Covariance of the gauge field in three-dimensional quantum electrodynamics

Carl M. Bender
Department of Physics, Washington University, St. Louis, Missouri 63130

Gerald V. Dunne
Department of Mathematics and Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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We consider a theory of electrodynamics in three-dimensional space-time in the presence of the topological (Chern-Simons) interaction term $(\kappa/2)\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$. It is appropriate to quantize electrodynamics in the radiation gauge because in that gauge the states in the Hilbert space have positive-definite norms. However, in that gauge the vector potential A^μ cannot transform as a vector under the Lorentz group. We show that the vector potential *does* transform covariantly as a representation of the Lorentz group, but the representation is infinite dimensional and, surprisingly, the specific representation depends on the value of the parameter κ.

I. INTRODUCTION

It was recognized first by Strocchi [1] that in quantum electrodynamics the field A^μ cannot transform as a Lorentz vector, as it does in classical electrodynamics. One can understand this as follows: One chooses to quantize electrodynamics in the radiation gauge to avoid the appearance of negative norm states in the Hilbert space. In this gauge the spatial part of the vector potential satisfies the condition of transversality:

$$\nabla\cdot\mathbf{A}=0\ . \tag{1.1}$$

Thus, the A field transforms in a peculiar way under the Lorentz group. To obtain the A field in a new frame of reference, we first transform A as if it were an ordinary Lorentz vector, but we must follow this transformation by retaining only the transverse parts of $\mathbf{A}$ and discarding the longitudinal part of $\mathbf{A}$. This transformation procedure ensures the transversality of the electric potential in all frames of reference and it also explains why this field is not a Lorentz vector.

The question of the covariance of the radiation gauge in free four-dimensional quantum electrodynamics has been investigated [2]. In Ref. [2] it was shown that in the radiation gauge the A field transforms as a pair of infinite-dimensional, irreducible representations of the Lorentz group (one representation for each helicity state). This study was followed by a number of others [3] and various additional research was conducted with regard to the transformation properties of interacting radiation gauge fields [4], infinite-dimensional stress tensors [5,6], and nonconserved charges [7].

The purpose of this paper is to examine the transformation law of the vector potential in $(2+1)$-dimensional electrodynamics. This theory is remarkable because, while it exhibits gauge invariance, the presence of a so-called topological (Chern-Simons) term in the Lagrangian gives a mass to the photon field. We find that the A field belongs to an infinite-dimensional representation of the three-dimensional Lorentz group SO(2,1) but that this representation is *not* irreducible. Rather it belongs to a remarkable class of representations known as *noncompletely reducible* representations. Furthermore, we find that the specific representation to which A belongs depends on the value of the mass parameter κ. Ordinarily, in a quantum field theory the fields transform according to well-defined representations of the Lorentz group and these representations are independent of the mass and coupling constant parameters that specify the theory. Here, we find that there are three regions in κ: (i) $\kappa=0$, (ii) $0<\kappa<\infty$, and (iii) $\kappa=\infty$. In each of these regions, A transforms according to a different representation of the group SO(2,1).

We have organized our presentation as follows. In Sec. II we give a brief and heuristic description of the representations of the Lorentz group [both for SO(2,1) and for the more familiar case of SO(3,1)] with special emphasis on noncompletely reducible representations. Then in Sec. III we review the properties of $(2+1)$-dimensional electrodynamics in the presence of a topological term. In the next two sections we determine the Lorentz transformation properties of the gauge field in $(2+1)$-dimensional quantum electrodynamics: Section IV presents the special case of the noninteracting theory (vanishing topological mass); we show that this theory resembles an integrable theory in that it has an infinite number of local conservation laws of the form $\partial_t U=\nabla\cdot\mathbf{V}$. Finally, in Sec. V we examine the general case of nonzero topological mass. This theory also has an infinite number of conservation laws, but these laws have a generalized form in which the time derivative is replaced by a more complicated differential operator.

II. NONCOMPLETELY REDUCIBLE REPRESENTATIONS

In the case of compact groups, once a representation has been block triangularized, it can then always be block diagonalized (decomposed into irreducible representations). The matrix that performs this block diagonalization can be expressed as an integral over the group space. This integral exists if the group is compact. However, for the case of noncompact groups this integral does not necessarily exist. Thus, noncompact groups can have *noncompletely reducible* representations.

A simple example of a noncompletely reducible representation can be given for the one-dimensional translation group. Consider the set of 2×2 matrices of the form

$$M(a)=\begin{bmatrix}1 & 0\\ a & 1\end{bmatrix} . \tag{2.1}$$

Clearly, these matrices form a two-dimensional triangular representation of the translation group because they satisfy the equation

$$M(a)M(b)=M(a+b) .$$

However, it is not possible to reduce this representation to two one-dimensional representations because the translation group is not compact.

Before examining the Lorentz group SO(2,1), which is relevant for our study of three-dimensional quantum electrodynamics, let us first consider the more familiar example of SO(3,1), the homogeneous Lorentz group in four-dimensional space-time. To characterize the irreducible representations of SO(3,1) it is first necessary to recall the irreducible representations of the rotation group SO(3). The irreducible representations of the rotation group are all finite dimensional and act on a base space of symmetric, traceless tensors of the form

$$Q^{[N]}_{a_1a_2\cdots a_N} , \tag{2.2}$$

where the indices a_i take the values 1, 2, or 3. [For purposes of brevity, from here on we will no longer distinguish between the *representations* of a group, which are matrices, and the *base space* upon which these matrices act, which is a tensor. For example, we will simply say that the tensor in (2.2) is a spin-N representation of the rotation group. Recall that this tensor has $2N+1$ independent components.]

We can now describe the irreducible representations of SO(3,1). Each irreducible representation Φ of SO(3,1) consists of a sequence of irreducible representations of the rotation group SO(3):

$$\Phi=\{Q^{[l_0]},Q^{[l_0+1]},Q^{[l_0+2]},\cdots\} . \tag{2.3}$$

In this sequence all the spins occur consecutively: each spin occurs once and only once; there are no degenerate spins or spins missing from the sequence. If the sequence of spins is finite the representation is said to be finite dimensional and if the sequence has no highest spin then the representation is termed infinite dimensional. In the notation [8] of Gel'fand, Minlos, and Shapiro l_0 is the lowest spin and, if the representation is finite dimensional l_1 is the highest spin plus one. The two numbers, (l_0,l_1) uniquely characterize all the irreducible representations of the Lorentz group. If the representation is infinite dimensional l_0 is still the lowest spin in the representation but l_1 is no longer an integer greater than l_0; rather, it is some other number, which can even be complex.

Here are some simple examples: A Lorentz scalar S consists only of spin 0. The lowest spin is 0 and the highest spin is also 0. This representation is labeled by the pair of numbers (0,1). A Lorentz vector V^μ contains two spin components, spin 0 and spin 1: $\{V^0,V^i\}$. Thus, the vector transforms as the (0,2) representation. A symmetric, traceless tensor $T^{\mu\nu}$ contains spins 0, 1, and 2: T^{00}, T^{0i}, $T^{ij}-\frac{1}{3}\delta^{ij}T^{00}$. Thus, it transforms as the (0,3) representation of SO(3,1). An antisymmetric tensor $F^{\mu\nu}$ transforms as the direct sum of two representations, each containing only spin 1: $F^{0j}\pm i\epsilon^{jkl}F^{kl}$. In general, if a representation is finite dimensional, it contains exactly $l_1^2-l_0^2$ independent components.

In this paper we will consider the effect of infinitesimal Lorentz transformations on a representation. Infinitesimal Lorentz transformations are performed using the generators of the Lorentz group. These are six generators of SO(3,1), three rotations J^{kl} and three Lorentz boosts J^{0k}. Given any single component of an irreducible representation of SO(3,1) it is possible to determine all of the others by performing infinitesimal transformations as follows. Under an infinitesimal rotation the components of a given spin representation of the rotation group mix among themselves; under an infinitesimal boost the different spin representations of the sequence in (2.3) mix among themselves. For example, under the boost J^{0k} the components of a vector V^μ mix as follows:

$$\begin{aligned} &J^{0k}{:}V^0\longrightarrow V^k ,\\ &J^{0k}{:}V^j\longrightarrow V^0\delta^{jk} . \end{aligned} \tag{2.4}$$

Thus, if we are given any component of the vector V^μ we can fill out the entire representation by performing a sequence of infinitesimal boosts. [The situation is the same for SO(2,1) except that there are then two boost generators and one rotation generator.]

The formula in (2.4) is a special case of a completely general formula that applies to *any* quantum field in four-dimensional space-time that transforms as an irreducible representation of the Lorentz group SO(3,1). Given a field $\Phi(x)$ that transforms as the (l_0,l_1) representation and whose spin content is displayed in (2.3), an infinitesimal Lorentz boost (obtained by commuting the field with the Lorentz boost operator) on any one of its spin components $Q^{[N]}_{a_1a_2a_3\cdots a_N}(x)$ gives [6]

$$\frac{1}{i}[Q^{[N]}_{a_1a_2\cdots a_N}(x),J^{0k}]=(x^k\partial^0-x^0\partial^k)Q^{[N]}_{a_1a_2\cdots a_N}(x)-\frac{(N+1-l_1)[(N+1)^2-l_0^2]}{(N+1)^2}Q^{[N+1]}_{a_1a_2\cdots a_Nk}(x)$$
$$+\frac{l_0l_1}{iN(N+1)}\sum_{i=1}^{N}\mathcal{E}_{a_ikq}Q^{[N]}_{a_1a_2\cdots\hat{a}_i\cdots a_Nq}(x)$$
$$+\frac{(N+l_1)}{(2N+1)}\left[\sum_{i=1}^{N}\delta_{a_ik}Q^{[N-1]}_{a_1a_2a_3\cdots\hat{a}_i\cdots a_N}(x)-\frac{1}{(2N-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\delta_{a_ia_j}Q^{[N-1]}_{a_1a_2\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_Nk}(x)\right], \tag{2.5}$$

where an index with a hat over it indicates that it is *absent* from the sequence of indices. The first term in (2.5) is the orbital contribution reflecting the functional dependence of the field on the spatial variable x_μ. Notice that a Lorentz boost J^{0k} on $Q^{[N]}$ gives the next higher spin component $Q^{[N+1]}$ and the next lower spin component $Q^{[N-1]}$ in the sequence in (2.3). Thus, repeated application of J^{0k} eventually fills out the entire sequence in (2.3).

Here is a simple example that indicates how repeated infinitesimal Lorentz boosts can be used to generate the full representation of the Lorentz group. Let x^μ be a null vector. [A null vector is one that lies in the light cone and whose components satisfy the constraint $(x^0)^2=(x^i)^2$.] Now we ask the question, how does $1/x^0$ transform under the group SO(3,1)? Clearly, $1/x^0$ is a rotational scalar so it must be the spin-0 component of some representation. The formula in (2.4) expresses the effect of an infinitesimal Lorentz boost J^{0k} on the components of the vector x^μ. Using this formula once gives

$$J^{0k}:\frac{1}{x^0}\longrightarrow-\frac{x^k}{(x^0)^2}\ . \tag{2.6}$$

This identifies the spin-1 component of the representation to which $1/x^0$ belongs. Next, we examine the effect of an infinitesimal Lorentz boost on this spin-1 component; using (2.4) we obtain

$$J^{0k}:\frac{x^j}{(x^0)^2}\longrightarrow\frac{\delta^{jk}}{x^0}-2\frac{x^jx^k}{(x^0)^3}\ . \tag{2.7}$$

The right-hand side of this formula contains both spin-0 and spin-2 representations of the rotation group SO(3). Recalling that irreducible representations of the rotation group must be totally symmetric and traceless, we separate and identify the spin content of the terms on the right-hand side of (2.7):

$$\frac{\delta^{jk}}{x^0}-2\frac{x^jx^k}{(x^0)^3}=\frac{1}{3}\frac{\delta^{jk}}{x^0}(\text{spin }0)$$
$$-2\left[\frac{x^jx^k}{(x^0)^3}-\frac{1}{3}\frac{\delta^{jk}}{x^0}\right](\text{spin }2)\ . \tag{2.8}$$

Note that the spin-0 part of (2.8) (apart from the *numerical* tensor δ^{jk}) is exactly the same as the spin-0 object we began with in (2.6). Thus, the spin-0 component of this representation is unique. We have also generated the spin-2 component. If we continue this process, we find that each spin component of the representation is unique (occurs once and only once): When we examine the effect of an infinitesimal transformation on the spin-N component, we recover the spin-$(N-1)$ component that we already have found and a new spin-$(N+1)$ component. Next, we compare (2.7) and (2.8) with the general formula in (2.5) (without the orbital part) and identify the representation: We have constructed the (0,0) irreducible representation of the Lorentz group SO(3,1). We can represent the above process of using repeated infinitesimal Lorentz boosts to generating all the components of the representation schematically as follows:

$$Q^{[0]}\leftrightarrow Q^{[1]}\leftrightarrow Q^{[2]}\leftrightarrow Q^{[3]}\cdots\ , \tag{2.9}$$

where each arrow represents an infinitesimal Lorentz boost.

Now let us reconsider the above example in *three*-dimensional space-time. In descending from $(3+1)$ dimensions to $(2+1)$ dimensions none of the formalism we have described so far needs to be changed. It is still true that an irreducible representation of SO(2,1) can be expressed as a sequence of irreducible representations of SO(2), each of which is a symmetric, traceless tensor of the form $Q^{[N]}_{a_1a_2a_3\cdots a_N}$. The only changes are that now all roman indices take the values 1 and 2 and we will call $Q^{[N]}$ a rank-N representation instead of a spin-N representation of the rotation group. Equation (2.4), which describes the effect of an infinitesimal Lorentz boost on a vector, remains unchanged, as do (2.6) and (2.7). However, an interesting change occurs in the interpretation of (2.7). The right-hand side is *already* symmetric and traceless. Hence, it is not necessary to add and subtract a term proportional to δ^{jk}. Apparently, an infinitesimal Lorentz boost of the rank-1 component of the representation produces a rank-2 component, but it does not reproduce the rank-0 component. We have a representation of the SO(2,1) Lorentz group which has one and only one component of rank 0, rank 1, rank 2, and so on. However, the representation is not irreducible because an infinitesimal Lorentz transformation will not reproduce the rank-0 component from the other components; this representation is block triangular and noncompletely reducible, like that in (2.1). To represent the effect of infinitesimal Lorentz boosts on the components of this representation we use a scheme similar to that in (2.9):

$$Q^{[0]} \longrightarrow Q^{[1]} \leftrightarrow Q^{[2]} \leftrightarrow Q^{[3]} \leftrightarrow \cdots . \quad (2.10)$$

Note that if we begin with $Q^{[N]}$, $N \neq 0$, repeated infinitesimal boosts fill out all of the components of the representation except for $Q^{[0]}$ because of the one-way arrow between $Q^{[0]}$ and $Q^{[1]}$. Thus, the representation whose rank-0 component is $1/x^0$ has a remarkable structure: It consists of an infinite-dimensional irreducible representation whose lowest-rank component is $N=1$ with a rank-0 component glued indecomposably to it [9]. We will see later on that the vector potential in the radiation gauge transforms as this representation when the gauge mass parameter κ is infinite.

III. ELECTRODYNAMICS IN 2+1 DIMENSIONS

Here is a brief review of electrodynamics in 2+1 dimensions [10]. The Lagrangian density describing this theory is conventionally written in the form

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\kappa}{2} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho , \quad (3.1)$$

where the field strength is given by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu . \quad (3.2)$$

The electric field is given by

$$E_i = F_{i0} = \partial_i A_0 - \partial_0 A_i \quad (3.3)$$

and the magnetic field is given by

$$B = F_{12} = \partial_1 A_2 - \partial_2 A_1 = \epsilon^{ij} \partial_i A_j , \quad (3.4)$$

where ϵ^{ij} is the antisymmetric symbol and $\epsilon^{12}=1$. Throughout this paper Greek indices $\mu,\nu \ldots$ denote space-time indices while Roman indices $i,j,\ldots$ denote spatial indices.

The theory described by the Lagrangian density $\mathcal{L}$ in (3.1) is the usual Maxwell theory of electrodynamics augmented by a so-called topological mass term (the Chern-Simons term) proportional to κ, a constant whose engineering dimensions are those of a mass. This theory is invariant under the local gauge transformation

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$

because under this gauge transformation

$$\mathcal{L} \longrightarrow \mathcal{L} + \text{total derivative} .$$

To obtain the field equations from (3.1) we vary with respect to the potential A^ν to obtain

$$\partial_\mu F^{\mu\nu} + \frac{\kappa}{2} \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0 . \quad (3.5)$$

The $\nu=0$ component of (3.5) gives

$$\nabla \cdot \mathbf{E} = \kappa B . \quad (3.6)$$

We now choose to work in the Coulomb gauge in which A^i is transverse:

$$A_i = A_i^T . \quad (3.7)$$

In this gauge

$$\nabla \cdot \mathbf{A} = 0 . \quad (3.8)$$

Thus, from (3.3) and (3.6) we have

$$\nabla^2 A_0 = \kappa^2 B . \quad (3.9)$$

The $\nu=i$ component of (3.5) gives

$$\Box A_i = \kappa^2 A_i , \quad (3.10)$$

where $\Box = \nabla^2 - \partial_0^2$. This equation is a massive Klein-Gordon equation, whose solutions evolve as free massive fields whose mass is $|\kappa|$.

The energy-momentum tensor $T^{\mu\nu}$ for this theory can be expressed in terms of the field-strength tensor in the usual way:

$$T^{\mu\nu} = F^{\mu\alpha} F^{\nu}{}_{\alpha} - \tfrac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} , \quad (3.11)$$

where the metric is defined by $g = \text{diag}[-1,1,1]$. Note that $T^{\mu\nu}$ has the same form as the conventional Maxwell energy-momentum tensor because the Chern-Simons term in the action is metric independent, and therefore does not contribute to the energy-momentum tensor. By virtue of the field equations in (3.5) the energy-momentum tensor obeys a local conservation law:

$$\partial_\mu T^{\mu\nu} = 0 .$$

In terms of the metric tensor we define the generators of translations

$$P^\mu = \int d^2x \; T^{0\mu}(x) \quad (3.12)$$

and the generators of the homogeneous Lorentz group SO(2,1)

$$J^{\mu\nu} = \int d^2x \, (x^\mu T^{0\nu} - x^\nu T^{0\mu}) . \quad (3.13)$$

In the quantum theory the canonically conjugate fields satisfy equal-time commutation relations

$$[A_i^T(x), E_j^T(y)] = i \left(\delta^{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta(x-y) . \quad (3.14)$$

Using the commutation relation in (3.14) we can verify that the operators $J^{\mu\nu}$ obey the Lie algebra of the SO(2,1) group. Indeed, one can verify that together the operators P^μ and $J^{\nu\rho}$ generate the full Poincaré group in 2+1 dimensions.

From the commutation relation in (3.14) and the formula for the generators of the Lorentz group in (3.11) and (3.13) we obtain the commutator of the gauge field A and J^{0k}. This commutator expresses the effect of an infinitesimal Lorentz transformation on the gauge field:

$$\frac{1}{i}[A_i(x), J^{0k}] = (x^k \partial^0 - x^0 \partial^k) A_i(x) + \frac{\partial_0 \partial_i}{\nabla^2} A_k - \frac{\kappa}{\nabla^2} \left(\delta^{ik} - \frac{\partial_i \partial_k}{\nabla^2} \right) B . \quad (3.15)$$

This is the formula that we will study for the remainder of this paper. We will iterate this formula to determine the Lorentz transformation properties of the gauge field.

IV. LORENTZ TRANSFORMATION LAW OF THE GAUGE FIELD: FREE-FIELD CASE

The problem addressed in this paper concerns the Lorentz transformation properties of the gauge field A. In this section we consider the special case for which $\kappa=0$. For this case the formula in (3.15) reduces to

$$\frac{1}{i}[A_i^{[1]}(x),J^{0k}]=(x^k\partial^0-x^0\partial^k)A_i^{[1]}(x)+\frac{\partial_0\partial_i}{\nabla^2}A_k^{[1]}\ , \tag{4.1}$$

where we take the rank-1 part of the representation $A_i^{[1]}$ to be just the spatial part of the gauge field:

$$A_i^{[1]}=A_i\ . \tag{4.2}$$

We now follow the pattern of the simple example considered in Sec. II after (2.6). We take the last term in (4.1) and decompose it into a rank-2 component and a rank-0 component:

$$\frac{\partial_0\partial_i}{\nabla^2}A_k^{[1]}=\tfrac{1}{2}A_{ik}^{[2]}+\tfrac{1}{2}\epsilon_{ik}A^{[0]}\ , \tag{4.3}$$

where the rank-2 component is

$$A_{ij}^{[2]}=\frac{\partial_0}{\nabla^2}(\partial_i A_k^{[1]}+\partial_k A_i^{[1]}) \tag{4.4}$$

and the rank-0 component is

$$A^{[0]}=\frac{\partial_0}{\nabla^2}B\ . \tag{4.5}$$

Note that this rank-0 component is *not* the same as A^0; in the Coulomb gauge (3.9) implies that $A^0=0$ when $\kappa=0$.

The commutator of $A^{[0]}$ with J^{0k} is very simple:

$$\frac{1}{i}[A^{[0]},J^{0k}]=(x^k\partial^0-x^0\partial^k)A^{[0]}\ . \tag{4.6}$$

This formula implies that $A^{[0]}$ is a Lorentz *scalar;* $A^{[0]}$ transforms *irreducibly* under the Lorentz group as a one-component representation.

Next we find the commutator of $A^{[2]}$ with J^{0k}:

$$\frac{1}{i}[A_{ij}^{[2]},J^{0k}]=(x^k\partial^0-x^0\partial^k)A_{ij}^{[2]}+A_{ijk}^{[3]}-\delta_{ik}A_j^{[1]}-\delta_{jk}A_i^{[1]}+\delta_{ij}A_k^{[1]}\ , \tag{4.7}$$

where $A^{[3]}$ is the rank-3 (symmetric and traceless) object

$$A_{ijk}^{[3]}=\frac{4}{3\nabla^2}(\partial_k\partial_i A_j^{[1]}+\partial_k\partial_j A_i^{[1]}+\partial_i\partial_j A_k^{[1]})-\tfrac{1}{3}(\delta_{ik}A_j^{[1]}+\delta_{jk}A_i^{[1]}+\delta_{ij}A_k^{[1]})\ . \tag{4.8}$$

We may continue this process and iteratively generate the rank-N component $A^{[N]}$ of the representation for $N=4,5,6,\ldots$. We find that, in general, under an infinitesimal Lorentz transformation this infinite-component gauge field

$$A=(A^{[0]},A^{[1]},A^{[2]},A^{[3]},\ldots) \tag{4.9}$$

transforms as

$$\frac{1}{i}[A_{a_1\cdots a_N}^{[N]},J^{0k}]=(x^k\partial^0-x^0\partial^k)A_{a_1\cdots a_N}^{[N]}+\frac{N}{2}A_{a_1\cdots a_N k}^{[N+1]}-\sum_{i=1}^{N}\delta_{ka_i}A_{a_1\cdots\hat{a}_i\cdots a_N}^{[N-1]}+\frac{1}{N-1}\sum_{\substack{i,j=1\\ i<j}}^{N}\delta_{a_ia_j}A_{a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_N k}^{[N-1]}. \tag{4.10}$$

We can represent the infinitesimal transformation formula in (4.10) pictorially as follows:

$$A^{[0]}\leftarrow A^{[1]}\leftrightarrow A^{[2]}\leftrightarrow A^{[3]}\leftrightarrow A^{[4]}\leftrightarrow\cdots\ . \tag{4.11}$$

This formula has a strong resemblance to that in (2.10) except that the one-way arrow between the $A^{[0]}$ and $A^{[1]}$ points in the opposite direction. The presence of this one-way arrow implies that the gauge field transforms as a *noncompletely reducible representation* of the SO(2,1) Lorentz group. This noncompletely reducible representation consists of an infinite-component irreducible representation (whose lowest rank is 1) glued indecomposably to a scalar.

The components $A^{[N]}$ of this infinite-dimensional representation are all fields satisfying the massless wave equation:

$$\Box A^{[N]}=0\ . \tag{4.12}$$

These tensors are *dependent* fields except for $A_i^{[1]}=A_i$, which is the usual transverse spatial part of the gauge field A^μ. Each of the components is connected by a simple system of differential equations:

$$\partial_k A_{a_1\cdots a_N k}^{[N+1]}=\partial_0 A_{a_1\cdots a_N}^{[N]},\quad (N>0) \tag{4.13}$$

and

$$\partial_k A_k^{[1]}=0\ . \tag{4.14}$$

Equation (4.13) is an infinite system of conservation laws that hold in all reference frames. Equation (4.14) states that in every frame of reference A_i is transverse. Equations (4.13) and (4.14) together should be thought of not as an infinite system of partial differential equations but rather as a *single covariant equation* expressing a local conservation law for an infinite-dimensional representation (in the same sense that we regard $\partial_\mu J^\mu=0$ or $\partial_\mu T^{\mu\nu}=0$ as a single tensor equation).

V. LORENTZ TRANSFORMATION LAW OF THE GAUGE FIELD: GENERAL CASE

In this section we repeat the calculation of the previous section for the case in which the Lagrangian (3.10) has a topological mass (Chern-Simons) term with $\kappa\neq 0$. We begin the analysis by reexpressing the terms on the right-hand side of the commutator in (3.15) so that they are symmetric and traceless tensors:

$$\frac{1}{i}[A_i^{[1]},J^{0k}]=(x^k\partial^0-x^0\partial^k)A_i^{[1]}+\tfrac{1}{2}A_{ik}^{[2]}+\tfrac{1}{2}\epsilon_{ik}A^{[0]}-\tfrac{1}{2}\delta_{ik}C^{[0]}\ , \tag{5.1}$$

where the tensors $A^{[0]}$, $C^{[0]}$, and $A^{[2]}$ are given by

$$A^{[0]}=\frac{\partial_0}{\nabla^2}B \ , \tag{5.2}$$

$$C^{[0]}=\frac{\kappa}{\nabla^2}B \ , \tag{5.3}$$

$$A_{ij}^{[2]}=\frac{\partial_0}{\nabla^2}(\partial_i A_j^{[1]}+\partial_j A_i^{[1]})+\frac{\kappa}{\nabla^2}\left[2\frac{\partial_i\partial_j}{\nabla^2}-\delta_{ij}\right]B \ . \tag{5.4}$$

Notice the surprising result that there are *two* rank-0 tensors in the infinitesimal Lorentz transformation law in (5.1); one of these tensors multiplies the numerical tensor δ_{ik} and the other multiplies ϵ_{ik}.

Next we examine the behavior of $A^{[2]}$ under an infinitesimal Lorentz transformation:

$$\frac{1}{i}[A_{ij}^{[2]},J^{0k}]=(x^k\partial^0-x^0\partial^k)A_{ij}^{[2]}+A_{ijk}^{[3]}+(\delta_{ij}A_k^{[1]}-\delta_{jk}A_i^{[1]}-\delta_{ik}A_j^{[1]})+(\delta_{ij}\epsilon_{kl}-\delta_{jk}\epsilon_{il}-\delta_{ik}\epsilon_{jl})C_l^{[1]} \ , \tag{5.5}$$

where

$$C_i^{[1]}=\frac{\kappa}{\nabla^2}\left[\partial_0 A_i^{[1]}+\frac{\kappa}{\nabla^2}\partial_i B\right] \tag{5.6}$$

and

$$A_{ijk}^{[3]}=\frac{4}{3}\left[\frac{1}{\nabla^2}-2\frac{\kappa^2}{\nabla^4}\right](\partial_i\partial_j A_k^{[1]}+\partial_j\partial_k A_i^{[1]}+\partial_k\partial_i A_j^{[1]})-\frac{1}{3}\left[1-2\frac{\kappa^2}{\nabla^2}\right](\delta_{ij}A_k^{[1]}+\delta_{jk}A_k^{[1]}+\delta_{ki}A_j^{[1]})+8\frac{\kappa\partial_i\partial_j\partial_k\partial_0}{\nabla^6}B-2\frac{\kappa}{\nabla^4}(\delta_{ij}\partial_k+\delta_{jk}\partial_i+\delta_{ki}\partial_j)\partial_0 B \ . \tag{5.7}$$

Notice that we have recovered the original one-index tensor $A^{[1]}$ but that we have discovered a *second* one-index tensor $C^{[1]}$.

Now we examine the transformation properties of the two zero-index tensors $A^{[0]}$ and $C^{[0]}$:

$$\frac{1}{i}[A^{[0]},J^{0k}]=(x^k\partial^0-x^0\partial^k)A^{[0]}+\epsilon_{kp}C_p^{[1]} \tag{5.8}$$

and

$$\frac{1}{i}[C^{[0]},J^{0k}]=(x^k\partial^0-x^0\partial^k)C^{[0]}+\tfrac{1}{2}C_k^{[1]} \ . \tag{5.9}$$

Note that the right-hand sides of (5.8) and (5.9) contain the one-index tensor $C^{[1]}$ that we have already identified; thus, we have an indication that the process of repeated commutation with J^{0k} is closing on itself. The presence of a tensorial term in addition to an orbital term on the right-hand side of (5.8) implies that $A^{[0]}$ is *not* a scalar. Recall that in the free-field ($\kappa=0$) case [see (4.6)] the zero-rank component $A^{[0]}$ *is* a scalar. Thus, the massive case $\kappa\neq 0$ is a significant change from the massless case.

Next, we determine the transformation properties of $C^{[1]}$:

$$\frac{1}{i}[C_i^{[1]},J^{0k}]=(x^k\partial^0-x^0\partial^k)C_i^{[1]}+C_{ik}^{[2]} \ , \tag{5.10}$$

where $C^{[2]}$ is a rank-two tensor of the form

$$C_{ij}^{[2]}=\left[\frac{\kappa}{\nabla^2}-2\frac{\kappa^3}{\nabla^4}\right](\partial_i A_j^{[1]}+\partial_j A_i^{[1]})+2\frac{\kappa^2}{\nabla^4}\left[2\frac{\partial_i\partial_j}{\nabla^2}-\delta_{ij}\right]\partial_0 B \ . \tag{5.11}$$

We now have found *two* rank-two tensors in the representation, $A^{[2]}$ and $C^{[2]}$.

Finally, let us look at the transformation properties of $C^{[2]}$:

$$\frac{1}{i}[C_{ij}^{[2]},J^{0k}]=(x^k\partial^0-x^0\partial^k)C_{ij}^{[2]}+\tfrac{3}{2}C_{ijk}^{[3]}+\tfrac{1}{2}(\delta_{ij}C_k^{[1]}-\delta_{jk}C_i^{[1]}-\delta_{ik}C_j^{[1]}) \ , \tag{5.12}$$

where

$$C_{ijk}^{[3]}=\frac{4}{3}\left[\frac{\kappa\partial_0}{\nabla^4}-4\frac{\kappa^2\partial_0}{\nabla^6}\right]\left[(\partial_i\partial_j A_k^{[1]}+\partial_j\partial_k A_i^{[1]}+\partial_k\partial_i A_j^{[1]})-\frac{\nabla^2}{4}(\delta_{ij}A_k^{[1]}+\delta_{jk}A_i^{[1]}+\delta_{ki}A_j^{[1]})\right]+4\left[3\frac{\kappa^2}{\nabla^6}-4\frac{\kappa^4}{\nabla^8}\right]\left[\partial_i\partial_j\partial_k B-\frac{\nabla^2}{4}(\delta_{ij}\partial_k+\delta_{jk}\partial_i+\delta_{ki}\partial_j)B\right] \ . \tag{5.13}$$

Apparently, there are *two* rank-three components in the representation, $A^{[3]}$ and $C^{[3]}$.

By iterating this process of repeated commutation we can identify the representation to which the vector potential in the Coulomb gauge belongs. This representation has a rather unusual structure in that it is *double valued:* There are two rank-zero components $A^{[0]}$ and $C^{[0]}$, two rank-one components, $A^{[1]}$ and $C^{[1]}$, two rank-two components, $A^{[2]}$ and $C^{[2]}$, and so on. Here are the general formulas for the transformation properties of this double-valued representation:

$$\frac{1}{i}[A^{[N]}_{a_1\cdots a_N},J^{0k}]=(x^k\partial^0-x^0\partial^k)A^{[N]}_{a_1\cdots a_N}+\frac{N}{2}A^{[N+1]}_{a_1\cdots a_N k}-\sum_{i=1}^{N}\delta_{ka_i}\left[A^{[N-1]}_{a_1\cdots\hat{a}_i\cdots a_N}+\frac{2}{N}\tilde{C}^{[N-1]}_{a_1\cdots\hat{a}_i\cdots a_N}\right]$$
$$+\frac{1}{N-1}\sum_{\substack{i,j=1\\ i<j}}^{N}\delta_{a_ia_j}\left[A^{[N-1]}_{a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_N k}+\frac{2}{N}\tilde{C}^{[N-1]}_{a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_N k}\right], \tag{5.14a}$$

where $\tilde{C}^{[N]}_{a_1\cdots a_N}\equiv\epsilon_{a_1l}C^{[N]}_{la_2\cdots a_N}$, and

$$\frac{1}{i}[C^{[N]}_{a_1\cdots a_N},J^{0k}]=(x^k\partial^0-x^0\partial^k)C^{[N]}_{a_1\cdots a_N}+\frac{N+1}{2}C^{[N+1]}_{a_1\cdots a_N k}$$
$$-\left[\frac{N-1}{N}\right]\sum_{i=1}^{N}\delta_{ka_i}C^{[N-1]}_{a_1\cdots\hat{a}_i\cdots a_N}+\frac{1}{N}\sum_{\substack{i,j=1\\ i<j}}^{N}\delta_{a_ia_j}C^{[N-1]}_{a_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_N k}\ . \tag{5.14b}$$

Figure 1 displays these infinitesimal transformation laws schematically. Figure 1 is a generalization of the diagram in (4.11), which describes the special case $\kappa=0$. Observe from the position of the one-way arrows that we have found an infinite-dimensional *irreducible* representation consisting of the components $C^{[1]},C^{[2]}$, $C^{[3]},C^{[4]},\ldots$. The remaining tensor components are glued indecomposably to this irreducible representation, thereby forming the peculiar double-valued indecomposable representation shown in Fig. 1.

The fields $A^{[N]}$ and $C^{[N]}$ satisfy field equations. We discuss these equations below. First, each component of the representation satisfies a massive Klein-Gordon wave equation:

$$\Box A^{[N]}_{a_1\cdots a_N}=\kappa^2A^{[N]}_{a_1\cdots a_N} \tag{5.15a}$$

and

$$\Box C^{[N]}_{a_1\cdots a_N}=\kappa^2C^{[N]}_{a_1\cdots a_N}\ . \tag{5.15b}$$

Equations (5.15) are the analogs of (4.12) for the case of nonzero κ.

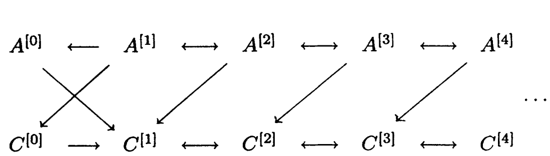

FIG. 1. A map of the infinite-dimensional representation of the Lorentz group to which the gauge field in (2+1)-dimensional quantum electrodynamics belongs. The arrows indicate how the components of this representation behave under an infinitesimal Lorentz transformation. The precise transformation properties of this representation are described in (5.14). From this figure one can see that in the Coulomb gauge the gauge field transforms as a double-valued, noncompletely reducible representation of SO(2,1).

The divergence conditions (4.13) and (4.14) generalize to the case of nonzero κ as follows. First, (4.14) remains unchanged:

$$\partial_i A^{[1]}_i=0\ . \tag{5.16a}$$

This equation states that $A^{[1]}_i=A_i$ is a transverse field. The companion condition on $C^{[1]}$ is

$$\partial_i C^{[1]}_i=\frac{\kappa^2}{\nabla^2}B\ . \tag{5.16b}$$

Equation (4.13) generalizes in an interesting way; the usual time derivative ∂_0 must be replaced by a slightly more complicated structure,

$$D_{ij}=\delta_{ij}\partial_0+\kappa\epsilon_{ij}\ , \tag{5.17}$$

which we call a generalized time derivative. In terms of this generalized time derivative we have

$$\partial_i A^{[N]}_{ia_1\cdots a_{N-1}}=D_{a_{N-1}i}A^{[N-1]}_{ia_1\cdots a_{N-2}}\quad (N>1) \tag{5.18a}$$

and

$$\partial_i C^{[N]}_{ia_1\cdots a_{N-1}}=D_{a_{N-1}i}C^{[N-1]}_{ia_1\cdots a_{N-2}}\quad (N>1)\ , \tag{5.18b}$$

which are the analogs of (4.13).

The generalized time derivative D_{ij} can be used to interchange the roles of the A and C fields:

$$C^{[N]}_{a_1\cdots a_N}=\frac{\kappa}{\nabla^2}D_{a_1p}A^{[N]}_{pa_2\cdots a_N} \tag{5.19a}$$

and

$$A^{[N]}_{a_1\cdots a_N}=\frac{1}{\kappa}D_{pa_1}C^{[N]}_{pa_2\cdots a_N}\ . \tag{5.19b}$$

Equation (5.19) expresses clearly the behavior of the theory in the limit of large and small κ. Apparently, as $\kappa\rightarrow0$ (5.19) implies that the C fields disappear and only the A fields survive. This is how our results reduce to the massless case discussed in Sec. IV. As $\kappa\to\infty$ the A fields disappear and only the C fields survive. In this limit, Fig. 1 simplifies to

$$C^{[0]}\to C^{[1]}\leftrightarrow C^{[2]}\leftrightarrow C^{[2]}\leftrightarrow C^{[3]}\leftrightarrow\cdots \ . \tag{5.20}$$

This is exactly the same indecomposable representation we saw in (2.10). Observe that the gauge field transforms as one of three possible representations depending on the value of κ: When $\kappa=0$ the gauge field transforms as in (4.11), when $\kappa=\infty$ the gauge field transforms as in (5.20), and for intermediate values of κ, $0<\kappa<\infty$, the gauge field transforms as in Fig. 1. Thus, we have discovered the surprising result that in this theory the Lorentz transformation properties of the field depend on the value of the mass parameter κ. To be more precise, we observe that it is not exactly the *representation* that changes with κ but rather the *glue* holding together the components of the indecomposable representation that depends on the parameter κ. Thus, as κ vanishes or becomes infinite the top and bottom rows of Fig. 1 become unglued with the top row surviving in the former case and the bottom row surviving in the latter [11].

In addition to the generalized divergence equations in (5.16) and (5.18) there are generalized curl equations,

$$\epsilon_{ij}\partial_i A^{[N+1]}_{ja_1\cdots a_N}=\epsilon_{a_1 i}D_{ij}A^{[N]}_{ja_2\cdots a_N} \quad (N>1)\ , \tag{5.21a}$$

$$\epsilon_{ij}\partial_i C^{[N+1]}_{ja_1\cdots a_N}=\epsilon_{a_1 i}D_{ij}C^{[N]}_{ja_2\cdots a_N} \quad (N>1)\ , \tag{5.21b}$$

$$\epsilon_{ij}\partial_i A^{[1]}_j=B\ , \tag{5.21c}$$

$$\epsilon_{ij}\partial_k C^{[1]}_j=\frac{\kappa\partial_0}{\nabla^2}B\ , \tag{5.21d}$$

and generalized symmetric gradient equations (valid for $N\geq 1$),

$$\sum_{i=1}^{N+1}\partial_{a_i}A^{[N]}_{a_1\cdots\hat{a}_i\cdots a_{N+1}} =\frac{N+1}{2}D_{pa_1}A^{[N+1]}_{pa_2\cdots a_{N+1}} +\frac{1}{N}\sum_{\substack{i,j=1\\ i<j}}^{N+1}\delta_{a_i a_j}\partial_p A^{[N]}_{pa_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_{N+1}} \tag{5.22a}$$

and

$$\sum_{i=1}^{N+1}\partial_{a_i}C^{[N]}_{a_1\cdots\hat{a}_i\cdots a_{N+1}} =\frac{N+1}{2}D_{pa_1}C^{[N+1]}_{pa_2\cdots a_{N+1}} +\frac{1}{N}\sum_{\substack{i,j=1\\ i<j}}^{N+1}\delta_{a_i a_j}\partial_p C^{[N]}_{pa_1\cdots\hat{a}_i\cdots\hat{a}_j\cdots a_{N+1}}\ . \tag{5.22b}$$

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[9] We say that one representation is *glued* to another when the representations are connected in a block-diagonal fashion as in (11). For a lengthy discussion of the structure of noncompletely reducible representations of SO(3,1), see Ref. [5]; I. M. Gel'fand and V. A. Ponamarev, Usp. Mat. Nauk **23**, 3 (1968) [Russ. Math. Surv. **23**, 1 (1968)]; I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Functions* (Academic, New York, 1966), Vol. V, Chap. III. It is interesting to note that the noncompletely reducible representations of other noncompact groups [for example, SO(2,1)] have not been the subject of such extensive mathematical investigation. To our knowledge, the results of this paper also exhibit the first *physical* application of noncompletely reducible representations of SO(2,1).
[10] For a more complete discussion of three-dimensional electrodynamics in the presence of a topological term, see S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[11] For a discussion of the difference between the massless ($\kappa=0$) and massive ($\kappa\neq 0$) cases with regards to the "spin" of the elementary fields, see Deser, Jackiw, and Templeton [10].