1}$, i.e., scattering in the SA system.

An analogous situation arises in the case where $\mu_1 = (\mu_1)_{\text{out}}$, $\mu_2 = (\mu_2)_{\text{out}}$. In this case we have scattering between a gravisoliton with the pole $\mu_{\text{in}} = \alpha^2\mu_1^{-1}$ and an antigravisoliton associated to the pole $\mu_{\text{out}} = \mu_2$.

A similar analysis shows that, in the case $\mu_1 = (\mu_1)_{\text{out}}$, $\mu_2 = (\mu_2)_{\text{in}}$ (or vice versa), solution (20) describes the scattering of two gravisolitons of the same charge, i.e., scattering in the SS or AA system. In this case, as follows from (24) and (25), σ is timelike and τ becomes spacelike. The asymptotics for large absolute values of σ, τ are the same as in the previous case, and at the initial ($\sigma \ll -1$) and final ($\sigma \gg 1$) stages of collision we have again a pair of free gravisolitons with "free poles" μ_2 and $\alpha^2\mu_1^{-1}$, but now both of them are in-poles [or out-poles for $\mu_1 = (\mu_1)_{\text{in}}$, $\mu_2 = (\mu_2)_{\text{out}}$].

There also exists the third kind of two-soliton solution (20). It corresponds to the case when μ_1 and μ_2 form a complex-conjugate pair. For the spacelike u_0, this solution becomes the gravitational analogue of the breather. In the case $\mu_2 = \bar{\mu}_1$ (the overbar means complex conjugation), the variable τ and quantity $\mu_1 - \mu_2$ become purely imaginary; however, the metrics (20) and (21) remain real with correct physical signature. After the substitution $\tau = i\tau'$, we get the real variable τ' which is timelike, as can be seen from (25), and two-soliton solutions (20) and (21) appear to be oscillating in time τ' (but not periodically). This solution corresponds to the purely imaginary values of the constant w in (26). Thus, the gravibreather can be considered as the analytic continuation in w of one of the two-soliton solutions with real μ_1 and μ_2. The main question now is, of which type, SS (AA) or SA?

It is a simple task to prove that the gravibreather emerges in this way from the gravisoliton-antigravisoliton state only. The analytic continuation we need comes from the general solution to Eqs. (26) for arbitrary complex values of the constant w. Let us define the function $F(s)$ by the equation $F^2 = s^2 - \alpha^2$, where s is complex valued and α is considered formally as a fixed real parameter. This function is analytic on the Riemann surface containing two sheets glued across each other at the cut between the points $s = \alpha$ and $-\alpha$. On the first sheet we have $F > 0$ for ($\mathrm{Im}s = 0$, $\mathrm{Re}s > \alpha$); $F < 0$ for ($\mathrm{Im}s = 0$, $\mathrm{Re}s < -\alpha$), and $\mathrm{Im}F > 0$ for ($\mathrm{Im}s > 0$, $\mathrm{Re}s = 0$); $\mathrm{Im}F < 0$ for ($\mathrm{Im}s < 0$, $\mathrm{Re}s = 0$). At the corresponding points of the second sheet, F has opposite signs. Using the function $F(s)$, we get two principally distinct pairs of the w analytical solutions to Eqs. (26):

$$\mu_1 = w - \beta + F(w - \beta), \quad \mu_2 = -w - \beta - F(w + \beta) \tag{27}$$

and

$$\mu_1 = w - \beta + F(w - \beta), \quad \mu_2 = -w - \beta + F(w + \beta) \,. \tag{28}$$

On the real w axis in the reality regions of μ_1 and μ_2, the choice (27) gives $\mu_1 = (\mu_1)_{\text{out}}$, $\mu_2 = (\mu_2)_{\text{out}}$ for the first w sheet and $\mu_1 = (\mu_1)_{\text{in}}$, $\mu_2 = (\mu_2)_{\text{in}}$ for the second. Contrary to this, in the same reality regions choice (28) corresponds to the "out-in" or "in-out" (depending on the sheet) pair μ_1 and μ_2.

Using the definition of F, it easy to prove that, on the imaginary w axis, this function has the property

$$\bar{F}(w + \beta) = -F(w - \beta) \,. \tag{29}$$

Consequently, (27) and only this choice give the complex-conjugate pair $\mu_2 = \bar{\mu}_1$ when w becomes purely imaginary. This means that the gravibreather follows from (20) by analytic continuation from the real to purely imaginary values of w only in those cases where, for the real values of w, the "interactive poles" μ_1 and μ_2 in (20) form an "in-in" or "out-out" pair, i.e., (as was already shown), just in those cases in which (20) represents the collision between a gravisoliton and an antigravisoliton.

Choice (28) corresponds to the collision of gravisolitons of the same charge. However, in these cases μ_1 and μ_2 for imaginary w cannot be complex conjugate to each other; i.e., we shall get in this w region an unphysical solution with complex-valued metric tensor.

The last thing we need to show is that the above w continuation is equivalent to the analytic continuation of solution (20) from the real to imaginary values of the relative collision velocity of the colliding gravisolitons. It has been shown before that the measure of the local solition velocity is $-2\alpha\mu(\alpha^2+\mu^2)^{-1}$. This expression is invariant under the interchange $\mu\rightarrow\alpha^2\mu^{-1}$, so it is the same for a gravisoliton and an antigravisoliton. Because of this property and the previous analysis, we conclude that solution (20) for any pair of real μ_1 and μ_2 describes the collision of two gravisolitons with the initial velocities

$$v_1=-2\alpha\mu_1(\alpha^2+\mu_1^2)^{-1}=\alpha(\beta-w)^{-1}$$

and

$$v_2=-2\alpha\mu_2(\alpha^2+\mu_2^2)^{-1}=\alpha(\beta+w)^{-1} .$$

Inside the triangle T, i.e., in the collision area, we have $v_1<0$, $v_2>0$. (The quantities α and β in these formulas should be referred to that symbolic point in the T interior where the world lines of the colliding solitons intersect.) The relativistic formula for the relative velocity gives

$$v_{\rm rel}=(v_2-v_1)(1+v_1v_2)^{-1}=2\alpha w(w^2-\alpha^2-\beta^2)^{-1} .$$

So, the purely imaginary values of w indeed correspond to the purely imaginary values of the relative velocity of the colliding gravisolitons.

Independently of the topological properties, the gravibreather can be interesting enough in its own right. Let us write this solution here in a more suitable form, simpler than general expressions (20) and (21). It is convenient to choose coordinates in such a way that $\alpha=q\sinh t\cosh z$ and $\beta=q\cosh t\sinh z$, where $q>0$ is an arbitrary constant. The spacelike solution u_0 to Eq. (7) we take as simple as possible, i.e., $u_0=2k\beta$, where $k=$const. Then the two-solitonic perturbation on the background (6) with the poles $\mu_1=q(\sinh z-i)(1-\cosh t)$, $\mu_2=q(\sinh z+i)(1-\cosh t)$ [they are solutions to Eqs. (26), in which $w=-iq$] gives the following exact solution to the vacuum Einstein equations:

$$\begin{aligned}-ds^2=&(q\sinh t\cosh z)^{-1/2}D\exp(k^2q^2\sinh^2 t\cosh^2 z)(-dt^2+dz^2)\\&+q\sinh t\cosh z D^{-1}\{[(\cosh t\cosh\sigma-\sinh\sigma)^2+(\sinh z\sin\tau-\cos\tau)^2]\exp(2kq\cosh t\sinh z)(dx^1)^2\\&\qquad+[(\cosh t\cosh\sigma+\sinh\sigma)^2+(\sinh z\sin\tau+\cos\tau)^2]\exp(-2kq\cosh t\sinh z)(dx^2)^2\\&\qquad+4(\cosh t\cosh\sigma\cos\tau-\sinh z\sinh\sigma\sin\tau)dx^1dx^2\} ,\end{aligned}\qquad(30)$$

where

$$\begin{aligned}&D=\sinh^2 t\cosh^2\sigma+\cosh^2 z\sin^2\tau ,\\&\sigma=2kq\sinh z+\sigma_0 ,\\&\tau=2kq(\cosh t-1)+q\tau_0 ,\end{aligned}\qquad(31)$$

and σ_0, τ_0 are arbitrary constants. [The substitution $t\rightarrow t/iw$, $z\rightarrow z/iw$, $q\rightarrow iw$ would replace (30) by the two-soliton solution related to the SA-scattering process inside the triangle T.]

The gravibreather (30) represents the inhomogeneous cosmological model which starts at the instant $t=0$ by the big bang with anisotropic Kasner-like asymptotics and, at $t\rightarrow+\infty$, approaches the background solution (6). In between, the model oscillates in time.

[1] V. Belinsky and V. Zakharov, Zh. Eksp. Teor. Fiz. **75**, 1953 (1978) [Sov. Phys. JETP **48**, 985 (1978)].

[2] Topological properties of the stationary axially symmetric gravisolitons (its construction procedure was given in Ref. [3]) should be treated from a different point of view with respect to the approach of this paper. Also, I shall not discuss here the nonstationary solutions with solitonic perturbations of tachyonic character.

[3] V. Belinsky and V. Zakharov, Zh. Eksp. Teor. Fiz. **77**, 3 (1979) [Sov. Phys. JETP **50**, 1 (1979)].

[4] The integration of (11) is equivalent to the integration of the equations for the background solution ψ_0 for the related linear spectral problem [1].

[5] The majority of the gravisolitonic solutions which can be found in the literature have been obtained for timelike u_0, which is related to tachyonic solitons in this sense.

[6] Within each of two causal domains, $\beta-w<-\alpha$ and $\beta-w>\alpha$, this assertion is evident, but there are no physical reasons why one cannot choose μ to be $\mu_{\rm in}$ in one of those domains and $\mu_{\rm out}$ in the other. A choice of this kind would have no influence on the physical results of this paper, but would violate some continuity properties of a purely mathematical nature, which were referred to earlier as "general continuity requirements." The best way to see it is to consider w in (10) as a complex-valued quantity and then to construct the phase diagram for the pole trajectories in the phase space (Reμ, Imμ) for some arbitrary fixed α, using β as a dynamical parameter and w as two real constants identifying each individual trajectory. With this diagram one can find the absolute separation (no con-

tacts even through the singular points) between the complex-generalized μ_{in} trajectories (for which $|\mu| < \alpha$) and μ_{out} curves (for $|\mu| > \alpha$). Passing continuously to the limit $\mathrm{Im} w \to 0$, one obtains from these two families the unique rule for choosing the solutions μ to Eq. (10) for real w. This rule demands that one keep the "in" (or "out") property for the trajectories "superglobally," i.e., even through the causally disconnected regions.

[7] With the notation $\sigma+\tau=\rho_1$ and $\sigma-\tau=\rho_2$, these equations read $\rho_{k,\zeta}=(\alpha+\mu_k)(\alpha-\mu_k)^{-1}u_{0,\zeta}$ and $\rho_{k,\eta}=(\alpha-\mu_k)(\alpha+\mu_k)^{-1}u_{0,\eta}$, where $k=1,2$. Thus, in view of (11), it would be more natural to use ρ_k instead of σ and τ. However, the latter variables are more convenient for the calculational purposes in what follows.