Isospin-violating radiative decays of the η meson

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The branching ratio for $\eta \to \pi^+\pi^-\pi^0\gamma$ is calculated using an effective low-energy Lagrangian. The bremsstrahlung part of the amplitude as well as the direct emission term, which is significant, are taken into account. The predicted value for the $\eta \to \pi^+\pi^-\pi^0\gamma$ decay width is below present experimental limit. A comment on the possible $\eta' \to \pi^+\pi^-\pi^0\gamma$ decay is made.

I. INTRODUCTION

Effective Lagrangians were rather successfully applied to the description of low-energy hadronic interactions. They were particularly successful as a large-N limit of QCD, where the $1/N$ expansion was considered as a simple technique to describe the qualitative features of the meson strong interactions [1, 2]. The spontaneous breakdown of the chiral SU(3)$\times$SU(3) symmetry can also be simply proven [3] and the effective nonlinear chiral Lagrangian for the light pseudoscalars [4] is fully justified. Weak and electromagnetic interactions can be introduced simply by standard gauging of the basic strong Lagrangian [5].

In this Brief Report we determine the branching ratios for $\eta \to \pi^+\pi^-\pi^0\gamma$ and $\eta' \to \pi^+\pi^-\pi^0\gamma$ decays. Experimental data for these processes are missing and only an upper bound on $\eta \to \pi^+\pi^-\pi^0\gamma$ exists in the literature [6,7].

II. EFFECTIVE LAGRANGIAN

The chiral $U(3)_L \times U(3)_R$ nonlinear Lagrangian is given by

$$\mathcal{L}^{(0)} = \frac{f^2}{4}\mathrm{Tr}(\partial_\mu U\, \partial^\mu U^\dagger) , \tag{1}$$

where

$$U = \exp(i\pi/f), \quad \pi = \sum_{(a=0)}^{8} \lambda_a \pi^a .$$

In order to reproduce observed mass splitting among the Goldstone-boson-nonet members one introduces the following symmetry-breaking terms [8]:

$$\mathcal{L}_n = \frac{f^2}{4}\left[r\,\mathrm{Tr}[M(U+U^\dagger)] + \frac{\beta}{4N}[\mathrm{Tr}(\ln U - \ln U^\dagger)]^2 \right] . \tag{2}$$

M is the real diagonal quark mass matrix, such that the dimensional parameter r is given by

$$r = \frac{2m_K^2}{m_s+m} = \frac{m_\pi^2}{m} , \tag{3}$$

where in the isospin-symmetry limit $m_u = m_d = m$.

The second term in (2) is a result of the gluon anomaly and solves the $U(1)_A$ problem in the $1/N$ approach [9]. The parameter β is determined by the trace condition

$$\beta = m_{\eta'}^2 + m_\eta^2 - 2m_K^2 \cong 0.72\ \mathrm{GeV}^2 . \tag{4}$$

The η-η' mixing angle θ can be determined by diagonalizing the η_8-η_0 mass matrix:

$$\begin{aligned} \eta &= \eta_8\cos\theta - \eta_0\sin\theta , \\ \eta' &= \eta_8\sin\theta + \eta_0\cos\theta , \end{aligned} \tag{5}$$

and it can be estimated in the large-N limit [10] to be

$$\theta \cong -22^\circ , \tag{6}$$

in agreement with the present experimental limit [11].

The Lagrangian in (1) and (2) is the standard nonlinear realization of the model. It has been shown that the weak interactions of pseudoscalars need additional next-to-leading-order terms in the chiral expansion [12]. The corrections in the chiral Lagrangian are simply given by [13]

$$\delta\mathcal{L} = \left[-\frac{r}{\Lambda_0^2}\mathrm{Tr}(\partial^2 U + \mathrm{H.c.}) + \frac{r}{4\Lambda^2}(MU^\dagger M U^\dagger + \mathrm{H.c.}) \right] . \tag{7}$$

The nonderivative term contributes to a chiral correction for the pseudoscalar-meson masses,

$$M_{K,\pi}^2 = m_{K,\pi}^2 \left[1 + \frac{m_{K,\pi}^2}{\Lambda^2} \right] , \tag{8}$$

and it also affects the Gell-Mann–Okubo relation:

$$M_8^2 = \frac{1}{3}(4M_K^2 - M_\pi^2) + \frac{4}{3}\left[\frac{M_K^2 - M_\pi^2}{\Lambda} \right]^2 . \tag{9}$$

The η-η' mixing angle can be expressed in the following way:

$$\theta = \arcsin\left[\frac{M_8^2 - M_\eta^2}{M_{\eta'}^2 - M_\eta^2} \right]^{1/2} . \tag{10}$$

The value $\theta=-22°$ is in agreement with the scale $\Lambda \cong 1$ GeV. The Λ_0 parameter can be extracted from measured π, K decays: $\Lambda_0 \cong 1.02$ GeV.

It has been shown in Ref. [14] that $\eta \to \pi\pi\pi$ can be approached by means of the leading chiral Lagrangian. The chiral corrections to the amplitude are of the order $m^2_{\eta(\eta')}/\Lambda^2_{1,2}$ and our prediction should be taken with extreme caution. The decay width resulting from the leading term in chiral Lagrangian [13,14] is about a factor of 2 smaller than the experimental value, obtained by using "weighted average" $\eta \to 2\gamma$ (the Primakoff effect experiment included) [15]. However, the coefficients of the Dalitz plot are in rather good agreement with the experimental results. The calculated value of $\eta' \to \pi\pi\pi$ decay width is above the present experimental limit [13]. In order to avoid this uncertainty we normalize the $\eta \to \pi^+\pi^-\pi^0\gamma$ decay width to $\eta \to \pi^+\pi^-\pi^0$ decay.

The electromagnetic interaction is realized replacing the partial derivatives in the Lagrangian by the covariant ones:

$$\partial_\mu U \to D_\mu U = \partial_\mu U + ie A_\mu [Q, U] \ . \tag{11}$$

The relevant piece of the effective Lagrangian is then given by

$$\mathcal{L}^{\text{em}} = ie\frac{f^2}{4} A^\mu \text{Tr}(\partial_\mu U[Q, U^\dagger] + [Q, U]\partial_\mu U^\dagger) \ . \tag{12}$$

In our calculations we neglect the terms coming from higher-order terms in the strong chiral Lagrangian.

III. DECAY AMPLITUDES

Inner bremsstrahlung gives the dominant part of the amplitude and its contribution is presented in Fig. 1. Explicitly, for the decay $\eta(P) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\gamma(k)$ one finds

$$\begin{aligned} A_{\text{IB}}(\eta \to \pi^+\pi^-\pi^0\gamma) &= -\frac{er}{6\sqrt{3}f^2}(m_u - m_d)(\cos\theta - \sqrt{2}\sin\theta) \\ &\quad \times \left[\frac{\epsilon \cdot p_+}{k \cdot p_+} - \frac{\epsilon \cdot p_-}{k \cdot p_-} \right] \left\{ 1 + \frac{m^2_{\pi^0}}{m^2_\eta} - \frac{m^2_{\pi^0}}{m^2_\eta - m^2_{\pi^0}}[1 - (\cos\theta - \sqrt{2}\sin\theta)^2] \right. \\ &\quad \left. - \frac{m^2_{\pi^0}}{m^2_{\eta'} - m^2_{\pi^0}}(\sin\theta + \sqrt{2}\cos\theta)^2 + \frac{1}{m^2_\eta - m^2_{\pi^0}}(2m^2_\eta - 6P \cdot p_0) \right\} \\ &\equiv \theta\epsilon \cdot \left[\frac{p_+}{k \cdot p_+} - \frac{p_-}{k \cdot p_-} \right](a - bP \cdot p_0) \ . \end{aligned} \tag{13}$$

The direct-emission contribution is given in Fig. 2. The leading contribution (lowest order in k) is calculated neglecting higher-order terms in m^2_π:

$$\begin{aligned} A_{\text{DE}}(\eta \to \pi^+\pi^-\pi^0\gamma) &= -\frac{er}{6\sqrt{3}f^2}(m_d - m_u)(\epsilon \cdot p_+ - \epsilon \cdot p_-) \\ &\quad \times (\cos\theta - \sqrt{2}\sin\theta) \left\{ (\cos^2\theta - 2\sqrt{2}\sin\theta\cos\theta + 2\sin^2\theta)\frac{1}{m^2_\eta - m^2_{\pi^0}} \right. \\ &\quad \left. + \frac{1}{m^2_\eta - m^2_{\pi^0}} + \frac{1}{m^2_{\eta'} - m^2_{\pi^0}}(\sqrt{2}\cos 2\theta - \sin\theta\cos\theta) \right\} \\ &= \theta\epsilon \cdot (p_+ - p_-)c \ . \end{aligned} \tag{14}$$

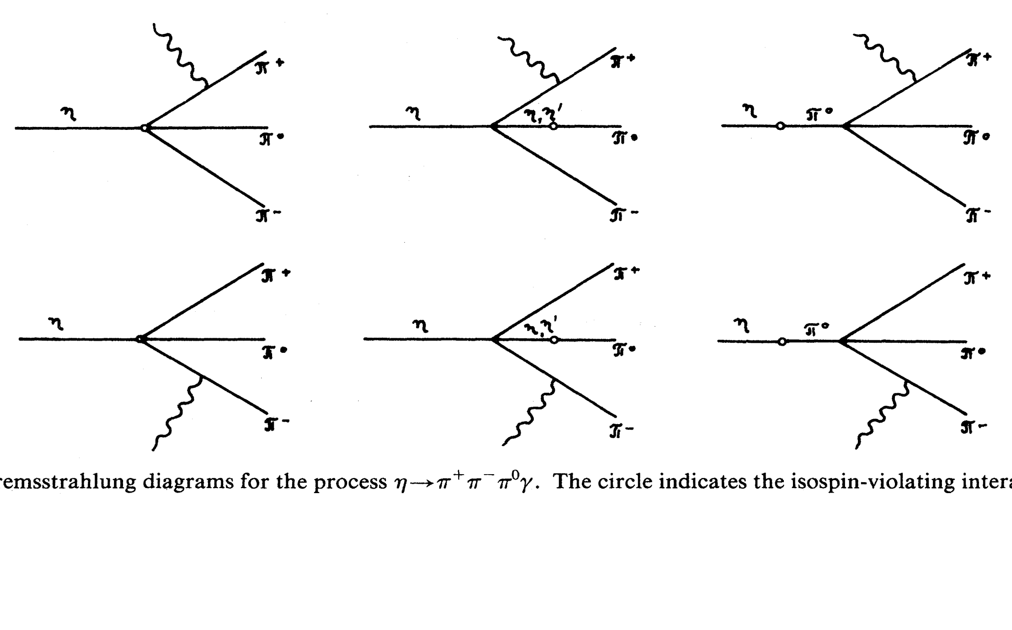

FIG. 1. Bremsstrahlung diagrams for the process $\eta \to \pi^+\pi^-\pi^0\gamma$. The circle indicates the isospin-violating interaction.

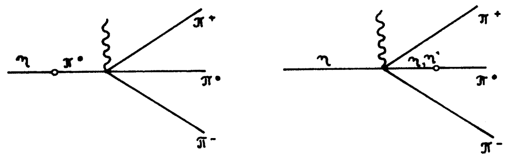

FIG. 2. Direct-emission diagrams for the process $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$. The circle indicates the isospin-violating interaction.

In the leading order of the chiral expansion the isospin-breaking mass term appearing in (13) and (14) is taken from Ref. [13]: $r(m_d - m_u) = 0.0106\ \mathrm{GeV}^2$. We note that since only pseudoscalar mesons are included both the bremsstrahlung and direct-emission terms are proportional to same factor $r(m_d - m_u)$, which is determined by the η-π mixing.

Vector and axial-vector mesons in the effective Lagrangian would lead to additional non-η-π-mixing contributions to the direct-emission amplitude, which have been neglected for this low-energy process $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$. The phase-space integration was performed numerically. Relevant formulas are given in the Appendix.

We investigate the importance of direct emission (DE), calculating separately total decay width and the width corresponding to the internal bremsstrahlung (IB) alone. In Table I the contributions of IB and DE are presented for subintervals of photon energy of 10 MeV each, centered around given values of k (branching ratios $\Delta\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0)$ are taken instead of widths in order to avoid the uncertainty coming from the calculation of $\eta \rightarrow \pi^+\pi^-\pi^0$ decay [13,14]).

This table verifies that direct emission becomes relatively more important as the photon energy increases, going as high as 60% at large k. But Γ_{total} is largest for small k, so overall effect of direct emission is about 13% (decrease, because interference is negative). The direct emission is not negligible even at small photon energies. We also analyze the $\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta \rightarrow \pi^-\pi^+\pi^0)$ dependence on the photon energy cut (k_{min}), taking three different values of mixing angle θ. These results are presented in Table II.

The $\eta' \rightarrow \pi^+\pi^-\pi^0\gamma$ decay rate can be found similarly as for $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay and we obtain, integrating over phase space,

$$R = \Gamma(\eta' \rightarrow \pi^+\pi^-\pi^0\gamma)/\Gamma(\eta' \rightarrow \pi^+\pi^-\pi^0)|_{k_{\mathrm{min}}=5\ \mathrm{MeV}}$$
$$= 1.89 \times 10^{-2} ,$$

or, taking $k_{\mathrm{min}} = 10$ MeV, $R = 1.38 \times 10^{-2}$. Unfortunately, there is no experimental evidence for this decay mode, since it is possible that vector-meson contributions could be significant in this process because of the large η' mass.

IV. SUMMARY

The $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ partial widths were calculated by introducing a gauge-invariant electromagnetic interaction into the effective Lagrangian describing $\eta \rightarrow \pi^+\pi^-\pi^0$. The direct emission was systematically included, in addition to the bremsstrahlung. The calculated values are below the present experimental upper bound. The direct emission is not negligible. We also predict the $\eta' \rightarrow \pi^+\pi^-\pi^0\gamma$ decay rate, as given by the same effective Lagrangian, but there are no data on this process yet.

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APPENDIX

In calculation of the $\eta \rightarrow \pi^+\pi^-\pi^0\gamma$ decay width the integration over two pions (π^+, π^-) momenta, as well as the angular variables of the third pion and photon, can be done analytically. For pions it is preferable to use the 3π center-of-mass system. For numerical integration there remains

$$\Gamma = c \int dk \int d\omega\, f(k,\omega) ,$$

where in k = photon energy (in the η rest system), $\omega = \pi^0$ energy (in the 3π c.m. system),

TABLE I. Branching fractions and relative direct-emission contribution at various photon energies.

k (MeV)	$\dfrac{\Delta\Gamma_{\mathrm{IB}}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)}{\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0)}$	$\dfrac{\Delta\Gamma_{\mathrm{tot}}(\eta \rightarrow \pi^+\pi^-\pi^0\gamma)}{\Gamma(\eta \rightarrow \pi^+\pi^-\pi^0)}$	$\dfrac{\Delta\Gamma_{\mathrm{IB}} - \Delta\Gamma_{\mathrm{tot}}}{\Delta\Gamma_{\mathrm{IB}}}$
10	2.86×10^{-3}	2.69×10^{-3}	6.5%
20	1.05×10^{-3}	9.08×10^{-4}	13.7%
30	5.33×10^{-4}	4.24×10^{-4}	20.3%
40	2.29×10^{-4}	2.17×10^{-4}	26.5%
50	1.66×10^{-4}	1.13×10^{-4}	32.3%
60	9.17×10^{-5}	5.71×10^{-5}	37.7%
70	4.76×10^{-5}	2.73×10^{-5}	42.8%
80	2.23×10^{-5}	1.17×10^{-5}	47.6%
90	8.63×10^{-6}	4.14×10^{-6}	52.1%
100	2.33×10^{-6}	1.02×10^{-6}	58.2%
110	2.69×10^{-7}	1.08×10^{-7}	59.8%

TABLE II. $R=\Gamma(\eta\to\pi^+\pi^-\pi^0\gamma)/(\eta\to\pi^+\pi^-\pi^0)$ dependence on photon energy cut, for several values of mixing angle θ.

$k_{\min}$ (MeV)	R ($\theta=-22°$)	R ($\theta=-15°$)	R ($\theta=-10°$)
5	4.46×10^{-3}	3.86×10^{-3}	3.38×10^{-3}
10	2.67×10^{-3}	2.32×10^{-3}	2.04×10^{-3}
15	1.76×10^{-3}	1.54×10^{-3}	1.36×10^{-3}
20	1.22×10^{-3}	1.06×10^{-3}	9.39×10^{-4}
25	8.56×10^{-4}	7.51×10^{-4}	6.63×10^{-4}
30	6.07×10^{-4}	5.34×10^{-4}	4.73×10^{-4}

$$c=\frac{\alpha}{(2\pi)^4 2m_\eta}\ ,$$

$$f(k,\omega)=\left[b^2qk+\frac{\varphi(k,\omega)^2}{Q^2k}q+2b\varphi(k,\omega)\operatorname{arctanh}\left[\frac{q}{E_k-\omega}\right]\right][(1+\chi^2)\operatorname{arctanh}(\chi)-\chi]$$

$$+2c\left[bq\omega+\varphi(k,\omega)\operatorname{arctanh}\left[\frac{q}{E_k-\omega}\right]\right](Q^2-4m^2)\operatorname{arctanh}(\chi)+\frac{c^2}{2}kq(Q^2-4m^2)\chi\ ,$$

$$\varphi(k,\omega)=a\frac{E_k}{m_\eta}b(m_\eta\omega+E_k\omega-2k\omega)\ ,$$

$$\chi=\left[\frac{Q^2-4m^2}{Q^2}\right]^{1/2},\quad Q^2=E_k^2-2E_k\omega+m_0^2\ ,$$

$$E_k=\sqrt{m_\eta(m_\eta-2k)},\quad q=\sqrt{\omega^2-m^2},\quad m=m_\pi\ ,$$

and a,b,c are relevant numerical factors from $A_{\rm IB}$ and $A_{\rm DE}$.

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