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Time ordering, anomalies, and chiral gauge theories

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For chiral gauge theories, singularity of the time-ordered products results in some ambiguity spoiling the Tomonaga-Schwinger equation; this is explicitly demonstrated in two- and four-dimensional Abelian theories. The ambiguity can be eliminated by the requirement of gauge invariance and, thus, nonpolynomially modifies the conventional quantization of chiral gauge theories. A correctly quantized theory should suffer no gauge anomalies, neither the perturbative "triangle" nor the nonperturbative Witten anomalies.

The pioneering Adler-Bell-Jackiw axial anomalies [1] and the subsequent anomalies of other symmetry have had a decisive, crucial role in quantum field theories. They simply are as fundamental as the symmetry principle concerned. The anomalous breaking of global axial invariance has provided the answers to certain problems of phenomenology (ranging from the neutral pion decay to, possibly, the proton-spin structure). On the other hand, when the symmetry in question is the local symmetry of some gauge theory, anomalies entail disastrous consequences and, in doing so, impose severe constraints. The standard model of the electroweak interactions is an example of this.

However, there are doubts about the validity of the conventional quantization of anomalous chiral gauge theories. These were raised in the context of a path integral in an *ad hoc* manner [2]. From the canonical operatorial quantization, I have found an argument against the widely accepted path integral and, equivalently, Feynman rules for chiral gauge theories [3]. The argument, however, is not illuminating in perturbation theory as the eigenstates of the *full* first-quantized Hamiltonian are required. The present work is the extension into perturbation theory [4].

Consider the classical Lagrangian density of Abelian chiral gauge theories [5], where only the left-handed current is coupled to the gauge fields:

$$\mathcal{L}=-\tfrac{1}{4}F^2+\bar{\psi}(i\partial\!\!\!/+A\!\!\!/P_L)\psi\ . \tag{1}$$

At this level it possesses the usual chiral gauge invariance, but the quantum version is well known to suffer from anomalies.

In the interaction picture the evolution operator of the second-quantized states satisfies the Tomonaga-Schwinger equation

$$\begin{aligned}&\frac{\partial}{\partial t}U(t,t_0)=-iH_{\rm int}(t)U(t,t_0)\ ,\\ &U(t_0,t_0)=\mathbb{1}\ ,\end{aligned} \tag{2}$$

where

$$H_{\rm int}(t)=\int d\mathbf{x}\ A_\mu\bar{\psi}\gamma^\mu P_L\psi(\mathbf{x},t)\ , \tag{3}$$

and A_μ,ψ are the *free-field* operators. Normal ordering with respect to the in-vacuum is implicit here. The S matrix can then be obtained as the limit

$$S=\lim_{t\to\infty}\ \lim_{t_0\to-\infty}\ U(t,t_0)\ . \tag{4}$$

The solution of (2) is found by iteration to be the Neumann series

$$\begin{aligned}U(t,t_0)=1&+(i)\int_{t_0}^{t}dt_1H_{\rm int}(t_1)+(-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2H_{\rm int}(t_1)H_{\rm int}(t_2)\\ &+(-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3H_{\rm int}(t_1)H_{\rm int}(t_2)H_{\rm int}(t_3)+\cdots\ .\end{aligned} \tag{5}$$

It is usually necessary to convert this into another form by the Dyson time-ordered product of operators, which can be defined pairwise as [6]

$$\begin{aligned}\mathcal{T}[A(t_1)B(t_2)]&=\theta(t_1-t_2)A(t_1)B(t_2)+\theta(t_2-t_1)B(t_2)A(t_1)\\ &=\tfrac{1}{2}[A(t_1),B(t_2)]+\tfrac{1}{2}\epsilon(t_1-t_2)[A(t_1),B(t_2)]\ .\end{aligned} \tag{6}$$

Consequently, the Feynman rules and/or the path integral are conventionally derived from the evolution operator

$$\begin{aligned}\tilde{U}(t,t_0)&=1+(-i)\int_{t_0}^{t}dt_1 H_{\rm int}(t_1)+\frac{(-i)^2}{2!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\,\mathcal{T}[H_{\rm int}(t_1)H_{\rm int}(t_2)]\\&\quad+\frac{(-i)^3}{3!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\int_{t_0}^{t}dt_3\,\mathcal{T}[H_{\rm int}(t_1)H_{\rm int}(t_2)H_{\rm int}(t_3)]+\cdots\\&=\mathcal{T}\exp\left[-i\int_{t_0}^{t}d\tau\,H_{\rm int}(\tau)\right],\end{aligned}\tag{7}$$

and also from which the various derivations of anomalies follow.

I now claim that the above step going from (5) to (7), is, in general, ambiguous, because in (7) there are further contributions relative to (5) when the time arguments of the integrands coincide. This statement is more than the usual statement of the indefiniteness of the time-ordered product at equal times. Here I wish to draw attention to the fact that the time-ordered products are highly singular at equal times. Consequently, the two expressions are not equivalent, despite whatever redefinition one may employ for the time-ordered products at coinciding times. That is, $\tilde{U}$ is capable of producing anomalies, which cannot be absorbed by local counterterms.

More precisely, it can be shown that $\tilde{U}$ is not a solution of (2) due to some ambiguity. Consider

$$\frac{\partial\tilde{U}(t)}{\partial t}=\lim_{\epsilon\to 0}\frac{\tilde{U}(t+\epsilon,t_0)-\tilde{U}(t,t_0)}{\epsilon}\ ;\tag{8}$$

if one ignores terms of *apparent* order ϵ or higher such as

$$\lim_{\epsilon\to 0}\frac{(-i)^2}{2!\epsilon}\int_{t}^{t+\epsilon}dt_1\int_{t}^{t+\epsilon}dt_2\,\mathcal{T}[H_{\rm int}(t_1)H_{\rm int}(t_2)]\ ,\tag{9}$$

then (2) is satisfied. Such ignorance is fatal if the integrand in (9) behaves singularly as a δ function killing off one integration, thus reducing the order $O(e^0)$ and contributing to the right-hand side (RHS) of the differential equation of (2).

The anticommutator in (6) does not contribute to (9), but the commutator term does and is dominated by the fermion currents. In two dimensions the *different-time* commutator for free-field currents is known [7]:

$$\begin{aligned}&[J_L^0(x,t),J_L^0(x',t')]=\frac{i}{2\pi}\frac{\partial}{\partial x}\delta(t-t'+x'-x)\ ,\\&J_L^1=-J_L^0\ .\end{aligned}\tag{10}$$

Expression (9) can now be evaluated by contour integration in momentum space with the use of, with $\eta\to 0^+$,

$$\begin{aligned}&\delta(x)=\frac{1}{2\pi}\int dp\,e^{ipx-\eta|p|}=\frac{1}{2\pi i}\left[\frac{1}{x+i\eta}-\frac{1}{x-i\eta}\right],\\&\theta(x)=\frac{1}{2\pi i}\int dp\frac{e^{ipx}}{p-i\eta}\ ,\end{aligned}\tag{11}$$

from which, for example,

$$\frac{1}{2}\epsilon(t-t')\delta(t'-t+x'-x)=\frac{1}{\partial_t-\partial_x}[\delta(t-t')\delta(x-x')]\ .$$

The result is that $\tilde{U}$ now satisfies an equation similar to (2), but with $H_{\rm int}$ replaced by

$$H_{\rm int}\to H_{\rm eff}\equiv H_{\rm int}(t)+Q_2(t)\ ,\tag{12}$$

where, modulo local counterterms,

$$\int dt\,Q_2(t)=\frac{c}{4\pi}\int dt\,dx(A_0-A_1)\times\frac{\partial_1}{\partial_0-\partial_1}(A_0-A_1)(x,t)\ .\tag{13}$$

c in (13) is ambiguous, owing to the well-known nonassociativity of products of distributions [8]. This ambiguity can also be seen in the contour integrals by taking the limits of different $\eta s'$ of (11), corresponding to different distributions in a product, via different routes to 0^+. In general, the interchanging of the order of such limits gives different results, hence the ambiguity.

c can be fixed by a gauge-invariance requirement, however. With the choice $c=1$, the integrand of the last expression under a gauge variation yields precisely the anomaly term in the fermion current divergence [9]. In this way the origin of gauge-symmetry anomalies is transparent: The underlying theory of (7) is not gauge invariant since it admits $H_{\rm eff}$, not $H_{\rm int}$ as naively expected, for its interaction sector. Also clarified is the connection between the Schwinger term in the current commutators and the anomalous divergence.

It can be shown that (9) is the only term in (8) that produces extra contributions to the RHS of (2). Then the S matrix which comes from an evolution operator that satisfies (2) exactly is easily envisaged,

$$S=\mathcal{T}\exp\left[-i\int dt(H_{\rm int}-Q_2)\right].\tag{14}$$

The theory of $c=1$ is now anomaly free; the anomalous fermion current divergence is canceled by the relative minus sign in (14). Gauge invariance has thus determined c.

In four dimensions the term that may violate the Tomonaga-Schwinger equation is, apart from (9),

$$\lim_{\epsilon\to 0}\frac{(-i)^3}{3!\epsilon}\int_{t}^{t+\epsilon}dt_1\int_{t}^{t+\epsilon}dt_2\int_{t}^{t+\epsilon}dt_3\times\mathcal{T}[H_{\rm int}(t_1)H_{\rm int}(t_2)H_{\rm int}(t_3)]\ ,\tag{15}$$

whose naive order is $O(\epsilon^2)$. Explicit calculation, however, shows that it survives the limit, but is once again, ambiguous. The singularity of the integrand of (15) now comes from the *triple commutators* of the free-field fermion currents at different times. This quantity is not available in the literature. Nevertheless, by an indirect method the counterpart of Q_2 in four dimensions can be derived, modulo local counterterms,

$$\int dt\, Q_4(t) = \frac{c}{48\pi^2}\int d^4x\, F_{\mu\nu}\tilde{F}^{\mu\nu}\frac{\partial_\alpha}{\partial^2}A^\alpha(x) , \qquad (16)$$

of which the gauge transformation of the first term, with $c=1$, corresponds to the well-studied expression of chiral anomalies.

The derivation of (16) is facilitated by the Wick theorem. The time ordering in (15) factorizes into boson and fermion factors and is Wick expressed in normal-ordered and propagator terms. Of these, the only terms that survive the limit $\epsilon \rightarrow 0$ are the c-number, highly singular products of fermion propagators, that is, the triangle diagrams. Closed expressions for these diagrams have been obtained a long time ago [10]; in using them we have to appeal to the Bose symmetry of external legs to fix the ambiguous accompanying numerical factors:

$$\Gamma_{\mu\nu\rho}(p|p') = \frac{i}{12\pi^2}\epsilon_{\mu\rho\alpha\beta}p^\alpha(p+p')^\beta \frac{p'_\nu}{p'^2+i0^+} + \text{symmetrization} . \qquad (17)$$

The result (16) is also connected to the violation of equal-time Jacobi identity of free-field fermion currents [11]; it is only another manifestation and a special case of the triple commutators at different times which renders (15) nonzero.

In summary, I have claimed that the anomalously broken local symmetry of a chiral gauge theory can be restored thanks to an ambiguity of the time derivative of the evolution operator for second-quantized states. The ambiguity spoils the Tomonaga-Schwinger equation; consequently, the conventional S matrix has to be modified. The modification is nonlocal because of the appearance of antiderivatives, but does not necessarily imply that the spacelike separated operators no longer commute. The enforcement of gauge invariance can then be called upon to eliminate the ambiguity (because, after all, it can be seen that the vacuum polarization and triangle diagrams, which are the potential cause of current anomalous nonconservation in two and four dimensions, respectively, are also the ones giving rise to extra, but ambiguous, contributions to the time derivative).

This work thus provides some motivation for the use of nonlocal "counterterms," which have been advocated in an *ad hoc* manner before [12], to cancel the gauge anomalies.

At least for Abelian theories, in four dimensions or otherwise, I am convinced that chiral gauge theories quantized in this way are consistent: anomaly free, renormalizable, and unitary. I hope to present this result elsewhere.

The adjective "anomaly-free" here is meant exclusively for gauge symmetry, since the consideration above has no effect on the existence of anomalies of rigid, ungauged symmetry. (The Noether current of this latter kind of symmetry, not being gauged, does not appear in the second-quantized Hamiltonian and thus is not subjected to the treatment above.)

I believe that these conclusions are applicable to not only non-Abelian theories, but also other type of (nonchiral) gauge anomalies.

In the path-integral representation of the vacuum-to-vacuum amplitude corresponding to the S matrix (14), the appearance of the $Q_{2,4}$ terms apparently resembles that of the Wess-Zumino (WZ) term advocated in [2], for example, in the four-dimensional Abelian case:

$$X_{\rm WZ} \approx \int d^4x\, \phi F_{\mu\nu}\tilde{F}^{\mu\nu} . \qquad (18)$$

But the two approaches are not equivalent. This point can be seen by performing the integration over the WZ field $\phi(x)$, forcing a constraint on the gauge fields:

$$\int \mathcal{D}\phi\, e^{iX_{\rm WZ}} \approx \delta(F_{\mu\nu}\tilde{F}^{\mu\nu}) . \qquad (19)$$

In contradistinction, no such constraint is encountered explicitly in the above.

Finally, with regard to the Witten anomalies [13] as in the example of SU(2) theory, I suspect that there are no such anomalies in the current quantization. I should point out first that my arguments are entirely different from those in Ref. [14]. Those authors criticize the mathematical credibility of the anomaly derivation. I, on the other hand, have no such criticism, but, instead, doubt that Witten's path-integral starting point is the appropriate one. My suspicion is based on arguments of, apart from that of the Berry's phase approach [3], the embedding of the SU(2)-Witten anomaly into the perturbative "triangle" anomaly of some bigger group G [$\supset$SU(2)] [15]. As the conventional path integral of the latter theory should be rectified accordingly, its restriction to the SU(2) subgroup may not be the one that was originally employed in [13].

Note added. Upon completing this work, in the literature search I came across Ref. [16] where some doubt about the equivalence of (5) and (7) was also raised.

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