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t-expansion calculation of the SU(3) axial and tensor glueballs

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We use the t-expansion method to calculate the masses of the 1^{+-} and 2^{++} states in the pure glue sector of the SU(3) lattice gauge theory. The masses are calculated to order t^6 and the physical predictions are derived from D-Padé analysis of the ratios of the square of the masses to the string tension. Our results indicate that the axial state is considerably heavier than the scalar (by at least 50% or so) while the tensor lies in between, much closer to the scalar state.

I. INTRODUCTION

Glueball states are predicted by QCD. The calculation of their spectrum in the pure gauge sector of lattice QCD is of major importance. The main reason is that a reliable estimate of glueball masses may allow an interesting comparison of QCD predictions derived from first principles with the experimental candidates for glueball states such as the $f_0(991)$ or the $\eta(1440)$. No doubt, pure gauge results cannot be considered as final, since they completely ignore the interactions of gluons with quarks. The inclusion of dynamical fermions seems to leave the spectrum of baryons and mesons unchanged [1–4] apart from renormalizing the overall scale. The same may hold also for glueball calculations, as long as no strong mixing occurs with $q\bar{q}$ states. In any case, coping successfully with the pure gauge sector is an important milestone on the way to producing conclusive results for full QCD.

Of particular importance are the masses of the 0^{++} and the 2^{++} glueballs, which were expected to be the lowest pure glue excitations. Early analytic and numerical SU(3) computations on spatially small volumes [5,6] argued that the two levels have very similar masses with the 2^{++} slightly lower. It has been suspected that these results might be dominated by finite-size effects and only further numerical investigations on spatially larger lattices [7–10] verified that this indeed was the case, and that the tensor is about 50% heavier than the scalar glueball.

In the Hamiltonian formulation, the 0^{++}, 2^{++}, and 1^{+-} glueballs were calculated by Hamer [11]. This calculation was an extension of the pioneering work of Kogut, Sinclair, and Susskind [12] who computed the first four orders of the strong-coupling expansion. Hamer disagrees with the third and fourth order and extends the calculation to seventh order. He concludes that the mass ratio $M(2^{++})/M(0^{++})$ is strictly greater than one for all finite couplings. The values reported by him [11], $M(2^{++})/M(0^{++})\simeq 1.5$ and $M(1^{+-})/M(0^{++})\simeq 2$ around $g^2=1$ ($\beta=6$), are extracted from various approximants that continue the strong-coupling series to the weak-coupling domain.

A variational ansatz combined with Hamiltonian Monte Carlo methods has been used by Chin, Long, and Robson [13] to estimate the tensor-scalar mass ratio. The result follows a similar trend to that of Hamer [11] but the actual values are substantially lower.

We investigate the tensor and axial glueballs by using the t-expansion [14] method for the Hamiltonian formulation of lattice QCD. This method proved its usefulness in the investigation of other glueballs in the past, including the scalar glueball in SU(2) [15,16] and SU(3) [17] and various odd charge-conjugation states in SU(3) [18].

After a short formulation of the problem and the method, we present the explicit t expansion of all the quantities we calculate. The axial glueball has already been calculated [18] by this method. Our new results correct minor errors in the old calculation. The t-expansion method has been applied to the tensor glueball of SU(2) [19] but this is the first time it is carried out for the tensor glueball in SU(3). We use D-Padé fits as approximants with which we extract physical results from the series. Our results indicate that the axial glueball is much heavier than the scalar one and the tensor glueball lies in between. In the crossover region we find $1.06<M(2^{++})/M(0^{++})<1.3$ and $1.4<M(1^{+-})/M(0^{++})<1.8$.

II. SU(3) t-EXPANSION

The Kogut-Susskind Hamiltonian for the SU(3) pure gauge theory is defined as

$$H=\frac{g^2}{2}\left[\sum_l \mathbf{E}_l^2+x\sum_p(6-\mathrm{tr}U_p-\mathrm{tr}U_p^\dagger)\right] \tag{2.1}$$

where $\mathbf{E}_l$ is the electric-color field on the link l, $\mathrm{tr}U_p$ is the magnetic-color field of a plaquette p, derived from the product of four link operators U_l around the plaquette p; g is the coupling, and $x=2/g^4$. We will also use the variable $y=\sqrt{2x}=2/g^2$.

For calculational purposes it is useful to work with

$$\bar{H}=\sum_l \mathbf{E}_l^2-x\sum_p \mathrm{tr}(U_p+U_p^\dagger) \tag{2.2}$$

and to start the t-expansion procedure from the strong-coupling vacuum, which is the state obeying

$$\mathbf{E}_l|0\rangle=0\ .$$

This means that one calculates the energy function

$$E(t,g^2)=\frac{\langle 0|He^{-t\bar{H}}|0\rangle}{\langle 0|e^{-t\bar{H}}|0\rangle}\ , \tag{2.3}$$

which in the limit $t\to\infty$ turns into the correct vacuum energy. Expanding this energy function in powers of t one performs a cluster expansion whose coefficients are connected matrix elements of $\bar{H}$:

$$E(t,g^2)=\frac{g^2}{2}\sum_n \frac{(-t)^n}{n!}\langle \bar{H}^{n+1}\rangle^c+\frac{6}{g^2}N_p\ . \tag{2.4}$$

N_p is the number of plaquettes, which is taken to infinity.

We have recalculated the vacuum energy density to order H^9 and discovered a trivial error in the result published in the literature [20]. The error has almost no effect on the various fits. The vacuum energy density per plaquette is

$$\begin{aligned}\mathcal{E}=\frac{6}{g^2}+\frac{g^2}{2}[&-t2x^2+\tfrac{1}{2}t^2(-2x^3+\tfrac{32}{3}x^2)-\tfrac{1}{6}t^3(-\tfrac{64}{3}x^3+\tfrac{512}{9}x^2)+\tfrac{1}{24}t^4(10x^5-\tfrac{512}{3}x^3+\tfrac{8192}{27}x^2)\\&-\tfrac{1}{120}t^5(-\tfrac{650}{27}x^6+\tfrac{800}{3}x^5+\tfrac{1472}{9}x^4-\tfrac{32\,768}{27}x^3+\tfrac{131\,072}{81}x^2)\\&+\tfrac{1}{720}t^6(-\tfrac{1498}{9}x^7-\tfrac{72\,800}{81}x^6+3584x^5+\tfrac{178\,048}{27}x^4-\tfrac{655\,360}{81}x^3+\tfrac{2\,097\,152}{243}x^2)\\&-\tfrac{1}{5040}t^7(\tfrac{18\,634}{9}x^8-\tfrac{671\,104}{81}x^7-\tfrac{14\,328\,832}{729}x^6\\&\qquad+\tfrac{416\,768}{27}x^5+\tfrac{13\,204\,352}{81}x^4-\tfrac{4\,194\,304}{81}x^3+\tfrac{33\,554\,432}{729}x^2)\\&+\tfrac{1}{40\,320}t^8(-\tfrac{19\,352}{3}x^9+\tfrac{1\,192\,576}{9}x^8-\tfrac{17\,368\,064}{81}x^7-\tfrac{652\,354\,304}{2187}x^6\\&\qquad-\tfrac{51\,144\,192}{81}x^5+\tfrac{259\,201\,152}{81}x^4-\tfrac{234\,881\,024}{729}x^3+\tfrac{536\,870\,912}{2187}x^2)+O(t^9)]\ .\end{aligned} \tag{2.5}$$

This expression can be handled by using D-Padé approximants. These are nondiagonal Padé approximants, applied to the t derivative of the expression and integrated out to $t\to\infty$. We are then left with a complicated polynomial function of the coupling constant $y=2/g^2$. It is advantageous [15] to apply this method to $d\mathcal{E}/dy$. The results for $\mathcal{E}$ that are presented in Fig. 1 are obtained by integration over y. The latter is stopped at the point where an approximant turns negative, since $d\mathcal{E}/dy$ should be a positive-definite quantity. The curves plotted in Fig. 1 were those for which this range was the largest. They are consistent with one another for quite some range inside the weak-coupling domain. In the weak-coupling limit in which the lattice size is held fixed (to be distinguished from the continuum limit in which a vanishes) the SU(3) problem turns into a gauge theory of eight independent gauge fields on each link. Using the harmonic approximation one can obtain therefore the value of $\mathcal{E}(y\to\infty)=6.368$. This number should serve as an upper limit on $\mathcal{E}(y)$. Figure 1 is consistent with it.

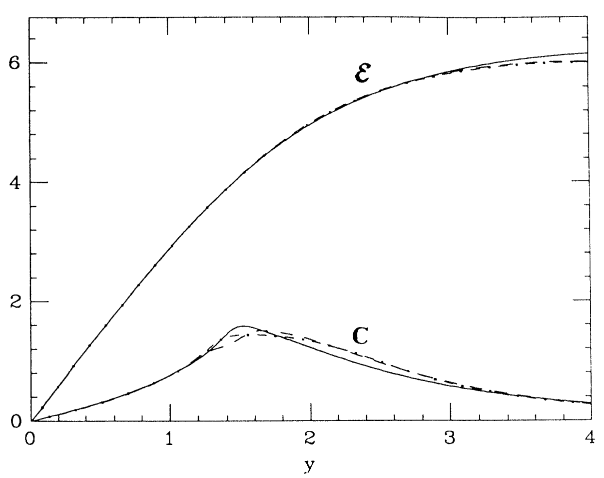

FIG. 1. Energy-density and specific-heat curves as derived from three D-Padé approximants which were applied to the y derivative of Eq. (2.5). The results of the approximants 0/6, 0/7, and 1/6 are represented by three different curves which overlap with one another for all practical purposes. The peak of the specific-heat curve specifies the crossover from strong to weak coupling.

On the same figure we plot also the specific-heat curves, defined by $C=-d^2\mathcal{E}/dy^2$. All curves peak between $y=1.5$ and 1.7. This is therefore identified with the crossover region between strong and weak coupling. Above these values of y we may expect our calculations of scaling ratios to approach the correct physical values.

From the energy-function one can deduce the t expansion for the scalar mass, since this state lies in the same sector of Hilbert space as the vacuum [15]:

$$M_S(t)=-\frac{g^2}{2}\frac{\partial}{\partial t}\ln\left[-\frac{\partial\mathcal{E}(t)}{\partial t}\right]\ . \tag{2.6}$$

Applying it to our energy function we obtain the following t expansion for the scalar glueball:

$$M_S(t)=\frac{g^2}{2}\Big[\tfrac{16}{3}-x+tx^2+\tfrac{1}{2}t^2 3x^3-\tfrac{1}{6}t^3(\tfrac{736}{9}x^2+\tfrac{80}{3}x^3+\tfrac{53}{27}x^4)+\tfrac{1}{24}t^4(\tfrac{10\,048}{9}x^2+\tfrac{608}{9}x^3+\tfrac{400}{81}x^4-\tfrac{1820}{27}x^5)$$
$$-\tfrac{1}{120}t^5(\tfrac{882\,112}{81}x^2-\tfrac{10\,880}{9}x^3-\tfrac{129\,728}{729}x^4-\tfrac{117\,440}{81}x^5+\tfrac{3643}{9}x^6)$$
$$+\tfrac{1}{720}t^6(\tfrac{831\,040}{9}x^2-\tfrac{1\,907\,392}{81}x^3+\tfrac{10\,447\,360}{2187}x^4-\tfrac{12\,991\,328}{729}x^5+\tfrac{1\,036\,000}{81}x^6+\tfrac{2284}{27}x^7)+O(t^7)\Big]\ . \quad (2.7)$$

III. AXIAL AND TENSOR GLUEBALLS

Nonscalar glueballs require separate calculations, starting from trial states which have the required quantum numbers. Thus the axial glueball was calculated [18] by using the one-plaquette strong-coupling state which corresponds to the T_1 representation of the cubic symmetry group [21]

$$|A\rangle=\sum(\mathrm{tr}U_{yz}-\mathrm{H.c.})|0\rangle \quad (3.1)$$

as the starting point of the calculation. It replaces the strong-coupling vacuum as the trial wave functional to which the t-expansion method is applied [18]. This leads to an estimate of the lowest energy in the axial sector of Hilbert space. By subtracting the vacuum energy one obtains the result for the axial mass. Similarly we solve the tensor problem by starting from a state defined in terms of the E representation:

$$|T\rangle=\sum(\mathrm{tr}U_{yz}-\mathrm{tr}U_{zx}+\mathrm{H.c.})|0\rangle\ , \quad (3.2)$$

where the summation extends over all the appropriate plaquettes of the lattice.

We have repeated the axial calculation which we will present here together with that of the tensor glueball. Both were calculated to order $H^7(t^6)$:

$$M_A(t)=\frac{g^2}{2}\Big[\tfrac{16}{3}+x+tx^2+\tfrac{1}{2}t^2(-\tfrac{8}{3}x^2+x^3)-\tfrac{1}{6}t^3(\tfrac{256}{9}x^2+\tfrac{16}{3}x^3+\tfrac{43}{27}x^4)$$
$$+\tfrac{1}{24}t^4(512x^2-\tfrac{256}{3}x^3+\tfrac{7760}{243}x^4-10x^5)-\tfrac{1}{120}t^5(\tfrac{146\,048}{27}x^2-2048x^3+\tfrac{360\,512}{729}x^4-\tfrac{19\,040}{81}x^5-\tfrac{415}{9}x^6)$$
$$+\tfrac{1}{720}t^6(\tfrac{11\,692\,672}{243}x^2-\tfrac{730\,240}{27}x^3+\tfrac{19\,757\,216}{2187}x^4-\tfrac{1\,946\,000}{729}x^5-\tfrac{40\,040}{27}x^6+\tfrac{25\,754}{27}x^7)+O(t^7)\Big]\ , \quad (3.3)$$

$$M_T(t)=\frac{g^2}{2}\Big[\tfrac{16}{3}-x+tx^2+\tfrac{1}{2}t^2 3x^3-\tfrac{1}{6}t^3(\tfrac{544}{9}x^2+\tfrac{80}{3}x^3-\tfrac{129}{81}x^4)+\tfrac{1}{24}t^4(\tfrac{7264}{9}x^2+\tfrac{992}{3}x^3-\tfrac{8080}{243}x^4-\tfrac{380}{27}x^5)$$
$$-\tfrac{1}{120}t^5(\tfrac{635\,872}{81}x^2-\tfrac{1088}{3}x^3-\tfrac{382\,880}{729}x^4-\tfrac{34\,240}{81}x^5+\tfrac{283}{9}x^6)$$
$$+\tfrac{1}{720}t^6(\tfrac{16\,303\,200}{243}x^2-\tfrac{1\,050\,208}{81}x^3+\tfrac{3\,999\,628}{2187}x^4-\tfrac{4\,615\,712}{729}x^5+\tfrac{136\,640}{81}x^6+\tfrac{9022}{27}x^7)+O(t^7)\Big]\ . \quad (3.4)$$

Comparing these expressions with the scalar mass (2.6) we see that all start out with the same value at $x=0$, because all these states were built out of single plaquettes. However, whereas the expansion of the tensor mass agrees with the scalar out to order t^2, the axial deviates from both already in the leading term. To see what this means for the physical masses we have to use appropriate extrapolations to the $t\to\infty$ limit. These will be discussed in the next section.

IV. MASS ESTIMATES

The method for estimating the glueball masses is based on calculating the scaling ratio

$$R=\frac{M(t)^2}{\sigma(t)}\ , \quad (4.1)$$

where σ is the string tension. The latter can be obtained by calculating the difference between the ground-state energies of the sector with a string of length L and the sector without any string. The tension $\sigma(t,g^2)$ is defined by dividing this difference by the length L of the string and taking the limit $L\to\infty$. Thus to calculate $\sigma(t,g^2)$ we compute

$$\sigma(t,g^2)=\lim_{L\to\infty}\frac{1}{L}\left[\frac{\langle 0|S^\dagger He^{-tH}S|0\rangle}{\langle 0|S^\dagger e^{-tH}S|0\rangle}-E(t,g^2)\right]\ , \quad (4.2)$$

where the operator S creates a straight infinite string along one axis: i.e.,

$$S=\prod_{l=(-L/2,0,0)}^{(L/2,0,0)}U_l\ . \quad (4.3)$$

The series of the string tension to order t^6 is

$$\sigma(t)=\frac{g^2}{2}\Big[\tfrac{4}{3}-\tfrac{1}{6}t^3\tfrac{64}{9}x^2+\tfrac{1}{24}t^4(\tfrac{928}{9}x^2-\tfrac{160}{9}x^3)-\tfrac{1}{120}t^5(\tfrac{3040}{3}x^2-\tfrac{9376}{27}x^3)$$
$$+\tfrac{1}{720}t^6(\tfrac{2\,044\,960}{243}x^2-\tfrac{348\,064}{81}x^3+\tfrac{3360}{9}x^4+\tfrac{2800}{9}x^5)+O(t^7)\Big]\ . \quad (4.4)$$

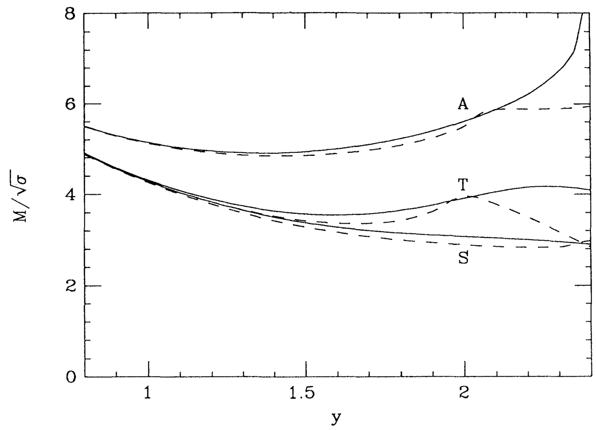

FIG. 2. Results of the 1/3 and 1/4 D-Padé approximants designated by solid and dashed lines respectively. The approximants were applied to the series of M^2/σ for the three different glueballs.

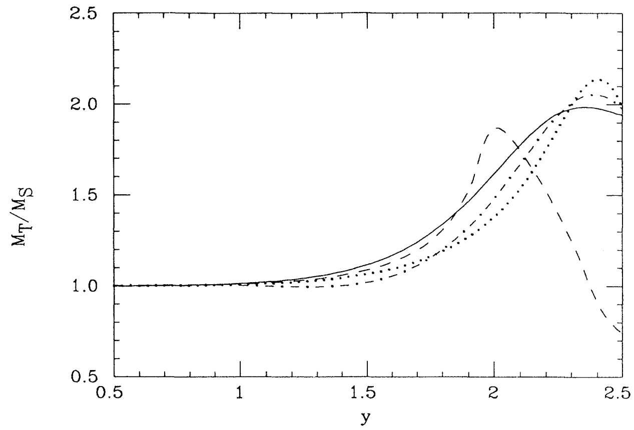

FIG. 3. Various estimates of the ratio M_T/M_S as derived from ratios of approximants of M^2/σ for two states. The full and dashed lines refer to results derived from 1/3 and 1/4 D-Padé approximants shown in Fig. 2. The dotted curve represents results of the diagonal 2/2 Padé fit and the dot-dashed curve follows from the 3/3 Padé.

Having obtained an algebraic series for both $\sigma(t)$ and $M(t)$ one is in a position to construct one also for R. This was used successfully both for the SU(2) theory [15] and the SU(3) one [17]. The numerical evaluation is based on applying D-Padé approximants to the physical ratios R.

Results for $\sqrt{R}$ vs y, as derived from the 1/3 and 1/4 D-Padé approximants for all three glueball states, are plotted in Fig. 2. We concentrate on these particular approximants because they are the ones that display stability in y. As expected, the tensor mass coincides with the scalar mass throughout the strong-coupling regime, becoming larger in the crossover region of $1.5<y<2$. The axial-mass is quite larger than the scalar one in the strong-coupling domain as well. Since the three different types of curves do not reach a flat minimum at the same value of y it is difficult to draw a precise prediction for the masses or their ratios. Within the crossover range M_S reaches a value of about $3\sqrt{\sigma}$. Using as the physical dimensional input $\sigma=(420\text{ MeV})^2$, as determined from slopes of Regge trajectories, we are led to a value of $M_S=1.3$ GeV. M_A reaches a shallow minimum at $y=1.5$ of about $5\sqrt{\sigma}$. Judging from the variability between $y=1.5$ and 2 we may estimate $1.4<M_A/M_S<1.8$. A similar "educated guess" for the tensor glueball leads to $1.06<M_T/M_S<1.3$. These variations reflect the qualitative conclusion that the 1^{+-} state should be considerably heavier than the 0^{++} with the 2^{++} lying in between, much closer to the scalar.

Finally we wish to point out that in the SU(3) tensor calculation one can make use also of diagonal Padé approximants. Using them in the same way as the nondiagonal ones, i.e., approximating first the M^2/σ ratios and then deriving from them the ratio of the masses, we obtain the results shown in Fig. 3. All show the same general trend of mild increase of M_T/M_S in the crossover region. The fact that the ratio increases within the crossover region does not allow us to draw a firm conclusion about its physical value. Thus, at $y=1.5$ the average of the four curves is $1.07\pm.04$ whereas at $y=2$ it reaches 1.6 ± 0.2. Basing our estimate on the ratios of the mass curves, rather than on their minima, we find the variation $1.07<M_T/M_S<1.6$ over the region $1.5<y<2$.

V. DISCUSSION

Our results for the axial glueball compare favorably with other evaluations of this quantity. The pioneering work of Kogut *et al.* [12], which was based on a Hamiltonian strong-coupling expansion, predicted a value of $M_A/M_S=1.58$ at $y\to\infty$. Using the same method Hamer [11] was led to a ratio of about 2. It is difficult to state what the margin of error of his calculation should be. From the 4-dimensional Lagrangian formulation we may quote the analytic results of Weisz and Ziman [5], which vary in the domain of 1.5 to 2, and the numerical simulations of Michael and Teper [7], which lead to a ratio of about 2.

The striking difference between the axial and tensor glueballs can be interpreted as an indication for the truth of this approach. The axial state, which is an odd charge-conjugation state constructed from one-plaquette operator in the strong-coupling regime, seems to rise to the level of other odd charge-conjugation glueballs, which can be constructed from two plaquettes in the strong-coupling limit [22]. In this way it indicates that states necessitating three gluons in the weak-coupling limit are heavier than two gluon states.

Our estimate for M_T/M_S is smaller than that of Hamer [11] but is higher than the results of the variational calculation of Chin, Long, and Robson [13]. They estimate the ratio to increase slowly with y, passing through 1.2 at $y=2.1$.

It is interesting to note that SU(2) QCD exhibits similar features; specifically, the SU(2) mass ratio of $M(2^{++})/M(0^{++})$ rises to a value of approximately 1.5

over the intermediate-volume regime. This observation is supported by various Monte Carlo computations [23,24] and analytic calculations [25,26] in the Lagrangian formulation. The SU(2) study in Ref. [24] that includes dynamical fermions reveals that the contributions of fermions can be absorbed into a shift of the coupling constant, leaving the mass ratio essentially unchanged. This is consistent with the observation in Ref. [2], which studied the SU(3) theory with rather heavy quarks.

The Hamiltonian calculations of SU(2) and SU(3) glueball masses accentuate the technical differences between the two theories. The series of the tensor glueball in the SU(2) calculation of Ref. [19] is very different from the SU(3) one presented here. The successful fitting procedures are also different. The *D*-Padé method worked for SU(2) but has shown some singular behavior. It was helpful therefore that one could also apply the exponential-fit method there, fitting the t expansion with a series of decreasing exponentials in t. Using the same method on the SU(3) series leads to rather poor results for the vacuum energy beyond the crossover region, and no reliable results can be extracted for the mass ratios. Nonetheless the *D*-Padé estimates of the tensor and scalar masses lead to similar physical conclusions in both theories.

ACKNOWLEDGMENTS

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[1] M. Fukugita, Y. Oyanagi, and A. Ukawa, Phys. Rev. D **36**, 824 (1987); Phys. Rev. Lett. **57**, 953 (1986).
[2] A. Gocksch, in *Field Theory on the Lattice*, Proceedings of the International Symposium, Seillac, France, 1987, edited by A. Billoire *et al.* [Nucl. Phys. B (Proc. Suppl.) **4**, 178 (1988)].
[3] A. Billoire and E. Marinary, Phys. Lett. B **184**, 381 (1987).
[4] M. Campostrini, K. J. M. Moriarty, J. Potvin, and C. Rebbi, Phys. Lett. B **193**, 78 (1987).
[5] P. Weisz and V. Ziemann, Nucl. Phys. **B284**, 157 (1986).
[6] B. Berg, A. H. Billoire, and C. Vohwinkel, Phys. Rev. Lett. **57**, 400 (1986).
[7] C. Michael and M. Teper, Phys. Lett. B **206**, 299 (1988); Nucl. Phys. **B314**, 347 (1989).
[8] C. Vohwinkel and B. A. Berg, Phys. Rev. D **40**, 584 (1989).
[9] M. Albanese *et al.*, Phys. Lett. B **192**, 163 (1987); **197**, 400 (1987); L. Fernandez and E. Marinary, Nucl. Phys. **B295**, 51 (1988); P. Bacilieri *et al.*, Phys. Lett. B **205**, 535 (1988).
[10] T. DeGrand, Phys. Rev. D **36**, 3522 (1987).
[11] C. Hamer, Phys. Lett. B **224**, 339 (1989).
[12] J. Kogut, D. K. Sinclair, and L. Susskind, Nucl. Phys. **B114**, 199 (1976).
[13] S. A. Chin, C. Long, and D. Robson, Phys. Rev. Lett. **60**, 1467 (1988).
[14] D. Horn and M. Weinstein, Phys. Rev. D **30**, 1256 (1984).
[15] D. Horn, M. Karliner, and M. Weinstein, Phys. Rev. D **31**, 2589 (1985).
[16] G. J. Mathews, N. J. Snyderman, and S. D. Bloom, Phys. Rev. D **36**, 2553 (1987).
[17] C. P. van den Doel and D. Horn, Phys. Rev. D **33**, 3011 (1986).
[18] C. P. van den Doel and D. Horn, Phys. Rev. D **35**, 2824 (1987).
[19] D. Horn and Y. Shadmi, in *Lattice '89*, Proceedings of the International Symposium, Capri, Italy, 1989, edited by R. Petronzio *et al.* [Nucl. Phys. B (Proc. Suppl.) **17**, 599 (1990)].
[20] C. P. van den Doel and R. Roskies, Phys. Rev. D **34**, 3165 (1986).
[21] R. C. Johnson, Phys. Lett. **114B**, 147 (1982).
[22] D. Horn, Int. J. Mod. Phys. A **4**, 2147 (1989).
[23] C. Michael and M. Teper, Nucl. Phys. **B305**, 459 (1988).
[24] J. Kripfganz and C. Michael, Nucl. Phys. **B314**, 25 (1989).
[25] C. Vohwinkel, Phys. Lett. B **213**, 54 (1988).
[26] P. van Baal, Nucl. Phys. **B307**, 274 (1988); CERN Report No. CERN-TH.5359/89 (unpublished).