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Prospects for observing subhorizon preinflation fluctuations in the cosmic microwave background

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We consider constraints on preinflation subhorizon fluctuations in a chaotic inflationary model. The requirement that the energy density in the fluctuating field not exceed the Planck scale limits the amplitude of the fluctuations. The requirement that the preinflation fluctuations have a presently observable amplitude that is at least comparable to the quantum fluctuations generated during inflation constrains the length scale of the fluctuations. We show that in order for preinflation fluctuations to be observable above the background due to quantum fluctuations generated during inflation, they must have a scale at least comparable to the preinflation Hubble length. Unless $\Omega_0 > 0.94$, such fluctuations could be observable in the present microwave background with a wavelength less than the present observed horizon.

I. INTRODUCTION

The observed cosmic microwave background appears remarkably isotropic [1–3]. This constitutes a horizon problem which is usually resolved by invoking an early inflationary epoch [4]. Nevertheless, at some level, fluctuations in the background radiation should appear due, for example, to quantum fluctuation generated during the inflationary epoch [5–10]. It is also possible that fluctuations in the scalar field and radiation energy density prior to inflation could appear at some level in the present microwave background. For this to happen requires that the present value for the closure parameter Ω_0 is beginning to deviate from unity. In such a circumstance, the observed microwave background corresponds to the preinflation Hubble scale or larger and preexisting fluctuations would be observable. Indeed, the fact that fluctuations even with wavelengths far in excess of the present observed horizon would be observable today is one of the arguments [11,12] that the amount of inflation should have been sufficient to guarantee that the present closure parameter is very close to unity. Nevertheless, there are a number of observations [13] which indicate that the present closure parameter may be consistent with a value slightly less than unity ($\Omega_0 = 0.1-0.5$). This is known as the omega problem [14]. If the closure parameter is indeed less than unity, then it may be possible to observe relic fluctuations from the preinflationary epoch.

The purpose of this paper is to specify the constraints on the prospects for observing such fluctuations based upon the requirement that their amplitude be at least comparable to the quantum fluctuations generated during inflation and that the energy density in the fluctuations not exceed the Planck scale prior to inflation. We construct numeric and analytical models for the evolution of chaotic [15] inflation with plane-wave fluctuations. We find that even for a fluctuation with the maximum amplitude consistent with the energy constraint, observability above the inflation-generated fluctuation background requires a wavelength comparable to the initial Hubble scale. The present analysis is only applicable to fluctuations with a length scale less than the Hubble length before inflation. We will consider superhorizon fluctuations in a subsequent paper.

II. EVOLUTION OF FLUCTUATIONS

The Hubble distance scale H^{-1} is sometimes referred to as the "horizon." In the present context, however, this would be confusing. In a radiation-dominated flat ($\Omega = 1$) Friedmann model, the light travel distance,

$$r_l = R(t)\int_0^t \frac{dt'}{R(t')} , \tag{1}$$

is indeed equal to the Hubble distance, $r_l = H^{-1}$, but in general this is not true. The matter-dominated flat Friedmann model has $r_l = 2H^{-1}$, and in an open ($\Omega < 1$) radiation-(matter-)dominated Friedmann model,

$$\frac{r_l}{R} = (2)\text{arcosh}\frac{1}{\sqrt{\Omega}} , \tag{2}$$

whereas $(HR)^{-1} = \sqrt{1-\Omega}$. For example, for a matter-dominated $\Omega = 0.1$ model, we have $r_l = 3.8H^{-1}$. Thus, in such a universe, we could see well beyond the Hubble distance. We call this the "observed horizon." Hence we shall avoid the use of the term "horizon" when referring to the Hubble scale.

During inflation, $(RH)^{-1}$ decreases, making $(R_bH_b)^{-1} > (R_0H_0)^{-1}$ possible. This is equivalent to $\Omega_0 > \Omega_b$, since, in an open universe ($k = -1$),

$$\Omega = 1 - 1/R^2H^2 . \tag{3}$$

(We use subscripts i for the Planck time, b for the beginning of inflation, x for the time during inflation at which a particular fluctuation exits the Hubble scale, and 0 for the present time.)

However, simply to have the present closure parameter exceed the preinflation value is neither a sufficient nor a necessary condition to solve the horizon problem [4]. This would make the preinflation Hubble length longer than the present Hubble length, in comoving coordinates.

But as we have just noted, the Hubble length does not give the range of causal contact.

A necessary condition is that the comoving light travel distance before inflation is larger than the light travel distance between the time of recombination and today. This is not too difficult to satisfy [14] even with $\Omega_0=0.1$, since before inflation a typical inflationary model can have a curvature-dominated, $\Omega \simeq 0$, period, during which light travels many Hubble distances, $r_l \gg H^{-1}$.

However, this is not a sufficient condition, since we would like inflation to explain the homogeneity and isotropy of the present Universe in the largest observable scales from "natural" initial conditions. Causal contact does not necessarily lead to homogeneity and isotropy.

We consider Linde's chaotic inflation [15] with, e.g.,

$$V(\phi)=\frac{1}{n}\mu\phi^n , \tag{4}$$

and prescribe inhomogeneous initial data at the Planck time, defined as

$$H_i^2=m_{\rm Pl}^2 . \tag{5}$$

We restrict ourselves in the present work to scales shorter than the Hubble scale, which allows us to ignore the gravitational reaction to the inhomogeneities. Such perturbations in radiation density and curvature (gravity waves) decay as R^{-4}.

The energy density of the inflaton field is

$$\rho_\phi=\frac{1}{2}\dot{\phi}^2+\frac{1}{2R^2}\nabla\phi^2+V(\phi) . \tag{6}$$

$V(\phi)$ can typically be ignored at earliest times, but becomes dominant as inflation begins. The gradient term decays as R^{-4} and so does the inhomogeneity in $\dot{\phi}^2$ at first. Later, however, when $V(\phi)$ becomes important, it only decays as R^{-2}. Therefore, $\dot{\phi}^2$ becomes the dominant inhomogeneity. The quantity of importance is the density perturbation

$$\delta\rho/(\rho+p) , \tag{7}$$

at the time the perturbation exits the Hubble scale, because it will reenter the Hubble scale with the same amplitude after inflation [10].

To obtain this quantity we have solved numerically the coupled equations

$$H^2=\frac{8\pi}{3m_{\rm Pl}^2}(\rho_{\rm rad}+\langle\rho_\phi\rangle)+\frac{1}{R^2} , \tag{8}$$

$$\ddot{\phi}=\frac{1}{R^2}\nabla^2\phi-3H\dot{\phi}-V'(\phi)-\xi\mathcal{R}\phi , \tag{9}$$

where $R=R(t)$, $H=H(t)=\dot{R}/R$, $\phi=\phi(t,\mathbf{x})$, and $\rho_{\rm rad}=\rho_{{\rm rad},i}(R_i/R)^4$. $\mathbf{x}$ is a comoving coordinate. The term $\xi\mathcal{R}\phi$, where $\mathcal{R}$ is the Ricci scalar, is a possible coupling to gravity. Minimal coupling is given by $\xi=0$ and conformal coupling by $\xi=\frac{1}{6}$. For a Robertson-Walker metric, $\mathcal{R}=6R^{-2}(\ddot{R}R+\dot{R}^2+k)$.

The initial inhomogeneity is constrained by

$$\langle\rho_{\phi,i}\rangle\equiv f\Omega_i\frac{3m_{\rm Pl}^4}{8\pi} , \quad 0<\Omega_i<1 , \quad 0<f<1 , \tag{10}$$

where f is the fraction of the total-energy density in the inflaton field. This comes from our definition of the initial time [see Eqs. (5) and (8)] and means that the fluctuation energy may not exceed the Planck energy.

In Eq. (9) the effect of gravitational perturbations on the inflaton evolution is ignored. This is only marginally justified when the fluctuation length is comparable to the Hubble length, i.e., $H\lambda\sim 1$. Because of this we are applying Eq. (9) only when $H\lambda<1$. At an initial time t_i, $H_i\lambda_i\equiv l<1$. After that, $H\lambda$ decreases until inflation begins. During inflation, $H\lambda$ grows, until $H_x\lambda_x=1$ at t_x. Thus, in most cases, $H\lambda\ll 1$ for most of the time from t_i to t_x. However, for $l\sim 1$, $H\lambda$ is close to 1, not only just before t_x, but also just after t_i. [But note that at first the density perturbation, coming from $\dot{\phi}^2$ and $\nabla\phi^2$, has only a wavelength of $\lambda/2$, and so even for $l=1$, it is only half the Hubble length initially. Later, when $V(\phi)$ begins to dominate, the density perturbation has a wavelength of λ.] How much smaller $H\lambda$ is during the time between t_i and t_x, depends on Ω_i. During a curvature-dominated phase, $H\lambda$ stays constant. If $\Omega_i\ll 1$, the curvature term begins to dominate the expansion soon and $H\lambda$ does not become very much smaller than it was at t_i. Thus our results are approximate. They are the least accurate for $l\sim 1$, $\Omega_i\ll 1$, for which we probably underestimate the final amplitude of the fluctuations.

For simplicity, let us consider plane-wave inhomogeneities in the inflation field:

$$\phi(t,z)=\phi_i+\delta\phi_i\sin\frac{2\pi}{\lambda_i}(R_iz-t) . \tag{11}$$

The wavelength of the fluctuations,

$$\lambda_i=lH_i^{-1}=\frac{l}{m_{\rm Pl}}=l\sqrt{1-\Omega_i}R_i , \quad 0<l<1 , \tag{12}$$

is specified as initial data.

The constraint (10) therefore becomes

$$\frac{\delta\phi_i}{m_{\rm Pl}}=\left[\frac{3}{16\pi^3}\right]^{1/2}\sqrt{f\Omega_i}\,l . \tag{13}$$

The shorter the wavelength, the smaller the amplitude must be for the energy density not to exceed the Planck density. The maximum initial amplitude we consider is therefore $(3/16\pi^3)^{1/2}m_{\rm Pl}=0.078m_{\rm Pl}$, for fluctuations initially of Hubble length. (The wavelength of the density perturbation is then half of this.) Fluctuations beyond the Hubble scale can of course have larger amplitudes, but for them our present approach is not valid; indeed, it is suspect for $l\sim 1$.

We are especially interested in the case where the length scales of these fluctuations were not expanded by inflation to be many orders of magnitude larger than the present observable scales; i.e., we have just the minimal amount of inflation, corresponding to $\phi_i\sim 5m_{\rm Pl}$. The results from a set of runs with $V(\phi)=\frac{1}{4}\mu\phi^4$ are shown in Table I.

TABLE I. Computed $\delta\rho/(\rho+p)|_x$ from the numerical runs (with minimal coupling, $\xi=0$) for different values of l and Ω_i. The other parameters were $f=1$, $\mu=10^{-14}$, and $\phi_i=5.0m_{\rm Pl}$. The corresponding approximate results from the analytic formula [Eq. (24)] are shown in parentheses.

	l			
Ω_i	0.01	0.1	0.5	0.9
0.1	2.4×10^{-9}	2.6×10^{-7}	1.1×10^{-5}	2.1×10^{-4}
	(2.6×10^{-9})	(2.6×10^{-7})	(1.0×10^{-5})	(1.5×10^{-4})
0.5	5.3×10^{-9}	5.6×10^{-7}	2.0×10^{-5}	1.3×10^{-4}
	(5.8×10^{-9})	(5.9×10^{-7})	(1.8×10^{-5})	(1.0×10^{-4})
0.9	6.9×10^{-9}	7.3×10^{-7}	2.0×10^{-5}	7.7×10^{-4}
	(7.8×10^{-9})	(7.8×10^{-7})	(2.0×10^{-5})	(7.2×10^{-4})

III. ANALYTIC MODEL

As specified above, our problem has five parameters Ω_i, l, f, ϕ_i, and μ (for a given form of the potential and the coupling to gravity). It is not practical to span this space with numerical results, and so let us supplement them with an analytical formula for $\delta\rho/(\rho+p)|_x$. For this we need some additional approximations. Let us first find out how much the universe has expanded when the fluctuation exits the Hubble scale:

$$H_x\lambda_x=H_xR_xl\sqrt{1-\Omega_i}=l\left[\frac{1-\Omega_i}{1-\Omega_x}\right]^{1/2}=1\ . \tag{14}$$

The expansion law (8) has an analytical solution if we approximate

$$\frac{8\pi}{3m_{\rm Pl}^4}\rho\simeq a\left[\frac{R_i}{R}\right]^4+c\ , \tag{15}$$

with a and c constants; i.e., we assume that $V(\phi)$ does not change much from t_i to t_x, and the rest of ρ decays as R^{-4}. The latter assumption is violated when $\dot\phi$ experiences a nonzero gradient in the potential, $V'(\phi)$, and the behavior of $\frac{1}{2}\langle\dot\phi^2\rangle$, changes to R^{-2}. However, in most cases this term will be smaller than the curvature term in (8), and so the solution for R is not much affected. With the further assumption $c\ll a$ [also noting that $V(\phi)$ is not important initially], the result is

$$\left[\frac{R_x}{R_i}\right]^2=\frac{1-l^2(1-\Omega_i)}{cl^2}\ , \tag{16}$$

where

$$c=\frac{8\pi}{3m_{\rm Pl}^4}V(\phi_i)\ . \tag{17}$$

The inflaton field contains a uniform part $\phi(t)$ and a wave part with amplitude $\lambda\delta\phi$. The wave part decays as $\delta\phi\propto R^{-1}$ and $\delta\dot\phi=\delta\phi2\pi/\propto R^{-2}$. This statement is exact only for a conformally coupled free field in a flat (i.e., $k=0$ or $\Omega=1$) Robertson-Walker model [16]. However, we found that it also held well in our numerical calculations with $k=-1$, minimal coupling, and $V(\phi)$, over most of the period evolved. This happens because the amplitude of the fluctuation $\delta\phi$ becomes small, and thus the last two terms in Eq. (9) become almost homogeneous, so that they are important only for the uniform part $\phi(t)$.

Initially, $\dot\phi$ is dominated by the wave part, but as inflation gets going, the uniform part approaches the slow-rolling solution

$$\dot\phi=-\frac{V'(\phi)}{3H}\ . \tag{18}$$

Since ϕ changes slowly, this is essentially constant, and soon the slow-rolling term begins to dominate $\dot\phi$. The energy-density term $\frac{1}{2}\delta(\dot\phi^2)=\dot\phi\delta\dot\phi$ thus switches from an R^{-4} behavior to R^{-2}. The fluctuation is described by

$$\delta\rho=\frac{1}{2}\delta(\dot\phi^2)+\frac{1}{2R^2}\delta(\nabla\phi^2)+\delta\rho_{\rm rad}+\delta V \tag{19}$$

and

$$\rho+p=\langle\dot\phi^2\rangle+\frac{1}{3R^2}\langle\nabla\phi^2\rangle+\frac{4}{3}\rho_{\rm rad}\ . \tag{20}$$

We can then ignore terms that decay as R^{-4}, leaving

$$\left.\frac{\delta\rho}{\rho+p}\right|_x\simeq\frac{\frac{1}{2}\delta(\dot\phi^2)_x+\delta V_x}{\dot\phi_x^2}\ . \tag{21}$$

Since

$$\frac{1}{2}\delta(\dot\phi^2)_x=\dot\phi_x\delta\dot\phi_x=\frac{V'(\phi_x)}{3H_x}\delta\phi_x\frac{2\pi}{\lambda_x}=\frac{2\pi}{3}\delta V_x\ , \tag{22}$$

we have

$$\begin{aligned}\left.\frac{\delta\rho}{\rho+p}\right|_x&\simeq\left[1+\frac{3}{2\pi}\right]\frac{\delta\dot\phi_x}{\dot\phi_x}\\&=\left[1+\frac{3}{2\pi}\right]\frac{6\pi\delta\phi_x}{V'(\phi_x)\lambda_x^2}\\&=\left[1+\frac{3}{2\pi}\right]\frac{6\pi\delta\phi_i}{V'(\phi_i)\lambda_i^2}\left[\frac{R_i}{R_x}\right]^3\ .\end{aligned} \tag{23}$$

Inserting (12), (13), (16), and (17) into (23) gives our approximate analytic result

$$\left.\frac{\delta\rho}{\rho+p}\right|_x\simeq A\frac{\sqrt{f}\sqrt{\Omega_i}l^2}{[1-l^2(1-\Omega_i)]^{3/2}}\ , \tag{24}$$

where

$$A=\left[1+\frac{3}{2\pi}\right]8\pi\sqrt{2}\frac{V(\phi_i)^{3/2}}{V'(\phi_i)m_{\rm Pl}^3} . \tag{25}$$

Setting $f=1$ gives us the upper limit for a fluctuation whose initial wavelength was $l<1$ times the Hubble scale. For $V(\phi)=\frac{1}{4}\mu\phi^4$, with $\mu=10^{-14}$ and $\phi_i=5m_{\rm Pl}$, we have $A=8\times10^{-5}$.

IV. QUANTUM FLUCTUATIONS

The above discussion has been completely classical; i.e., we have ignored quantum fluctuations. We should now compare the amplitude of these preinflation fluctuations to the inflation-generated quantum fluctuations [8]:

$$\delta\phi\sim\frac{H}{2\pi} , \tag{26}$$

at t_x. For them

$$\left.\frac{\delta\rho}{\rho+p}\right|_x\simeq\frac{V'(\phi_x)\delta\phi}{\dot\phi_x^2}\simeq24\left[\frac{2\pi}{3}\right]^{1/2}\frac{V(\phi_i)^{3/2}}{V'(\phi_i)m_{\rm Pl}^3}\simeq A . \tag{27}$$

V. RESULTS

In Fig. 1 we compare the present amplitude from preinflation fluctuations to the amplitude from inflation-generated quantum fluctuations. Shown are contours of constant ratio of the former to the latter. These are plotted in the fluctuation length scale l versus preinflation closure parameter Ω_i plane. The point to note is that the preinflation fluctuations are not comparable to the quantum fluctuations unless the length scale becomes comparable to the Hubble scale $l\sim1$ for any initial curvature. Thus we see that $l\ll1$ preinflation fluctuations will be swamped by these quantum fluctuations. For $l\sim1$ our calculation indicates that both fluctuations could be of comparable magnitude. Since for these scales our calculation is likely to underestimate the preinflation fluctuations, they could actually be larger.

Finally we can relate Ω_0 to the present scale of any anisotropy in the observed microwave background. The fluctuation that is just now reentering the Hubble scale is

$$l=\left[\frac{1-\Omega_0}{1-\Omega_i}\right]^{1/2} , \tag{28}$$

i.e., $l<1$ for $\Omega_0>\Omega_i$. However, the largest observable scale, that of the cosmic microwave background, has the comoving scale $2r_l/R_0=4\,\text{arcosh}1/\sqrt{\Omega_0}$. Our fluctuation has the comoving scale $l\sqrt{1-\Omega_i}$. These are equal for a fluctuation with

$$l=\frac{4}{\sqrt{1-\Omega_i}}\text{arcosh}\frac{1}{\sqrt{\Omega_0}} . \tag{29}$$

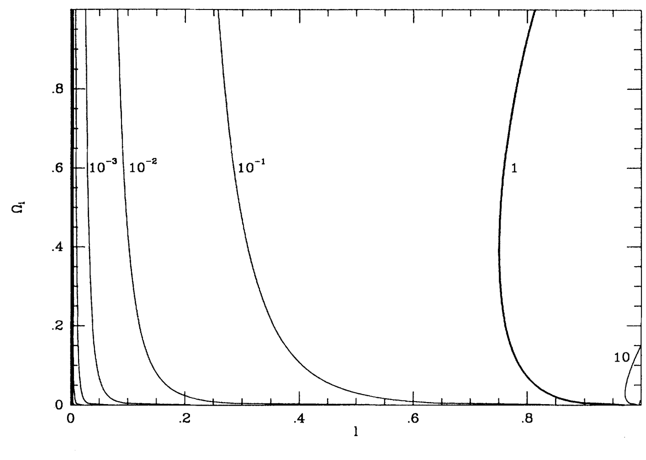

FIG. 1. Contour levels of the ratio of the maximum preinflation fluctuations to quantum fluctuations according to Eqs. (24) and (27). Ω_i is the closure parameter at the Planck time ($H=m_{\rm Pl}$), and l is the comoving length scale of the fluctuation in units of the Hubble length at the Planck time. When l is close to 1, our calculation underestimates the preinflation fluctuations.

For these fluctuations to be observable above the quantum fluctuations requires $l\gtrsim1$, which implies

$$\Omega_0\lesssim\frac{1}{\cosh^2(\sqrt{1-\Omega_i}/4)}\simeq0.94+0.06\Omega_i . \tag{30}$$

For $\Omega_0<0.94+0.06\Omega_i$ the present observable scale is larger than the Hubble scale at the Planck time and preinflation fluctuations with a wavelength less than the present observable scale could be seen. On the other hand, for $0.94+0.06\Omega_i<\Omega_0<1$, the present observable preinflation fluctuations would have wavelengths greater than the present observable scale.

VI. CONCLUSION

We have shown, for a chaotic inflationary universe with plane-wave fluctuations, that only those fluctuations with a length scale comparable to or greater than the Hubble scale before inflation could appear as fluctuations in the cosmic microwave background. We have also shown that, for $0.94+0.06\Omega_i<\Omega_0<1$, such fluctuations have a longer wavelength than the presently observable scale. Thus they could only be observable in the lower multipole moments of the anisotropy such as the quadrupole or even the dipole [12] moment.

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