Radiatively induced Chern-Simons terms on the torus

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A Chern-Simons term is induced by integrating over fermions in the effective action of a (2+1)-dimensional field theory on a torus. It is shown that the Chern-Simons coefficient is functionally dependent on the nonintegrable phases of the torus as well as on the finite temperature and density. In the non-Abelian case such nonintegrable phases inhibit the generation of a Chern-Simons term in the most general case.

I. INTRODUCTION

In 2+1 dimensions, the properties of two-component spin matrices are such that the discrete symmetries of parity and time reversal are not respected by massive Dirac fermions. In a gauge theory, this is reflected by the appearence of a Chern-Simons term in the fermion vacuum-polarization diagrams which contribute to the effective action. This radiatively induced Chern-Simons term has been discussed in a number of papers [1, 2] and recently its coefficient has been shown to depend on the thermodynamic variables: finite temperature and finite density [3]. In this paper, the functional dependence of the Chern-Simons coefficient is considered on toroidal spacetimes.

It has been emphasized, in a somewhat different connection that, on such a nonsimply connected spacetime, a quantized gauge theory may be significantly affected by Wilson loops or nonintegrable phases, which wrap around noncontractable closed loops. These may be thought of as arising from background gauge fields which, although locally pure gauge, are not globally so. The phenomenon is much the same as that which is manifest in Wilczek's flux-tube model of anyons [4] and the Aharanov-Bohm effect [5]. It is of particular interest to consider what effect these nonintegrable phases might have on the induced Chern-Simons term on a torus, which represents the usual mathematical model of the periodic boundary conditions of solid-state heterostructures. In a purely classical theory, there is no way to determine the values of these nonintegrable phases; however, it is important to realize [6–8] that they are determined dynamically in the quantum theory by the minimization of the vacuum energy, so they are not really arbitrary at all. (This phenomenon has been studied in the context of Chern-Simons theory in Refs. [9–11].) There is still a possibility of varying such phases by introducing nonzero field strengths into the physical system, so it is worthwhile knowing their effect, quite apart from the obvious importance of formulating the problem on the torus. From the way the phases enter the problem, also from previous wisdom in connection with spontaneous symmetry breaking, one would expect the coefficient to depend on each of the phases in a periodic way. A calculation of the coefficient, for the Abelian gauge theory, confirms this.

In the non-Abelian theory nonintegrable phases play a much more important role for the quantum theory. Specifically, the inclusion of nonintegrable phases leaves some doubt as to whether a Chern-Simons term can be induced at all. Field theory is greatly complicated if the nonintegrable phases are not proportional to the identity matrix. If one performs the relevant calculations on the torus, it may be seen that there are, as usual, extra contributions to the vacuum-polarization graph in 2+1 dimensions, but that they are in fact not of the form of a Chern-Simons term unless the phases map into the center of the gauge group. Only in the Abelian limit is the induced Chern-Simons term recovered, in general.

In this paper some calculations of the Chern-Simons coefficient on the circle and the Euclidean three-torus are discussed.

II. PERTURBATIVE FORMALISM

Consider the Dirac action in three-dimensional Euclidean spacetime:

$$S_D = \int dV_x\, \overline{\psi}(\gamma^\mu D_\mu + m)\psi \; . \tag{1}$$

We may take the Dirac matrices to be $\gamma^i = \sigma^i$ where σ^i are the Pauli matrices. The covariant derivative is defined by $D_\mu = \partial_\mu + gA_\mu(x)$. The metric is simply $\delta_{\mu\nu}$. ψ and $\overline{\psi}$ are two-component Dirac spinors. The Abelian Chern-Simons action is defined by

$$S_{\rm CS} = \frac{i}{2} k g^2\, \epsilon^{\mu\nu\lambda} \int dV_x\, A_\mu \partial_\nu A_\lambda \; . \tag{2}$$

On a multiply connected spacetime it is necessary to consider the effect of pure gauge contributions to A_μ; thus we expand $A_\mu \to A_\mu + \overline{A}_\mu$, where $\overline{A}_\mu = \lambda^i H^i$, a constant linear combination of the matrices which generate the Cartan subalgebra of the gauge group G. In order to take account of this pure gauge part, the simplest step now is to gauge transform the action, so that $\overline{A}_\mu$ is set to zero. Because there are nonshrinkable closed curves, this implies a modified boundary condition for the fields as they wrap around each circle on the torus. For example [12],

$$\psi(x_\mu + L_\mu) = e^{2\pi i\delta_\mu} W(L_\mu)\psi(x_\mu) , \quad (3)$$

$$A_\mu(x_\mu + L_\mu) = W(L_\mu)A_\mu(x_\mu)W^{-1}(L_\mu) , \quad (4)$$

where L_μ is the toroidal circumference, δ_μ an Abelian phase, and $W(L_\mu)$ is the Wilson loop in the direction μ of the torus. The Wilson loop may be defined by $W(L_\mu) = P\exp(g\oint \overline{A}_\mu dx^\mu) = e^{2\pi i\Theta}$, where Θ is a matrix-valued constant. These boundary conditions become trivial in the R^n limit, i.e., as the L_μ are sent to infinity. The components of the momenta become $k_\mu = (2\pi/L_\mu)(l_\mu + \delta_\mu + \Theta_\mu)$.

The remaining gauge term $\overline{\psi}A_\mu\psi$ can be treated as an interaction. The only term in the perturbative expansion which can give rise to a Chern-Simons term is that of order g^2. (It is not necessary to calculate the g^3 contribution in the non-Abelian case for our purposes.) Formally, it may be written

$$\Gamma_{g^2} = \frac{1}{2}g^2\int dV_x dV_{x'}[\gamma^\mu]_{\alpha\beta}[\gamma^\nu]_{\rho\sigma}\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3} \times e^{i[(k_1+k_3-k_4)x+(k_2-k_3+k_4)x']}A_\mu(k_1)A_\nu(k_2)\frac{(\gamma^\lambda k_{3\lambda}+im)_{\alpha\sigma}}{k_3^2+m^2}\frac{(\gamma^\tau k_{4\tau}+im)_{\rho\beta}}{k_4^2+m^2} , \quad (5)$$

where α,β,ρ,σ are spinor matrix indices, which run from 1 to 2.

In all cases except the most general non-Abelian case, this expression may be simplified by integrating over one spacetime volume dV_x and one momentum (e.g., k_4). This results, formally, in a expression of the form [2]

$$\Gamma_{g^2} = \frac{1}{2}g^2\int dV_x \int dp\, dq\, e^{i(p+q)x}A_\mu(p)A_\nu(q)\,\Pi^{\mu\nu}(p) , \quad (6)$$

where

$$\Pi^{\mu\nu}(p) = \int dk[\gamma^\mu]_{\alpha\beta}[\gamma^\nu]_{\rho\sigma} \times \frac{(\gamma^\lambda k_\lambda + im)_{\alpha\sigma}[\gamma^\tau(k+p)_\tau + im]_{\rho\beta}}{(k^2+m^2)[(k+p)^2+m^2]} \quad (7)$$

and dp and dq are the momentum measures which are to be specified for a given spacetime. Expanding this quantity and noting that $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda) = -2i\epsilon^{\mu\nu\lambda}$ in three dimensions, a Chern-Simons term may be identified. The aim here is to calculate this term for various theories on toroidal spacetimes.

III. ABELIAN CASE: $R^2 \times S^1$

Consider first the Abelian theory: here no major complications arise. The simplest case is to take a spacetime of the form $R^2 \times S^1$, which includes the case of finite temperature, then the denominators in (7) may be combined with the help of a Feynman parameter. After some manipulation the coefficient may be written

$$k = 2\int_0^1 dz\int \frac{d^nk}{(2\pi)^n}\frac{1}{L}\sum_{l=-\infty}^{+\infty}\left[k^2 + \left(\frac{2\pi}{L}\right)^2(l+\delta)^2 + m^2(z)\right]^{-2} , \quad (8)$$

where $n = 2$ (it is convenient to leave this general for the evaluation). $m^2(z) = m^2 + p^2z(1-z)$ involves the Feynman parameter, which should not be ignored *a priori*. This may be integrated using the standard techniques of dimensional regularization:

$$I = \int \frac{d^nk}{(2\pi)^n}\frac{1}{L}\sum_{l=-\infty}^{+\infty}[\mathbf{k}^2 + (2\pi/L)^2(l+\delta)^2 + m^2(z)]^{-2} \quad (9)$$

$$= \frac{\Gamma\left(2-\frac{n}{2}\right)}{L(4\pi)^{-n/2}}\left(\frac{2\pi}{L}\right)^{2(n/2-2)} S(n) , \quad (10)$$

where

$$S(n) = \sum_{l=-\infty}^{+\infty}[(l+\delta)^2+\nu^2]^{n/2-2} . \quad (11)$$

This summation may be tackled by a technique used by Ford [6], who has shown that

$$\sum_{l=-\infty}^{+\infty}[(l+b)^2+a^2]^{-\lambda} = \frac{\pi^{1/2}(a^2)^{1/2-\lambda}\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma(\lambda)} + 4\sin(\pi\lambda)\int_{|a|}^\infty du(u^2-a^2)^{-\lambda}\,\mathrm{Re}\left(\frac{1}{\exp[2\pi(u+ib)]-1}\right) \quad (12)$$

which is a finite result in our $n \to 2$ limit. However, there is still a pole in the integrand of (12). At $n = 2$, $(u^2-a^2)^{n/2-2}$ is singular at $u = |a|$. The sine function is vanishing on the other hand, so there is no divergence. Expanding around the simple pole at $n = 2$ and noting that $\sin(\pi\lambda)(u^2-a^2)^{-\lambda} \sim \frac{1}{2}\pi\delta(u-|a|)$ (the factor of $\frac{1}{2}$ arises because the limit of the integral ends on the center of the delta function) it is found that

$$S(2) = \frac{\pi}{|\nu|}\left(1 + 2\,\frac{e^{2\pi|\nu|}\cos(2\pi\delta)-1}{e^{4\pi|\nu|}-2\cos(2\pi\delta)e^{2\pi|\nu|+1}}\right) . \quad (13)$$

After some manipulation, the result for k is found to be

$$k=\int_0^1 dz\frac{\nu(z)}{|\nu(z)|}\frac{1}{4\pi}\frac{\sinh[2\pi\nu(z)]}{\cosh[2\pi\nu(z)]-\cos(2\pi\delta)}\;, \tag{14}$$

where $\nu(z)=m(z)L/2\pi$. By expanding around $z=0$, and dropping higher derivatives (which are not Chern-Simons terms), it may be shown that the only contribution to the coefficient is that which is obtained by effectively dropping z from the integrand. The Feynman parameter integration is then trivial. When the circle represents finite temperature and density we must choose antisymmetric boundary conditions for the fermion. Setting $\delta=\frac{1}{2}-i\mu L/2\pi$ and $L=\beta=1/kT$ gives a suitable analytic continuation of Ford's result, in the presence of the chemical potential μ. This gives

$$k_{\rm FT,FD}=\frac{m}{|m|}\frac{1}{4\pi}\frac{\sinh(|m|L)}{\cosh(|m|L)+\cosh(\mu L)} \tag{15}$$

which may be verified numerically, since the summation is absolutely convergent. This agrees with the result found in Ref. [3].

IV. ABELIAN CASE: THREE-TORUS

On the three-torus, a similar result may be derived in the Abelian case, to obtain the functional dependence in terms of temperature, density and the two nonintegrable phases in the spatial directions:

$$k=2\int_0^1 dz\frac{1}{L_1L_2L_3}\sum_{l_0=-\infty}^{+\infty}\sum_{l_1=-\infty}^{+\infty}\sum_{l_2=-\infty}^{+\infty}\left[\sum_\nu\left(\frac{2\pi}{L_\nu}\right)^2(l_\nu+\delta_\nu)^2+m^2(z)\right]^{-2}\;. \tag{16}$$

The summations required are of the form of those tackled in Refs. [11, 13]. Although it is not possible to obtain a functional result for the torus, the summation can be reexpressed in a slightly less meaningless form. The procedure for evaluating the summations is sketched out below.

First we define a zeta function $\zeta(s)$, such that the coefficient is given by $k=2m\zeta(2)$ [see Eq. (2)]:

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\frac{1}{L_0L_1L_2}\sum_{\boldsymbol{l}}e^{-t\lambda_{\boldsymbol{l}}}\;, \tag{17}$$

where

$$\lambda_{\boldsymbol{l}}=m^2+\sum_{\nu=0}2\left(\frac{2\pi}{L_\nu}\right)^2(l_\nu+\delta_\nu)^2\;. \tag{18}$$

Using the Poisson summation technique, it is straightforward to show that [13]

$$\sum_n e^{-t(2\pi/L)^2(n+\delta)^2}=\frac{L}{2\sqrt{\pi t}}\sum_l e^{il2\pi\delta-l^2L^2/4t}\;. \tag{19}$$

There are three such terms for the torus, thus it is possible to write

$$\zeta(s)=\frac{1}{(4\pi)^{3/2}\Gamma(s)}\int_0^\infty dt\,t^{s-5/2}\sum_{\boldsymbol{l}}\exp\left(2\pi i\sum_\nu l_\nu\delta_\nu-\sum_\nu l_\nu^2L_\nu^2/4t-m^2t\right)\;. \tag{20}$$

Taking the limit $s\to 2$ and noting the identity

$$\int_0^\infty x^{\nu-1}e^{-\beta/x-\gamma x}dx=2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{\beta\gamma}) \tag{21}$$

with $\gamma=m^2$, $\beta=\sum_\nu l_\nu^2L_\nu^2/4$ and $\nu=\frac{1}{2}$, the Chern-Simons coefficient may be written

$$k=\frac{m}{|m|}\frac{1}{4\pi}\left[1+\left(\frac{2}{\pi}\right)^{1/2}\sum_{\boldsymbol{l}\neq\boldsymbol{0}}\exp\left(2\pi i\sum_\nu l_\nu\delta_\nu\right)\sqrt{R(m,\boldsymbol{l},\mathbf{L})}K_{1/2}[R(m,\boldsymbol{l},\mathbf{L})]\right]\;, \tag{22}$$

where $R^2(m,\boldsymbol{l},\mathbf{L})=m^2(l_0^2L_0^2+l_1^2L_1^2+l_2^2L_2^2)$ and the notation $\boldsymbol{l}\neq\boldsymbol{0}$ is used to mean that the zero mode $\boldsymbol{l}=(0,0,0)$ is excluded. (In fact, it has been separated off by hand and is given in the first term, i.e., the leading 1.) The expression for k can be rendered more familiar by noting that the Bessel function has a particularly simple form $K_{1/2}(z)=\sqrt{\pi/2z}\,e^{-z}$. Thus the general form of the Chern-Simons coefficient in this case is

$$k_{\text{torus}} = \frac{m}{|m|}\frac{1}{4\pi}\left[1+\sum_{l\neq 0}\exp\left(2\pi i\sum_\nu l_\nu\delta_\nu\right)e^{-|m|(l_0^2L_0^2+l_1^2L_1^2+l_2^2L_2^2)^{1/2}}\right] . \tag{23}$$

It is clear from the form of this expression that the limit $L_0, L_1, L_2 \to \infty$ gives the correct result $k = (m/|m|)(1/4\pi)$ in the infinite-volume limit. It is rather less clear that it reduces to expression (15) in the limit $L_1, L_2 \to \infty$ with $\delta_0 = \frac{1}{2} - i\mu L/2\pi, \delta_1, \delta_2 = 0$. At first glance it would appear to give the wrong limit: as $L_1, L_2 \to \infty$ the exponential goes to zero and we are left with the infinite-volume limit result, which is wrong. However, this is incorrect, since there is a set of modes in the summation for which $l_1 = l_2 = 0$ and l_0 takes all values *except* zero (which was separated off by hand). The calculation can now be performed for general δ_0 by separating off these terms explicitly and taking the infinite-volume limit for the two spacelike directions:

$$k_{L_1,L_2\to\infty} = \frac{m}{|m|}\frac{1}{4\pi}\left(1+2\sum_{l_0=1}^{\infty}\cos(2\pi i\delta_0)e^{-|m|l_0L_0}+0\right) . \tag{24}$$

Now, using the fact that

$$\sum_{l=1}^{\infty}\cos(2\pi\delta l)e^{|m|Ll} = \frac{1}{2}\sum_{l=1}^{\infty}e^{l(2\pi i\delta-|m|L)} +\frac{1}{2}\sum_{l=1}^{\infty}e^{-l(2\pi i\delta+|m|L)} , \tag{25}$$

these are geometric series which can be summed exactly:

$$=\frac{1}{2}\frac{e^{2\pi i\delta-|m|L}}{1-e^{2\pi i\delta-|m|L}}+\frac{1}{2}\frac{e^{-(2\pi i\delta+|m|L)}}{1-e^{-(2\pi i\delta+|m|L)}} \tag{26}$$

$$=\frac{\cos(2\pi\delta)e^{|m|L}-1}{e^{2|m|L}-2\cos(2\pi\delta)e^{mL}+1} . \tag{27}$$

Inserting this result back into Eq. (24) yields 14 exactly. See also (13) for comparison. δ_0 may then be continued to include finite density, etc., as required. Note that in the general torus case, it is not correct to substitute $\delta_0 = \frac{1}{2} - i\mu L/2\pi$ directly, since this gives explicitly divergent summations if μ becomes large. Some manner of regularization prescription is required first. A numerical investigation is possible, but also difficult owing to slowness of convergence of the summation (16).

V. NON-ABELIAN CASE

Finally, we come to the non-Abelian case. Here the Chern-Simons term is no longer guaranteed unless the generalized non-Abelian phase matrices lie in the center of the gauge group (making them Abelian). When this *is* the case, the Abelian expressions apply, up to a numerical factor of $-d_R/N\, C_2(G_R)$, where d_R is the dimension of the matrix representation for ψ and A_μ, N is the dimension of the group and $C_2(G_R)$ is the quadratic Casimir invariant in the representation G_R. In general, however, it is not possible to identify a term of the form $\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$. The problem lies in the nature of the boundary conditions for the momenta, as was noted in Ref. [12]. In the Fourier transform we have, formally, for that part of Γ which gives the Chern-Simons term,

$$\Gamma_{\text{CS}} = g^2\int dV_x dV_{x'}\int dk_1dk_2dk_3dk_4\exp\{i[(k_1+k_3-k-4)x+(k_2-k_3+k_4)x']\}\left(\frac{A_\mu(k_1)k_{3\nu}A_\lambda(k_2)\epsilon^{\mu\nu\lambda}m}{(k_3^2+m^2)(k_4^2+m^2)}\right) . \tag{28}$$

In simply connected spacetime, there are no Θ phases, and one of the exponents may be identified as the Fourier transform of a Dirac delta function in the momenta. This implies a very important constraint on the four different momentum variables. However, on the torus, no such constraint is implied. The situation is most transparent in the case of a single circle ($R^2 \times S^1$). In this case we have two freely translatable, continuous momenta and one set of discrete eigenvalues. Making the eigenvalue replacements in (28), and integrating over one spacetime volume it is found that the integral which must be dealt with is, in fact,

$$I = \int_0^L dx^0\exp\left(\frac{2\pi i}{L}(l_1+l_2-l_3+\theta_1+\theta_2-\theta_3)x^0\right) , \tag{29}$$

where l_1, l_2, l_3 are integers coming from the discrete nature of the momentum in the S^1 dimension. θ_ν are just the components of the diagonal Θ matrices, since any Abelian phase δ must cancel between the fermion lines, by conservation of momentum at the vertices. Because the calculation requires $\theta_1, \theta_2, \theta_3$ to be general, evaluation of the above integral I results in

$$I = \frac{L}{2\pi i}\frac{\exp[2\pi i(\theta_1+\theta_2-\theta_3)]-1}{l_1+l_2-l_3+\theta_1+\theta_2-\theta_3} . \tag{30}$$

A delta function $\delta(l_1+l_2-l_3)$ only results in the case when $\Theta = 0$. The Θ matrix elements do not cancel each other unless they lie in the center of the group. The integral over one of the space coordinates does not therefore result in a delta function, over all space and time components, but instead in

$$\Gamma = g^2 \int dV_x \int \frac{d^2\mathbf{k_0}}{(2\pi)^2}\frac{d^2\mathbf{k_1}}{(2\pi)^2}\frac{d^2\mathbf{k_2}}{(2\pi)^2}\delta(\mathbf{k}_2 - \mathbf{k}_3 + \mathbf{k}_4)$$
$$\times \sum_{l=-\infty}^{+\infty} \frac{L}{2\pi i}\frac{e^{2\pi i\Theta_g} - 1}{l_2 - l_3 + l_4 + \Theta_g}$$
$$\times \frac{e^{i(k_1+k_3-k_4)x}\; m\, \epsilon^{\mu\nu\lambda} A_\mu(k_1) k_{3\nu} A_\lambda(k_2)}{[\mathbf{k}_3^2 + (2\pi/L)^2(l_3+\delta+\Theta_f)^2 + m^2][\mathbf{k}_4^2 + (2\pi/L)^2(l_4+\delta+\Theta_f)^2 + m^2]} . \quad (31)$$

Θ_f is a phase arising from a fermion boundary condition and Θ_g is a phase arising from the boundary condition for the gauge field. As $\Theta \to 0$ the usual results are obtained. It is, of course, still possible to integrate over the continuous momenta and obtain the usual constraint for them. We might also wonder whether any progress could be made by attempting to perform the discrete summations: perhaps we might actually find the Chern-Simons term concealed in this mess. However, an attempt to perform the sums quickly leads to the conclusion that there is no term of the simple form of (2). Thus we are forced to the conclusion that the presence of nonintegrable phases actually inhibits the production of a CS term in this general case.

It should be reemphasized that the actual values of the Θ_ν matrices are determined dynamically by the minimization of the effective action, so it is not clear *a priori* that their values should be other than zero at this stage. The effects of nonintegrable phases on the torus have been considered in a number of papers [13, 14], also in connection with Chern-Simons theory[11]. In a wide class of models it has been shown that this is indeed the case; thus, there is at least a possibility that the Chern-Simons term is absent from these. However, in reality these investigations are insufficient to determine its absence, since they do not take account of nonzero field strength — a rather difficult problem on these spacetimes. To obtain the full picture one must also know about the interaction between such phases and real electric and magnetic fields.

Another question worthy of mention is the following: how should one interpret the infinite-volume limit? Clearly the effects of these phases should become negligible in this limit and there should be a Chern-Simons term regardless of the initial topology. Indeed, this is certainly the case, but the limit (at least mathematically) is not a smooth one in this case. (The delta function is not a smooth distribution.) It is rather hard to imagine a set of terms gradually approaching a Chern-Simons form in the infinite-volume limit, but this is effectively the picture.

VI. DISCUSSION

The results summarized above show that the coefficient k of the induced Chern-Simons term, in $2+1$ toroidal dimensions is a function of both the thermodynamic variables and the nonintegrable phases of the torus $k(\delta, \Theta, \mu, \mathbf{L}, m)$. If the nonintegrable phases commute with the entire gauge group, this coefficient is well defined. In any other case, a coefficient cannot be evaluated since there is no identifiable Chern-Simons term arising from loop corrections.

Since a Chern-Simons form is not guaranteed, an obvious question is the following: if the extra terms which arise in $2+1$ dimensions are not a Chern-Simons term then what are they? This is particularly difficult to answer, owing to the difficulty of the calculations. It is unlikely that any presentable analytical answer can be found. However, if we restrict attention to the three-torus at $\mu = 0$ a numerical analysis might feasibly be carried out, to determine the behavior of such a system.

Since the question hinges on dynamical issues, one should look at the vacuum structure of models on toroidal backgrounds. Previous work carried out on the torus [13, 14] would seem to indicate that there is a wide range of models in which one would not expect to find an induced CS term, but these results can only serve as a guide. It is essential to look at models in which the nonintegrable phases are nonzero in the presence of nonzero field strength. Calculations of this kind are in progress [15]. Another important question to answer is the following: does the theory still become simply a massive gauge theory, or is the behavior even more complicated in this case?

It might be argued that loop corrections to the Chern-Simons coefficient should not play a role in Chern-Simons theory at all [16]. One argument for this is the necessity for k to be quantized in the non-Abelian case, in order to have a gauge-invariant quantum theory. The procedure of ignoring the loop corrections seems somewhat difficult to justify however. Had the coefficient been a constant, it would have been easy to absorb it into the definition of the Chern-Simons action. Since it is a function, it is not clear that the correction can be ignored. Lykken, Sonnenschein, and Weiss have emphasized that the corrected Chern-Simons coefficient is in fact the physical value [17]. On the torus, at least, it may not affect the quantization argument. In the Abelian theory there is no necessity for quantization of the coefficient, so there is no contradiction. In the non-Abelian theory, there may be no correction anyway. In a recent paper [18, 19], Hosotani has argued that, on the torus, time-dependent nonintegrable phases could lead to a quantization of the Chern-Simons coefficient, even in the Abelian case, if gauge invariance is to be respected. According to the calculations above, these phases would also affect the value of the coefficient, correcting coefficient with a function of time. This would seem to suggest that the consistency of such an idea is a nontrivial problem. Moreover if one is interested in models of anyonic superconductivity, formulated on a torus as was passingly mentioned in [20], the nonintegrable phases discussed here could well spoil the necessary cancellations required to produce a pole in the current-current correlation functions.

Finally, if there is no induced Chern-Simons term for a prescribed model, is it always appropriate to add the Chern-Simons action in $(2+1)$-dimensional field theory? The behavior of a system under parity and time-reversal transformations is of particular interest in theoretical models of high-T_c superconductors. Parity is not preserved in 2+1 dimensions regardless of whether a Chern-Simons terms appears by radiative corrections, but many models make explicit use of the Chern-Simons construction [17, 20, 21]. This feature may need to be reconsidered in any non-Abelian generalizations of such theoretical models, should they become of interest. Some of these issues will be considered in further work.

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[1] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).
[2] A. Redlich, Phys. Rev. D **29**, 2366 (1984).
[3] E. R. Poppitz, Phys. Lett. B **252**, 417 (1990).
[4] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982); **48**, 1144 (1981).
[5] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 484 (1959).
[6] L. Ford, Phys. Rev. D **21**, 933 (1980).
[7] Y. Hosotani, Phys. Lett. **126B**, 309 (1983).
[8] D. J. Toms, Phys. Lett. **126B**, 445 (1983).
[9] M. Burgess and D. J. Toms, Phys. Rev. Lett. **64**, 1639 (1990).
[10] M. Burgess and D. J. Toms, Phys. Lett. B **252**, 596 (1990).
[11] M. Burgess, A. McLachlan, and D. J. Toms, Phys. Rev. D **43**, 1956 (1991).
[12] M. Burgess and D. J. Toms, Ann. Phys. (N.Y.) (to be published).
[13] J. E. Hetrick and C. L. Ho, Phys. Rev. D **49**, 4085 (1989).
[14] A. McLachlan, Nucl. Phys. **B388**, 188 (1990).
[15] M. Burgess, A. McLachlan, and D. J. Toms (work in preparation).
[16] Y. Kao and M. Suzuki, Phys. Rev. D **31**, 2137 (1985).
[17] J. D. Lykken, J. Sonnenschein, and N. Weiss, Int. J. Mod. Phys. A **6**, 1335 (1991).
[18] Y. Hosotani, Phys. Rev. Lett. **62**, 2785 (1989); **64**, 1691 (1990).
[19] C.R. Hagen and E. Sudarshan, Phys. Rev. Lett. **64**, 1690 (1990).
[20] Y. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).
[21] T. Banks and J. Lykken, Nucl. Phys. **B366**, 500 (1990).