PHYSICAL REVIEW D VOLUME 44, NUMBER 10 15 NOVEMBER 1991

Semilocal cosmic strings

Tanmay Vachaspati
Tufts Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155

Ana Achúcarro
Department of Mathematics, Tufts University, Medford, Massachusetts 02155

(Received 22 May 1991)

We show that the presence of global symmetries in a Lagrangian can lead to U(1) flux-tube solutions even when the vacuum manifold is simply connected. We provide a model in which these flux tubes, called "semilocal" strings, occur and explicitly prove the existence of stable string solutions by Bogomol'nyi's method. The formation of semilocal strings in the early Universe and the evolution of the network is discussed. We also show that semilocal magnetic monopoles cannot exist.

I. INTRODUCTION

Topological defects in field theories and cosmology have been the subject of intense investigation for the last two decades. These defects are solutions to the classical field equations and topological reasons guarantee the existence and stability of the solutions. In the early Universe, the defects would form during a phase transition in which the symmetry of the Universe breaks down to a smaller symmetry. A wide variety of defects is now known and the properties of some of them are very well documented [1].

The topological defects generally considered occur in field-theoretic models in which the Lagrangian is invariant under the transformations of a gauged symmetry group but in which no global symmetry is present. This gives rise to local (gauged) defects. The simplest example of this kind of model is the Abelian Higgs model originally studied by Nielsen and Olesen [2]. Global defects, arising from the breaking of a global symmetry, have received less attention in the literature. The primary reason for this is that grand unified theories are based on gauged symmetries and global symmetries usually have to be included by hand without any compelling motivation for their inclusion. This, however, does not preclude the existence of global symmetries in the Universe and such symmetries may well play a role in the grand unification scheme.

In this paper, we will study the formation of defects and in particular the formation of cosmic strings, when there are *both* global and local symmetries present in the Lagrangian. We find the result that some of these field theories admit stable string solutions with finite energy per unit length even when the vacuum manifold is simply connected. In other words, strings can form even if the hypersurface given by the minimum of the potential V is simply connected. Alternatively, if the symmetry group G of the Lagrangian breaks down to H, there can be string solutions even if the first homotopy group $\pi_1(G/H)$ is trivial. The string solutions we find are very similar to the ordinary U(1) local strings but they have additional novel features that have some resemblance to global defects. For this reason, we have decided to call these strings "semilocal."

In some ways, semilocal strings are similar to the "frustrated" cosmic strings discussed by Hill, Kagan, and Widrow [3]. A frustrated string may be thought of as consisting of two distinct strings laid on top of one another. In Ref. [3] it is argued that, even though the (composite) string solution is a stable solution to the field-theoretic equations, the string will not be able to form in a cosmological scenario and will be "frustrated." This frustration will come about because one of the two distinct strings first forms in one location and then the other forms in some other location but the stable string solution is only the one with both strings forming at the same location. In this way, even if the field theory admits string solutions, the formation of these strings in the early Universe is "frustrated."

A similarity between the semilocal and the frustrated string is in the ingredients that go in the Lagrangian. Both kinds of strings, at least in the simplest models, involve two complex scalar fields and only one gauge field. But it should be pointed out that the existence of string frustration depends crucially on the smallness of some of the coupling constants in the Lagrangian. No such assumption is required in the case of semilocal strings and the form of the potential is fully protected by the symmetries. The frustrated string exists because the vacuum manifold is not simply connected and for this reason these strings cannot terminate whereas the existence of the semilocal string is not purely topological and it can terminate in a "cloud of energy."

We expect the cosmological role of semilocal strings to be similar to the role of frustrated strings. Even though the field theory admits strings, a string network of the type found by numerical simulations [4] would probably not form during a phase transition in the early Universe. Semilocal strings can end in a cloud of energy and the region between distant strings will be filled with gradient energy. This might make the evolution of semilocal strings much more complicated than that of local U(1) strings.

In Sec. II we explicitly demonstrate a model in which semilocal strings form and prove their existence. We also discuss some of the properties of these solutions and consider their cosmological role. In Sec. III we show that the semilocal property can hold for strings but not for magnetic monopoles. We conclude in Sec. IV.

II. SEMILOCAL STRINGS

The model we consider is a direct generalization of the Abelian Higgs model. The only difference is that the complex scalar field is replaced by an SU(2) doublet $\Phi=(\phi,\psi)$. Then the action is

$$S=\int d^4x\left[\frac{1}{2}|(\partial_\mu-ieA_\mu)\Phi|^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{\lambda}{4}(\Phi^\dagger\Phi-\eta^2)^2\right]. \tag{2.1}$$

This action is invariant under $G=\mathrm{SU}(2)_g\times\mathrm{U}(1)_l$ transformations, where, the subscripts g and l stand for "global" and "local." Under $\mathrm{SU}(2)_g$, we have $\Phi\to\exp(i\alpha_a\tau^a)\Phi$ where α_a are constants and under $\mathrm{U}(1)_l$, $\Phi\to\exp[i\beta(x)I]\Phi$ where I is the 2×2 identity matrix. Once Φ acquires a vacuum expectation value (VEV), the symmetry breaks down to $H=\mathrm{U}(1)_g$ exactly as in the Weinberg-Salam model for the electroweak interactions. And as in the electroweak model, here too $\pi_1(G/H)=1$. However, if we were to only consider the gauge symmetries, we would need to look at $\pi_1(\mathrm{U}(1)_l)$ which is nontrivial, and indicates the presence of gauge strings.

It is useful to study the shape of the potential in Eq. (2.1). The potential is minimum when $\Phi^\dagger\Phi=\eta^2$. Since Φ is a complex doublet, the minimum of the potential is a three-sphere and is simply connected. This is in contrast with the situation in the Abelian Higgs model where the potential minimum is a circle and the string solution correspond to a solution which winds around the circle. In our case, there is no such circle if we only look at the minimum of the potential. The crucial observation, however, is that one must also consider the gradient energy in order to find a minimum energy solution. Then, if we pick a point on the three-sphere, the $\mathrm{U}(1)_l$ transformation generates a circle on the three-sphere. For every point on the three-sphere there is a corresponding circle. Since the points on the circle are connected by local symmetry transformations, there is no energy cost, neither potential nor kinetic, in moving along the circle. However, it costs gradient energy to go off the circle because the only way we can transform to a point off the circle is to use the *global* SU(2) transformation. So it may be better to think of the three-sphere as being composed of an infinite set of circles and that there is a string corresponding to a winding around each of those circles. (We prove the existence of these string solutions below.) The string solution around any one of these circles is a minimum of the energy for nontrivial (winding) boundary conditions. String solutions that involve transitions from one circle to the other, however, are not minimum energy solutions and can either unwind or relax to a minimum energy string that winds around a single circle.

While the above arguments suggest the presence of string solutions in the model in Eq. (2.1), they do not constitute a proof. The reason is that the vacuum manifold is simply connected and so a field configuration that winds at infinity may unwind without ever leaving the vacuum manifold. In other words, we know that the field vanishes at the center of the Nielsen-Olesen vortex, but there is no guarantee that it will vanish in our case. To check this, we must actually construct the string solution.

We now show that the model in Eq. (2.1) has stable string solutions by a simple generalization of Bogomol'nyi's proof for the Abelian Higgs model [5]. For this, we consider the energy per unit length of a static, stringlike configuration along the z axis in Minkowski space:

$$E=\int d^2x\left[\frac{1}{4}F_{mn}^2+\frac{1}{2}(D_m\phi_a)^2+\frac{1}{2}(D_m\psi_a)^2+\frac{\lambda}{4}(\phi_a\phi_a+\psi_a\psi_a-\eta^2)^2\right], \tag{2.2}$$

where ϕ_a and ψ_a, $a=1,2$ are the real and imaginary parts of ϕ and ψ, respectively, $D_m\phi_a=\nabla_m\phi_a+e\epsilon_{ab}A_m\phi_b (m=1,2)$ is the U(1)-covariant derivative (similarly for ψ_a), and $\epsilon_{12}=-\epsilon_{21}=1$. We start by rescaling the charge and VEV of the Higgs doublet to unity:

$$\phi_a=\eta Q_a,\quad \psi_a=\eta R_a,\quad A_m=\frac{\eta}{\sqrt{2}}v_m,\quad x^m=\frac{\sqrt{2}}{e\eta}y^m. \tag{2.3}$$

The energy per unit length becomes

$$E=\frac{\eta^2}{2}\int d^2y\left[\frac{1}{4}f_{mn}^2+(D_mQ_a)^2+(D_mR_a)^2+\frac{\beta}{2}(Q_aQ_a+R_aR_a-1)^2\right], \tag{2.4}$$

where $\beta=2\lambda/e^2$ and $f_{mn}=\partial_mv_n-\partial_nv_m$. This is easily shown to equal

$$E=\frac{\eta^2}{2}\int d^2y\left[\frac{1}{4}[f_{mn}-\epsilon_{mn}(1-Q_aQ_a-R_aR_a)]^2+\frac{1}{2}(\epsilon_{mn}D_nQ_a+\epsilon_{ab}D_mQ_b)^2+\frac{1}{2}(\epsilon_{mn}D_nR_a+\epsilon_{ab}D_mR_b)^2+\frac{\beta-1}{2}(Q_aQ_a+R_aR_a-1)^2+[\tfrac{1}{2}f_{mn}\epsilon_{mn}(1-Q_aQ_a-R_aR_a)-\epsilon_{mn}\epsilon_{ab}D_nQ_aD_mQ_b-\epsilon_{mn}\epsilon_{ab}D_nR_aD_mR_b]\right]. \tag{2.5}$$

The terms in square brackets in the last line are the divergence of the vector

$$S_p=\epsilon_{pm}(Q_a D_m Q_b \epsilon_{ab}+R_a D_m R_b \epsilon_{ab}+v_m) \tag{2.6}$$

and therefore the energy has a contribution from the boundary:

$$\int d^2y\ \nabla_p S_p=\int_{C_\infty} ds_p \epsilon_{pm}(Q_a D_m Q_b \epsilon_{ab}+R_a D_m R_b \epsilon_{ab}+v_m)\ , \tag{2.7}$$

where C_∞ is the circle at infinity. Assuming the boundary conditions

$$D_n Q_a \to 0 \quad \text{as } r\to\infty,\quad D_n R_a \to 0 \quad \text{as } r\to\infty\ , \tag{2.8}$$

we find that the boundary term is proportional to the circulation of the gauge field, which in turn has to be an integer multiple of 2π in order for Φ to be single valued on C_∞:

$$\int d^2y\ \nabla_p S_p=\int_{C_\infty} ds_p v_m \epsilon_{pm}=2\pi n\ . \tag{2.9}$$

(The integer n is called the winding number of the string.) Thus,

$$\begin{aligned}\frac{E}{\eta^2\pi}=n+\frac{1}{2\pi}\int d^2y \Bigg[&\frac{1}{4}[f_{mn}-\epsilon_{mn}(1-Q_aQ_a-R_aR_a)]^2\\ &+\frac{1}{2}(\epsilon_{mn}D_nQ_a+\epsilon_{ab}D_mQ_b)^2\\ &+\frac{1}{2}(\epsilon_{mn}D_nR_a+\epsilon_{ab}D_mR_b)^2\\ &+\frac{\beta-1}{2}(Q_aQ_a+R_aR_a-1)^2\Bigg]\ .\end{aligned} \tag{2.10}$$

Let us concentrate on the case $\beta=1$: the energy is minimized when

$$\begin{aligned}&f_{mn}-\epsilon_{mn}(1-Q_aQ_a-R_aR_a)=0\ ,\\ &\epsilon_{mn}D_nQ_a+\epsilon_{ab}D_mQ_b=0\ ,\\ &\epsilon_{mn}D_nR_a+\epsilon_{ab}D_mR_b=0\ .\end{aligned} \tag{2.11}$$

The ansatz

$$Q\to a_1(r)e^{i\theta_1},\quad R\to a_2(r)e^{i\theta_2} \quad \text{as } r\to\infty \tag{2.12}$$

is compatible with the above conditions provided that, as $r\to\infty, (a_1)^2+(a_2)^2=1$ and the phases θ_1 and θ_2 differ by a constant, c. The correlation of the phases is due to their coupling to the U(1) gauge field v_m since the condition that D_mQ_a goes to zero at the boundary implies

$$(\partial_m-iv_m)(a_1e^{i\theta_1})=0 \Longrightarrow v_m=\partial_m\theta_1 \tag{2.13}$$

and this, in turn, means that

$$\begin{aligned}(\partial_m-iv_m)(a_2e^{i\theta_2})&=(i\partial_m\theta_2-i\partial_m\theta_1)(a_2e^{i\theta_2})\\ &=0\Longrightarrow\partial_m(\theta_2-\theta_1)=0\end{aligned} \tag{2.14}$$

so $\theta_2=\theta_1+c$ as $r\to\infty$. Introducing

$$a^2=(a_1)^2+(a_2)^2 \quad \text{and} \quad v_\phi=\frac{n}{r}v(r),\quad v_r=0 \tag{2.15}$$

the Bogomol'nyi equations become

$$\frac{da}{dr}=\frac{n}{r}(1-v)a,\quad \frac{dv}{dr}=\frac{r}{n}(1-a^2) \tag{2.16}$$

with the boundary conditions $a\to 1, v\to 1$ as $r\to\infty$.

These equations were analyzed in Ref. [5] where it was found that the solutions are stable for any value of n. Notice that they are identical to the Nielsen-Olesen vortex equations, and therefore the field Φ must vanish at the center of the string. Furthermore, the string solutions can be labeled by the continuous parameters $a_1(r\to\infty)$ [or $a_2(r\to\infty)$] and c. In this sense, the model contains an infinite number of strings corresponding to the infinite number of U(1) circles on the three sphere.

The case with $\beta\neq 1$ can be analyzed in an identical manner. With the ansatz in Eq. (2.12), the equations are identical to the equations found by Bogomol'nyi for the Nielsen-Olesen string and hence we simply state his results: the strings with unit winding number are stable for all values of β and for $|n|\geq 2$ the strings are unstable if $\beta>1$. These results should apply to the semilocal string also.

This completes the proof of the existence and stability of string solutions in the model (2.1).

In the early Universe, semilocal strings would form if a suitable phase transition took place. During the phase transition Φ would get a VEV that was uncorrelated at long distances. This corresponds to a random selection of points on the three-sphere at every point in space. In traversing some large contour in space, it is possible that we will wind around a $\mathrm{U(1)}_l$ circle on the three-sphere. This means that there will be a semilocal string passing through the large contour. In general, it will also happen that in two distant regions of space, we will get two "different" semilocal strings (that is, with different values of a_1 and c). The different strings are not connected by any gauge transformation and so there is energy expense in the gradients of the parameters a_1 and c. This means that the region between the two different strings will be filled with gradient energy. The evolution of the string network will depend on the string tension and on the dynamics of this gradient energy. The gradient energy may also be thought of as providing a long-range interaction between different strings.

Next consider a straight isolated semilocal string. We have already seen that the vacuum manifold is simply connected and so the winding of the field configuration may disappear. This means that the semilocal string can end. However, the field configuration can only unwind by using the global transformations and we know that global transformations are costly in terms of gradient energy. Hence, when the string ends, it must end in a cloud of gradient energy. This energy expense is infinite for a single isolated string but may be finite when one has a whole network of strings.

We expect that the evolution of the semilocal string network will be quite different from the evolution of the Nielsen-Olesen string network. However, a definitive word on this matter can only be given after more extensive studies, some of which may have to involve numerical simulations.

III. SEMILOCAL MAGNETIC MONOPOLES—A NEGATIVE RESULT

In this section we will show that it is not possible to have semilocal monopoles within the context of grand unified theories.

In a general phase transition we can write the initial symmetry group G as $G_l \times G_g$ and the final symmetry group as $H_l \times H_g$. Let us denote the generators of G_l by T^a, and the generators of G_g by τ^i. We will assume minimal coupling and so the kinetic term of the scalar field depends on the covariant derivative, $D_\mu\Phi \equiv (\partial_\mu - iW^a_\mu T^a)\Phi$. Here, W^a_μ are the gauge fields corresponding to the local transformations generated by T^a. Then, the action is invariant under G_g only if

$$[T^a,\tau^i]=0 \ . \tag{3.1}$$

Magnetic monopoles exist in the field theory if $\pi_2(G_l/H_l)$ is nontrivial. However, we have

$$\pi_2(G_l/H_l)=\pi_1(H_l) \tag{3.2}$$

provided that $\pi_2(G_l)=\pi_1(G_l)=1$. (It is generally believed that the grand unified group should satisfy these conditions.) For $\pi_1(H_l)$ to be nontrivial, H_l must necessarily be nontrivial and contain at least one generator that annihilates the VEV of Φ ($=\Phi_0$) at a given point in space-time. Without any loss of generality we can call this generator T^1:

$$T^1\Phi_0=0 \ . \tag{3.3}$$

The magnetic monopole will be semilocal if the following holds.

(i) Given any constant field Φ and any constant local transformation $e^{i\beta^a T^a}$, there is a corresponding global transformation $e^{i\alpha^k\tau^k}$ which has the same effect on Φ:

$$e^{i\alpha^k\tau^k}\Phi=e^{i\beta^a T^a}\Phi \ . \tag{3.4}$$

The α^k will depend on the choice of Φ and β^a.

(ii) The vacuum manifold, that is, the hypersurface $V=0$, lies on an orbit of G_g; in other words, given any Φ_0 in $V=0$, any other point Φ in $V=0$ can be written as

$$\Phi=e^{i\alpha^k\tau^k}\Phi_0 \tag{3.5}$$

for some choice of α^k.

(iii) Given any local symmetry transformation, there is *at least one* Φ in $V=0$ which is not invariant under it.

The first condition is crucial because it will permit any magnetic-monopole configuration to be unwound using global transformations. As seen in the semilocal string case, this is the key feature of semilocal defects. A direct outcome of this condition is that the global symmetry must be larger than the local symmetry. The second condition is reasonable because any accidental degeneracy of the potential would tend to be lifted by quantum corrections, so we expect $V=0$ to lie on an orbit of the *full* symmetry group, but then condition (i) implies that G_g will be enough to cover it completely. The third condition means that the local symmetry transformations are not trivial on the entire vacuum manifold.

Now, from Eq. (3.1), it is clear that

$$0=[T^1,\tau^i]\Phi_0=T^1(\tau^i\Phi_0)=0 \quad \text{for all } i \ . \tag{3.6}$$

But this means that every Φ in $V=0$ is annihilated by T^1 which is contrary to condition (iii). So H_l is trivial and semilocal monopoles cannot exist.

As a corollary, this proof shows that the only topological defects that can have the semilocal property are those for which the relevant local symmetry breaks completely.

IV. CONCLUSIONS

We have shown that the formation of U(1) gauge strings can be complicated by the presence of global symmetries. It may be possible to have strings even if the vacuum manifold is simply connected, that is, $\pi_1(G/H)=1$. For this reason we propose that it is better to consider the homotopy group $\pi_1(G_l/H_l)$ where the subscript l refers to only the local (gauged) parts of the initial and final symmetries of the Lagrangian. If this group is nontrivial, then gauge strings will form. If, in addition, we find that $\pi_1(G/H)$ is trivial, then we may conclude that the strings in the theory are not genuine local strings but are of the semilocal variety.

In Sec. II, we described a model with $SU(2)_g \times U(1)_l$ symmetry—essentially the Weinberg-Salam model for the electroweak interactions with the SU(2) charge and gauge fields set equal to zero. For this model, we used Bogomol'nyi's method to explicitly prove the existence and stability of the semilocal string solution. The model can easily be generalized by using larger global symmetry groups.

The cosmology of semilocal strings might be very different from that of ordinary U(1) gauge strings. A peculiar feature of semilocal strings is that they can end in a cloud of energy and the space between strings may be filled with energy. For an isolated string, the energy expense in terminating a string is infinite. In the cosmological context, we do not have an isolated string and the energy expense is large but finite. The network of strings will evolve under the tension in the strings as well as the gradient energy in the global field.

Finally, we have considered the possibility of semilocal magnetic monopoles. Had this possibility been realized we may have had a natural solution to the monopole overabundance problem since there would be energy filling the region between monopoles and this could enhance the annihilation rate (as happens in the global monopole case [6]). However, under some reasonable conditions, we proved the negative result that semilocal monopoles cannot be formed.

Note added in proof. It has been brought to our attention by Mark Hindmarsh that our ansatz can be generalized. He then finds that semilocal string solutions with $n=1$ are stable for $\beta<1$, neutrally stable for $\beta=1$, and unstable for $\beta>1$.

ACKNOWLEDGMENTS

We would like to thank John Van Bon, Gary Gibbons, and Mark Hindmarsh for discussion. T.V. is grateful to the NSF for support.

[1] A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[2] H. B. Nielsen and P. Olesen, Nucl. Phys. **B61**, 45 (1973).
[3] C. T. Hill, A. L. Kagan, and L. M. Widrow, Phys. Rev. D **38**, 1100 (1988).
[4] T. Vachaspati and A. Vilenkin, Phys. Rev. D **30**, 2036 (1984).
[5] E. B. Bogomol'nyi, Yad. Fiz. **24**, 861 (1976) [Sov. J. Nucl. Phys. **24**, 449 (1976)].
[6] D. P. Bennett and S. H. Rhie, Phys. Rev. Lett. **65**, 1709 (1990).