Time-dependent Chern-Simons solitons and their quantization

R. Jackiw
Center for Theoretical Physics, Laboratory for Nuclear Science, and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

So-Young Pi
Department of Physics, Boston University, 590 Commonwealth Avenue, Boston, Massachusetts 02215
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Interaction with an external harmonic or magnetic field can be introduced into the planar, gauged, nonlinear Schrödinger equation by a coordinate transformation. Correspondingly, the known vortex-soliton solutions are transformed into periodic solutions in the presence of these external fields. This shows that vortex solitons bind to the external fields and their binding energy may be quantized semiclassically.

I. THE MODEL AND ITS SYMMETRIES

The planar, gauged, nonlinear Schrödinger equation is an interesting generalization to (2+1)-dimensional space-time of the completely integrable (1+1)-dimensional nonlinear Schrödinger equation. Moreover, viewed as a quantum-field-theoretic Heisenberg equation of motion, it arises when the nonrelativistic N-body anyon system is second quantized. The equation for the "matter" field ψ reads [1]

$$iD_t\psi=-\frac{1}{2m}\mathbf{D}^2\psi-g(\psi^*\psi)\psi \ . \tag{1.1}$$

The last nonlinear term corresponds in the second-quantized N-body problem to a two-body δ-function attraction of strength g. The covariant derivatives

$$D_t\equiv\partial_t+iA^0,\quad \mathbf{D}\equiv\nabla-i\,\mathbf{A} \tag{1.2}$$

involve gauge potentials $A^\mu=(A^0,\mathbf{A})$ that satisfy the equations of an Abelian Chern-Simons theory,

$$B\equiv\nabla\times\mathbf{A}=-\frac{1}{\kappa}\rho \ , \tag{1.3a}$$

$$E^i=-\partial_t A^i-\partial_i A^0=\frac{1}{\kappa}\epsilon^{ij}j^j \ , \tag{1.3b}$$

where κ is the coupling strength, while the matter density and current $j^\mu=(\rho,\mathbf{j})$ are given by

$$\rho=\psi^*\psi,\quad \mathbf{j}=\frac{1}{m}\mathrm{Im}\,\psi^*\mathbf{D}\psi \ . \tag{1.4}$$

They obey a continuity equation, by virtue of (1.1) and (1.3):

$$\partial_t\rho+\nabla\cdot\mathbf{j}=0 \ . \tag{1.5}$$

The system (1.1), (1.3) arises from a Lagrangian and/or Hamiltonian that can be written solely in terms of ψ:

$$L=\int d^2\mathbf{r}\, i\psi^*\partial_t\psi-H \ , \tag{1.6}$$

$$H=\int d^2\mathbf{r}\left[\frac{1}{2m}|\mathbf{D}\psi|^2-\frac{g}{2}\rho^2\right] , \tag{1.7}$$

$$i\partial_t\psi=\frac{\delta H}{\delta\psi^*} \ . \tag{1.8}$$

The vector potential $\mathbf{A}$, which is included in the covariant derivative, is not an independent variable but is expressed in terms of ρ as

$$\mathbf{A}(t,\mathbf{r})=\frac{1}{\kappa}\int d^2\mathbf{r}'\mathbf{G}(\mathbf{r}-\mathbf{r}')\rho(t,\mathbf{r}') \ , \tag{1.9a}$$

where $\mathbf{G}(\mathbf{r})=(1/2\pi)\nabla\times\ln r$, so that $\mathbf{A}$ solves (1.3a). A^0 does not occur in L or H; it appears in the equation of motion when $\mathbf{A}$, present in H and given by (1.9a), is varied with respect to ψ^*. A^0 arises as

$$A^0(t,\mathbf{r})=\frac{1}{\kappa}\int d^2\mathbf{r}'\mathbf{G}(\mathbf{r}-\mathbf{r}')\cdot\mathbf{j}(t,\mathbf{r}') \tag{1.9b}$$

which can be easily shown to solve (1.3b).

Note that in the action formulation (1.6)–(1.8) the gauge freedom of the system (1.1), (1.3) is fixed: the potentials are uniquely prescribed by their integral representations (1.9); equivalently $\nabla\cdot\mathbf{A}=0$ and (1.9) imply boundary conditions on the differential equations (1.3): the potentials are regular at the origin while at a large distance they satisfy

$$\lim_{r\to\infty} A^0(t,\mathbf{r})=0 \ , \tag{1.10a}$$

$$\lim_{r\to\infty} rA^i(t,\mathbf{r})=\frac{1}{2\pi\kappa}\epsilon^{ij}\hat{r}^jN \ . \tag{1.10b}$$

Here $N\equiv\int d^2\mathbf{r}\,\rho$ is the "number", but it need not be an integer.

The identity

$$|\mathbf{D}\psi|^2=|(D_1-i\epsilon(\kappa)D_2)\psi|^2-\epsilon(\kappa)(B\rho+m\nabla\times\mathbf{j}) \tag{1.11}$$

allows presenting the Hamiltonian (1.7) as

$$H=\int d^2\mathbf{r}\left[\frac{1}{2m}|(D_1-i\epsilon(\kappa)D_2\psi|^2-\frac{1}{2}\left[g-\frac{1}{m|\kappa|}\right]\rho^2\right] \tag{1.12}$$

where $\epsilon(\kappa)=\kappa/|\kappa|$ and $\int d^2\mathbf{r}\nabla\times\mathbf{j}$ has been dropped (with the hypothesis that **j** is sufficiently well behaved).

The system possesses a dynamical symmetry that permits SO(2,1) (conformal) redefinitions of the coordinates [1]. The transformations comprise (i) time translation: $t \to t-a, \psi(t,\mathbf{r}) \to \psi(t+a,\mathbf{r})$, generated by H; (ii) dilation:

$$t \to at, \quad \mathbf{r} \to \sqrt{a}\,\mathbf{r}, \quad \psi(t,\mathbf{r}) \to (1/\sqrt{a}\,)\psi(t/a, r/\sqrt{a}\,) ,$$

generated by

$$D = tH - \frac{m}{2}\int d^2\mathbf{r}\cdot\mathbf{j} \tag{1.13}$$

and (iii) a conformal redefinition of time:

$$\frac{1}{t} \to \frac{1}{t} + a, \quad \mathbf{r} \to \frac{r}{1+at},$$

$$\psi(t,\mathbf{r}) \to \frac{1}{1-at}\exp\left[-\frac{imar^2}{2(1-at)}\right]\psi\left[\frac{t}{1-at}, \frac{\mathbf{r}}{1-at}\right]$$

generated by

$$K = -t^2H + 2tD + \frac{m}{2}\int d^2\mathbf{r}\, r^2\rho$$
$$= -t^2H + 2tD + \tfrac{1}{2}M\langle r^2\rangle , \tag{1.14}$$

where

$$\langle r^2\rangle = \frac{1}{N}\int d^2\mathbf{r}\, r^2\rho , \tag{1.15}$$

$$M = Nm . \tag{1.16}$$

The three generators close under commutation on the SO(2,1) Lie algebra, with

$$\mathcal{R} = \frac{1}{2\omega}(H+\omega^2K) \tag{1.17}$$

generating the compact SO(2) rotations. Here ω is an arbitrary parameter, but note from (1.14) that $2\omega R = H + \omega^2K$ (at $t=0$) has the form of a "Hamiltonian" for our system embedded in an external harmonic-oscillator potential of strength $k = m\omega^2$. This fact will be enlarged upon below.

Other symmetry generators of interest are the momentum

$$\mathbf{P} = m\int d^2\mathbf{r}\,\mathbf{j} \tag{1.18}$$

and the angular momentum

$$J = m\int d^2\mathbf{r}\,\mathbf{r}\times\mathbf{j} . \tag{1.19}$$

Also the system is invariant against Galileo boosts.

$$\mathbf{r} \to \mathbf{r} + \mathbf{v}t ,$$

$\psi(t,\mathbf{r}) \to \exp[im(\mathbf{v}\cdot\mathbf{r} - \tfrac{1}{2}v^2t)]\psi(t,\mathbf{r}-\mathbf{v}t)$ generated by

$$\mathbf{B} = t\mathbf{P} - m\int d^2\mathbf{r}\,\mathbf{r}\rho$$
$$= t\mathbf{P} - M\langle\mathbf{r}\rangle . \tag{1.20}$$

We have introduced the center of mass

$$\langle\mathbf{r}\rangle = \frac{1}{N}\int d^2\mathbf{r}\,\mathbf{r}\rho \tag{1.21}$$

which satisfies

$$\frac{d}{dt}\langle\mathbf{r}\rangle = \frac{1}{M}\mathbf{P} . \tag{1.22}$$

Because of (1.5) and (1.18) the last equality holds even in the absence of translation or Galileo invariance.

Since all the above generators are constant in time, it follows from (1.13),(1.14) and (1.20) that on *static* solutions, for which $\int d^2\mathbf{r}\,\mathbf{r}\cdot\mathbf{j}$, $\int d^2\mathbf{r}\, r^2\rho$, and $\int d^2\mathbf{r}\,\mathbf{r}\rho$ are time independent, D, H and $\mathbf{P}$ vanish—in particular, as a consequence of the SO(2,1) symmetry, *static solutions must carry zero energy* [2].

The dynamical symmetry has another consequence that is important for our analysis. Because of time translation invariance, time can be separated in the differential equation (1.1), (1.8). However, the higher symmetry provides other coordinate systems in which separation is possible. In particular upon defining [3]

$$\psi(t,\mathbf{r}) = \frac{1}{\sqrt{1+\omega^2t^2}}\exp\frac{i}{2}\frac{m\omega^2r^2t}{1+\omega^2t^2}$$
$$\times\Psi\left[\frac{1}{\omega}\arctan\omega t, \mathbf{r}/\sqrt{1+\omega^2t^2}\right] \tag{1.23}$$

and substituting this in (1.1), one finds that $\Psi(T,\mathbf{R})$, $T \equiv \frac{1}{\omega}\tan^{-1}\omega t$, $\mathbf{R} = \mathbf{r}/\sqrt{1+\omega^2t^2}$, satisfies

$$iD_T\Psi(T,\mathbf{R}) = -\frac{1}{2m}\mathbf{D}_R^2\Psi(T,\mathbf{R}) - g\rho(T,\mathbf{R})\Psi(T,\mathbf{R})$$
$$+\frac{m\omega^2}{2}R^2\Psi(T,\mathbf{R}) , \tag{1.24}$$

where the gauge potentials, as well as the charge and current densities are constructed as in (1.4) and (1.9) but now from Ψ. Note further that (1.24) may be presented as

$$i\partial_T\Psi = \frac{\delta 2\omega\mathcal{R}}{\delta\Psi^*} \tag{1.25}$$

with $\mathcal{R}$ taken at $T=0$. Equations (1.24), (1.25) show that the T coordinate can be separated.

II. SOLUTIONS

From (1.12) it is seen that nontrivial zero-energy solutions exist only for $g \geq 1/m|\kappa|$. Henceforth we take $g = 1/m|\kappa|$; a choice which renders the static system explicitly integrable, and also is "natural" in the sense that it possesses a hidden supersymmetry and corresponds to a minimal magnetic interaction in a spinorial formulation of the problem [1].

When the last term in (1.12) is absent, static and therefore zero-energy solutions require that ψ obey a self-dual equation

$$[D_1 - i\epsilon(\kappa)D_2]\psi = 0, \quad \mathbf{D}\psi = i\epsilon(\kappa)\mathbf{D}\times\psi . \tag{2.1}$$

Upon defining

$$\psi = \rho^{1/2}e^{i\Omega} \tag{2.2}$$

we further find that ρ satisfies the Liouville equation away from the zeros of ρ,

$$\nabla^2 \ln\rho = -\frac{2}{|\kappa|}\rho \tag{2.3}$$

all of whose solutions are known:

$$\rho(\mathbf{r}) = \frac{4|\kappa|\,|f'(z)|^2}{[1+|f(z)|^2]^2}, \quad z = re^{i\theta} . \tag{2.4}$$

Here $f(z)$ is an arbitrary holomorphic function, so chosen that ρ is nonsingular, thus describing vortex solitons.

It also follows that

$$\mathbf{j} = \frac{1}{2m}\epsilon(\kappa)\nabla\times\rho , \tag{2.5}$$

$$A^0 = \frac{1}{2m|\kappa|}\rho , \tag{2.6a}$$

$$\mathbf{A} = \nabla\Omega - \tfrac{1}{2}\epsilon(\kappa)\nabla\times\ln\rho . \tag{2.6b}$$

There is no equation for Ω; it is chosen to be a harmonic function that ensures $\mathbf{A}$ to be nonsingular at the zeros of ρ.

From the previous general result about static solutions, or by explicit calculation, we learn that H, $\mathbf{P}$, and D vanish. The angular momentum is nonvanishing:

$$J = \epsilon(\kappa)N = -|\kappa|\Phi , \tag{2.7}$$

where Φ is the flux, $\Phi \equiv \int d^2\mathbf{r}\, B$. Also for the conformal and Galileo boost generators we obtain the expressions

$$K = \frac{m}{2}\int d^2\mathbf{r}\, r^2\rho(\mathbf{r}) = \tfrac{1}{2}M\langle r^2\rangle , \tag{2.8}$$

$$\mathbf{B} = -m\int d^2\mathbf{r}\,\mathbf{r}\rho(\mathbf{r}) = -M\langle\mathbf{r}\rangle . \tag{2.9}$$

The last two integrals cannot be evaluated without an explicit form for ρ, viz. for f in (2.4).

The only time-dependent solutions to (1.1) that are known at present are Galileo or conformal boosts of the static solutions [1]. Thus, when $\psi(\mathbf{r})$ is a static solution to (1.1), the following time-dependent function also solves that equation

$$\psi_b(t,\mathbf{r}) = \frac{1}{1-at}e^{i(mv^2/2a)}e^{-i[ma/2(1-at)](\mathbf{r}-\mathbf{v}/a)^2}\psi(\mathbf{r}_b) , \tag{2.10}$$

$$\mathbf{r}_b \equiv \frac{\mathbf{r}-\mathbf{v}t}{1-at} . \tag{2.11}$$

The subscript b denotes quantities that are boosted from their static forms. For this solution, the gauge potentials as well as the charge and current densities can be related to the time-independent expressions by substituting (2.10) into (1.4) and (1.9):

$$\rho_b(t,\mathbf{r}) = \frac{1}{(1-at)^2}\rho(\mathbf{r}_b) , \tag{2.12a}$$

$$\mathbf{j}_b(t,\mathbf{r}) = \frac{1}{(1-at)^3}(\mathbf{j}(\mathbf{r}_b) + (\mathbf{v}-a\mathbf{r})\rho(\mathbf{r}_b)) , \tag{2.12b}$$

$$A_b^0(t,\mathbf{r}) = \frac{1}{(1-at)^2}(A^0(\mathbf{r}_b) + (\mathbf{v}-a\mathbf{r})\cdot\mathbf{A}(\mathbf{r}_b)) , \tag{2.13a}$$

$$\mathbf{A}_b(t,\mathbf{r}) = \frac{1}{1-at}\mathbf{A}(\mathbf{r}_b) . \tag{2.13b}$$

These formulas describe a geometric transformation rule: With

$$x^\mu = (t,\mathbf{r}) \quad X^\mu = \left[\frac{t}{1-at}, \mathbf{r}_b\right]$$

and A_μ the covariant vector $(A^0, -\mathbf{A})$, (2.13) may be presented as

$$A_{\mu b}(x) = A_\nu(\mathbf{r}_b)\frac{\partial X^\nu}{\partial x^\mu} \tag{2.14}$$

while the current $j^\mu = (\rho, \mathbf{j})$ transforms in (2.12) as a contravariant density [as a consequence of the Chern-Simons field-current identity $j^\mu = \kappa\epsilon^{\mu\alpha\beta}\partial_\alpha A_\beta$],

$$j_b^\mu(x) = \frac{1}{\Delta}j^\nu(\mathbf{r}_b)\frac{\partial x^\mu}{\partial X^\nu} \tag{2.15}$$

with $\Delta \equiv \det(\partial x^\mu/\partial X^\mu)$. Note that $\int d^2\mathbf{r}\,\rho_b(t,\mathbf{r}) = \int d^2\mathbf{r}\,\rho(\mathbf{r})$ and thus remains unchanged.

Constants of motion take the following expressions. Momentum and energy no longer vanish:

$$\mathbf{P}_b = M\mathbf{v}_a , \tag{2.16}$$

$$E_b = \tfrac{1}{2}Mv_a^2 + \tfrac{1}{2}Ma^2(\Delta\mathbf{r})^2$$
$$= \frac{P_b^2}{2M} + \tfrac{1}{2}Ma^2(\Delta\mathbf{r})^2 . \tag{2.17}$$

Here $\mathbf{v}_a \equiv \mathbf{v} - a\langle\mathbf{r}\rangle$ and $(\Delta\mathbf{r})^2 \equiv \langle r^2\rangle - \langle\mathbf{r}\rangle^2$. With a pure Galileo boost ($a=0$), we see that the soliton moves as a particle with mass M, while a conformal boost ($a\neq 0$) shifts the velocity, $\mathbf{v}\to\mathbf{v}_a$, and shows that there is internal structure since $(\Delta\mathbf{r})^2\neq 0$. The angular momentum is also boosted:

$$J_b = \langle\mathbf{r}\rangle\times M\mathbf{v}_a + \epsilon(\kappa)N$$
$$= \langle\mathbf{r}\rangle\times\mathbf{P}_b + \epsilon(\kappa)N . \tag{2.18}$$

The dilation generator reads

$$D_b = -\tfrac{1}{2}\langle\mathbf{r}\rangle\cdot\mathbf{P}_b + \tfrac{1}{2}Ma(\Delta\mathbf{r})^2 \tag{2.19}$$

while the conformal and Galileo generators retain their previous, static forms (2.8) and (2.9). [In the Lie algebra these two generators commute.]

Up to now, the discussion has not relied on the explicit form of the solution (2.4). But to give definite values to the various dynamical quantities we must calculate $N\equiv\int d^2\mathbf{r}\,\rho$, $\langle\mathbf{r}\rangle\equiv(1/N)\int d^2\mathbf{r}\,\mathbf{r}\rho$ and $\langle r^2\rangle\equiv(1/N)\int d^2\mathbf{r}\,\mathbf{r}^2\rho$. Since an integral over the plane is involved, there is also the question of convergence. As we shall see, ρ decreases as r^{-4} or faster for large r, so only $\langle r^2\rangle$ is afflicted by a possible divergence. (At finite $\mathbf{r}$, ρ is regular).

The general rotationally symmetric solution is obtained by choosing $f(z)=(z_0/z)^n$, $n=1,2,\ldots$. Then [1]

$$\rho(\mathbf{r})=\frac{4|\kappa|n^2}{r^2}\left[\left[\frac{r}{r_0}\right]^n+\left[\frac{r_0}{r}\right]^n\right]^{-2} \tag{2.20}$$

which behaves at large r as r^{-2n-2}. An explicit evaluation gives

$$N=4\pi n|\kappa| \tag{2.21}$$

$$\langle\mathbf{r}\rangle=0\;, \tag{2.22}$$

$$\langle r^2\rangle=r_0^2\left[\frac{\pi/n}{\sin\pi/n}\right]\;. \tag{2.23}$$

Note that $\langle r^2\rangle$ diverges for $n=1$ where ρ decreases at r^{-4}, converges for $n>1$ since ρ decreases faster than r^{-4}, and approaches r_0^2 as $n\to\infty$. The solution (2.20) describes n vortex solitons, superimposed on each other at the origin, with identical scales (r_0) and no relative phases. The general n-vortex solution makes use of $f(z)=\sum_{i=1}^n c_i/(z-z_i)$ and describes n vortices at location r_i, with different scales and phases coded in the c_i's. One still finds N as in (2.21), $\langle\mathbf{r}\rangle=(1/n)\sum_{i=1}^n\mathbf{r}_i$, where $z_i=x_i+iy_i$, but $\langle r^2\rangle$ diverges for generic parameters because at large distances $f(z)$ behaves as $\sum_{i=1}^n c_i/z$, i.e., like the one-soliton solution with ρ falling as r^{-4}. Convergence requires that all single soliton-vortices "screen" each other, in the sense that their scales and phases sum to zero, $\sum_i c_i=0$. [There may be double- or higher-pole contributions to $f(z)$, describing superimposed solitons, but these do not interfere with convergence of $\langle r^2\rangle$]. In the general case $\langle r^2\rangle$ cannot be expressed in terms of global data, but we can assert that an unscreened single vortex soliton acquires infinite energy when conformally boosted.

III. EXTERNAL HARMONIC FORCE

Analysis of the N-body anyon problem is frequently carried out in the presence of external harmonic forces. This is done so that the continuous anyon energy spectrum may be discretized, and level filling can be discussed [4]. When the quantum-mechanical problem is second quantized one obtains the field-theoretic Lagrangian and Hamiltonian

$$L_\omega=\int d^2\mathbf{r}\, i\psi_\omega^*\partial_t\psi_\omega-H_\omega\;, \tag{3.1}$$

$$H_\omega=\int d^2\mathbf{r}\left[\frac{1}{2m}|\mathbf{D}\psi_\omega|^2-\frac{1}{2m|\kappa|}\rho_\omega^2+\frac{m}{2}\omega^2r^2\rho_\omega\right]\;. \tag{3.2}$$

and the equation of motion reads

$$\begin{aligned}i\partial_t\psi_\omega&=\frac{\delta H_\omega}{\delta\psi_\omega^*}\\&=-\frac{1}{2m}\mathbf{D}^2\psi_\omega+\left[A_\omega^0-\frac{1}{m|\kappa|}\rho_\omega+\frac{m}{2}\omega^2r^2\right]\psi_\omega\;.\end{aligned} \tag{3.3}$$

We have included the nonlinearity (δ- function interaction in the quantum-mechanical problem) with the preferred coupling strength, and the subscript ω indicates quantities in the harmonic-oscillator well of strength $k=m\omega^2$. Gauge potentials as well as the charge and current densities are constructed as before, but now from ψ_ω.

A. Classical Periodic Solutions

Solutions to (3.3) can be obtained by reference to the conformal symmetry of the problem at $\omega=0$. We recognize, as remarked previously, that $H_\omega=2\omega\mathcal{R}$. Moreover the coordinate redefinition (1.23) takes Eq. (1.1) to (1.24), (1.25), which is precisely of the form (3.3). We now reinterpret that result in the following useful manner. Viewing the previous coordinate *redefinition* as a coordinate transformation, the $\omega=0$ problem is mapped onto the problem with harmonic forces. Therefore, by effecting this coordinate transformation on a solution $\psi(t,\mathbf{r})$ to (1.1), we obtain a solution $\psi_\omega(t,\mathbf{r})$ to (3.3) [5].

Referring to (1.23), we may thus assert that given $\psi(t,\mathbf{r})$ solving (1.1), direct substitution verifies that a solution to (3.3) is

$$\psi_\omega(t,\mathbf{r})=\frac{1}{\cos\omega t}\exp\left[-i\frac{m}{2}\omega r^2\tan\omega t\right]\psi(T,\mathbf{r}_\omega)\;, \tag{3.4}$$

$$T=\frac{1}{\omega}\tan\omega t,\quad \mathbf{r}_\omega\equiv\mathbf{r}/\cos\omega t\;. \tag{3.5}$$

[The transformation in Eq. (3.3) is the inverse of that in (1.23), with $T,\mathbf{R}$ renamed at $t,\mathbf{r}$.]

Further it follows that

$$\rho_\omega(t,\mathbf{r})=\frac{1}{\cos^2\omega t}\rho(T,\mathbf{r}_\omega)\;, \tag{3.6a}$$

$$\mathbf{j}_\omega(t,\mathbf{r})=\frac{1}{\cos^3\omega t}(\mathbf{j}(T,\mathbf{r}_\omega)-\omega\mathbf{r}\sin\omega t\,\rho(T,\mathbf{r}_\omega))\;, \tag{3.6b}$$

$$A_\omega^0(t,\mathbf{r})=\frac{1}{\cos^2\omega t}(A^0(T,\mathbf{r}_\omega)-\omega\sin\omega t\,\mathbf{r}\cdot\mathbf{A}(T,\mathbf{r}_\omega))\;, \tag{3.7a}$$

$$\mathbf{A}_\omega(t,\mathbf{r})=\frac{1}{\cos\omega t}\mathbf{A}(T,\mathbf{r}_\omega)\;. \tag{3.7b}$$

These in fact are the same geometric transformation rules as (2.14) and (2.15) except now $X^\mu\equiv(T,\mathbf{r}_\omega)$:

$$A_{\mu\omega}(x)=A_\nu(X)\frac{\partial X^\nu}{\partial x^\mu}\;, \tag{3.8}$$

$$j_\omega^\mu(x)=\frac{1}{\Delta}j^\nu(X)\frac{\partial x^\nu}{\partial X^\nu}\;. \tag{3.9}$$

The coordinate transformation (3.4) is an instance of general time reparametrization:

$$\psi(t,\mathbf{r})\to\sqrt{\dot T(t)}e^{-imr^2\ddot T/4\dot T}\psi(T(t),\sqrt{\dot T}\,\mathbf{r}) \tag{3.10}$$

With $T(t)=t+a,t/a$, and $t/(1-at)$, one obtains the previously mentioned time translation, dilation and conformal transformations, which are symmetries of the dynamics and leave our system invariant. When $T(t)$ is arbitrary, the dynamics is not invariant and our system is mapped onto a different system. In (3.4), with $T(t)=(1/\omega)\tan\omega t$, the mapping introduces a harmonic interaction [3,5].

The energy and angular momentum are still conserved. Using (3.4) to evaluate (3.2) gives

$$E_\omega=(1+\omega^2T^2)E-m\omega^2T\int d^2\mathbf{r}_\omega\mathbf{r}_\omega\cdot\mathbf{j}(T,\mathbf{r}_\omega)+\tfrac{1}{2}m\omega^2\int d^2\mathbf{r}_\omega r_\omega^2\rho(T,\mathbf{r}_\omega)\ . \tag{3.11a}$$

where the current and charge density correspond to a (possibly time-dependent) solution without harmonic forces. The integration variable has been changed from $\mathbf{r}$ to $\mathbf{r}_\omega=\mathbf{r}/\cos\omega t$ and $T=(1/\omega)\tan\omega t$. We may now use the conformal symmetry of dynamics at $\omega=0$ to express the integrals in terms of conformal generators. Equations (1.13) and (1.14) give

$$E_\omega=E+\omega^2K \tag{3.11b}$$

which should be compared to (1.17). For the boosted self-dual solution (2.10), E_ω is evaluated from (2.8) and (2.17) as

$$E_\omega=\tfrac{1}{2}Mv_a^2+\tfrac{1}{2}M\omega^2\langle r^2\rangle+\tfrac{1}{2}Ma^2(\Delta\mathbf{r})^2\ . \tag{3.11c}$$

To evaluate the angular momentum $J_\omega=m\int d^2\mathbf{r}\,\mathbf{r}\times\mathbf{j}_\omega$, we use (3.6b) and find that the harmonic force is invisible:

$$J_\omega=J\ . \tag{3.12a}$$

This is given by (2.18) for the boosted self-dual solution (2.10):

$$J_\omega=\langle\mathbf{r}\rangle\times M\mathbf{v}_a+\epsilon(\kappa)N\ . \tag{3.12b}$$

Although linear momentum is not conserved, it is interesting to evaluate it. We get, with the help of (3.6b),

$$\mathbf{P}_\omega(t)=m\int d^2\mathbf{r}\,\mathbf{j}_\omega(t,\mathbf{r})=\frac{m}{\cos\omega t}\int d^2\mathbf{r}_\omega\mathbf{j}(T,\mathbf{r}_\omega)-m\omega\sin\omega t\int d^2\mathbf{r}_\omega\rho(T,\mathbf{r}_\omega)\ . \tag{3.13a}$$

The first integral is the momentum (1.18), and the second involves a contribution to the boost generator (1.20), both time-independent in the absence of harmonic forces. Thus one has

$$\mathbf{P}_\omega(t)=\frac{1}{\cos\omega t}\mathbf{P}+\omega\sin\omega t(\mathbf{B}-T\mathbf{P})=\mathbf{P}\cos\omega t+\omega\mathbf{B}\sin\omega t\ . \tag{3.13b}$$

For the boosted self-dual solution (2.10), this further becomes, according to (2.9) and (2.16),

$$\mathbf{P}_\omega(t)=M\mathbf{v}_a\cos\omega t-M\langle\mathbf{r}\rangle\omega\sin\omega t\ . \tag{3.13c}$$

It also follows that

$$\langle\mathbf{r}\rangle_\omega\equiv\frac{1}{N}\int d^2\mathbf{r}\,\mathbf{r}\rho_\omega(t,\mathbf{r})=\langle\mathbf{r}\rangle\cos\omega t+\frac{1}{\omega}\mathbf{v}_a\sin\omega t \tag{3.14}$$

so that

$$E_\omega=\frac{P_\omega^2}{2M}+\tfrac{1}{2}M\omega^2\langle\mathbf{r}\rangle_\omega^2+\tfrac{1}{2}M(a^2+\omega^2)(\Delta\mathbf{r})^2\ . \tag{3.15}$$

Thus apart from the internal energy, proportional to $(\Delta\mathbf{r})^2$, the kinematical relations are those of a point particle in a harmonic well; our soliton executes simple harmonic motion.

While explicit forms for the solutions have not been used in the development, nor will they be needed for the semiclassical quantization described below, we note that the simplest solution, the unboosted ($\mathbf{v}_a=0=a$), radially symmetric n-soliton solution of (2.20), gives

$$E_\omega=\tfrac{1}{2}M\omega^2\langle r^2\rangle=\tfrac{1}{2}M\omega^2r_0^2\left[\frac{\pi/n}{\sin\pi/n}\right] \tag{3.16}$$

Hence a single soliton-vortex has an infinite interaction energy with an harmonic oscillator—compare with the discussion at the end of Sec. II.

B. Semiclassical quantization

Regardless of whether the solution in the absence of harmonic forces is static, boosted into a time dependence, or of some other (so far unknown) time-dependent form, its transform (3.4), which solves the same problem with harmonic forces, is always periodic with period $2\pi/\omega$. Therefore semiclassical quantization can be carried out. To this end we integrate the canonical one-form $\int d^2\mathbf{r}\,i\psi_\omega^*\partial_t\psi_\omega$ over the period $2\pi/\omega$, and equate this to $2\pi\mathcal{N}$, where $\mathcal{N}$ is a "principal quantum number," thereby obtaining from (3.1), (3.2) an energy quantization condition [6].

$$E_\omega=\omega\left[\mathcal{N}-\frac{1}{2\pi}\int_0^{2\pi/\omega}dt\,L_\omega\right]\ . \tag{3.17}$$

When the Lagrangian is evaluated on a solution to (3.4) only nonquadratic terms survive:

$$L_\omega=\int d^2\mathbf{r}\left[A_\omega^0-\frac{1}{2m|\kappa|}\rho_\omega\right]\rho_\omega\ . \tag{3.18a}$$

Substituting for A_ω^0 and ρ_ω their expressions (3.6) and (3.7) in terms of the solution at $\omega=0$ and changing the integration variable $\mathbf{r}$ to $\mathbf{r}_\omega=\mathbf{r}/\cos\omega t$ leaves

$$L_\omega=(1+\omega^2T^2)\int d^2\mathbf{r}_\omega\left[A^0(T,\mathbf{r}_\omega)-\frac{1}{2m|\kappa|}\rho(T,\mathbf{r}_\omega)\right]\rho(T,\mathbf{r}_\omega)-\omega^2T\int d^2\mathbf{r}_\omega\mathbf{r}_\omega\cdot\mathbf{A}(T,\mathbf{r}_\omega)\rho(T,\mathbf{r}_\omega)\ . \tag{3.18b}$$

The only solution known at present to the $\omega=0$ problem is the boosted, self-dual one (2.10). Therefore we use (2.12) and (2.13) to express (3.18b) as

$$L_\omega=\frac{1+\omega^2T^2}{(1-aT)^2}\int d^2\mathbf{r}\left[A^0(\mathbf{r})-\frac{1}{2m|\kappa|}\rho(\mathbf{r})\right]\rho(\mathbf{r})+\frac{1}{1-aT}\int d^2\mathbf{r}(\mathbf{v}-a\mathbf{r}-\omega^2T\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\rho(\mathbf{r})\ . \tag{3.18c}$$

The integration variable has been changed from $\mathbf{r}_\omega$ to $\mathbf{r}=(\mathbf{r}_\omega-\mathbf{v}T)/(1-aT)$. In fact (3.18c) vanishes on the self-dual solution: the first integrand is zero, according to (2.6a); the second involves $\int d^2\mathbf{r}\,\mathbf{A}(\mathbf{r})\rho(\mathbf{r})$ and $\int d^2\mathbf{r}\,\mathbf{r}\cdot\mathbf{A}(\mathbf{r})\rho(\mathbf{r})$, which from (1.9a) equal

$$\frac{1}{2\pi\kappa}\int d^2\mathbf{r}\,d^2\mathbf{r}'\frac{\epsilon^{ij}(r-r')^j}{|\mathbf{r}-\mathbf{r}'|^2}\rho(\mathbf{r})\rho(\mathbf{r}')$$

and

$$-\frac{1}{2\pi\kappa}\int d^2\mathbf{r}\,d^2\mathbf{r}'\frac{\mathbf{r}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2}\rho(\mathbf{r})\rho(\mathbf{r}')\ ,$$

respectively, but the integrals vanish due to antisymmetry of the integrand. Thus we conclude that

$$L_\omega=0 \tag{3.18d}$$

and the semiclassical quantization rule (3.17) becomes

$$E_\omega=\omega\mathcal{N} \tag{3.19}$$

which of course coincides with that for a point-particle in a harmonic oscillator potential on the plane.

IV. EXTERNAL MAGNETIC FIELD

The N-body anyon problem in an external magnetic field is another widely studied system [7]. The second-quantized Lagrangian Hamiltonian is

$$L_\omega=\int d^2\mathbf{r}\,i\psi_\omega^*\partial_t\psi_\omega-H_\omega\ , \tag{4.1}$$

$$H_\omega=\int d^2\mathbf{r}\left[\frac{1}{2m}|\mathbf{D}_\omega\psi_\omega|^2-\frac{1}{2m|\kappa|}\rho_\omega^2\right]\ . \tag{4.2}$$

The gauge-covariant derivative now includes a contribution from the external magnetic field $\mathcal{B}$, for which the following vector potential is chosen:

$$\mathcal{A}^i=-\frac{1}{2}\epsilon^{ij}r^j\mathcal{B}=-\frac{m\omega}{2e}\epsilon^{ij}r^j\ , \tag{4.3}$$

$$\omega\equiv\frac{e\mathcal{B}}{m}\ , \tag{4.4}$$

$$\mathbf{D}_\omega\equiv\nabla-i\,\mathbf{A}_\omega-ie\mathcal{A}\ . \tag{4.5}$$

The subscript ω now denotes quantities in the presence of the external magnetic field $\mathcal{B}=m\omega/e$. The equation of motion becomes

$$i\partial_t\psi_\omega=\frac{\delta H_\omega}{\delta\psi_\omega^*}=-\frac{1}{2m}\mathbf{D}_\omega^2\psi_\omega+\left[A_\omega^0-\frac{1}{m|\kappa|}\rho_\omega\right]\psi_\omega\ , \tag{4.6}$$

where A_ω^μ is still given by (1.9), with sources j_ω^μ constructed from ψ_ω as in (1.4).

A. Classical periodic solutions

It has recently been shown that Eq. (4.6) admits time-dependent solutions that are constructed from solutions $\psi(t,\mathbf{r})$ at $\mathcal{B}=0$ by a periodic coordinate transformation. The following $\psi_\omega(t,\mathbf{r})$ solves (4.6) [5]:

$$\psi_\omega(t,\mathbf{r})=\frac{1}{\cos\frac{\omega}{2}t}e^{-im\omega(r^2/4)\tan(\omega/2)t}e^{i(N/4\pi\kappa)\omega t}\psi(T,\mathbf{R}), \tag{4.7}$$

$$T=\frac{2}{\omega}\tan\frac{\omega}{2}t,\quad R^i(t)=\frac{1}{\cos\frac{\omega}{2}t}R^{ij}\left[\frac{\omega}{2}t\right]r^j\ . \tag{4.8}$$

The coordinate transformation involves the same rescaling as in (3.3), but with half the frequency, followed by rotation through the angle $(\omega/2)t$:

$$R^{ij}(\alpha)=\delta^{ij}\cos\alpha-\epsilon^{ij}\sin\alpha\ . \tag{4.9}$$

Moreover, it is necessary to perform a gauge transformation; this gives the additional phase factor $e^{i(N/4\pi\kappa)\omega t}$ in (4.7), which is needed because our formalism is gauge fixed.

Other dynamical quantities take the form

$$\rho_\omega(t,\mathbf{r})=\frac{1}{\cos^2\frac{\omega}{2}t}\rho(T,\mathbf{R})\ , \tag{4.10a}$$

$$j_\omega^i(t,\mathbf{r})=\frac{1}{\cos^3\frac{\omega}{2}t}\left[j^j(T,\mathbf{R})+\frac{\omega}{2}\epsilon^{jk}r^k\rho(T,\mathbf{R})\right]\times R^{ji}\left[\frac{\omega}{2}t\right]\ , \tag{4.10b}$$

$$A_\omega^0(t,\mathbf{r})=\frac{1}{\cos^2\frac{\omega}{2}t}\left[A^0(T,\mathbf{R})-\frac{\omega}{2}\mathbf{r}\times\mathbf{A}(T,\mathbf{R})\right]-\frac{\omega N}{4\pi\kappa}\ , \tag{4.11a}$$

$$A_\omega^i(t,\mathbf{r})=\frac{1}{\cos\frac{\omega}{2}t}A^j(T,\mathbf{R})R^{ji}\left[\frac{\omega}{2}t\right]\ . \tag{4.11b}$$

While the current transformation rule is still geometric as in (3.9) with

$$X^\mu=\left[\frac{2}{\omega}\tan\frac{\omega}{2}t,\frac{1}{\cos\frac{\omega}{2}t}R^{ij}(\omega t/2)r^j\right]\equiv(T,\mathbf{R})\ ,$$

the gauge potentials undergo an additional gauge transformation, which complements the above-mentioned phase factor of ψ_ω:

$$A_{\mu\omega}(x)=A_\nu(X)\frac{\partial X^\nu}{\partial x^\nu}-\frac{\partial}{\partial x^\mu}\left[\frac{\omega}{4\pi\kappa}Nt\right]\ . \tag{4.12}$$

The kinematical properties of the solution are coded in the constants of motion, which here comprise the energy, momentum, and angular momentum. The energy is given by (4.2), which may also be written as

$$E_\omega=\frac{1}{2m}\int d^2\mathbf{r}|(D_{\omega1}-i\epsilon(\kappa)D_{\omega2})\psi_\omega|^2-\epsilon(\kappa)\frac{\omega}{2}N \tag{4.13}$$

while the other two are given by expressions that include explicit contribution from the external magnetic field:

$$P^i_\omega = m\int d^2\mathbf{r}\, j^i_\omega - m\omega\int d^2\mathbf{r}\,\epsilon^{ij} r^j \rho_\omega\ , \tag{4.14}$$

$$J_\omega = m\int d^2\mathbf{r}\,\mathbf{r}\times\mathbf{j}_\omega + \tfrac{1}{2}m\omega\int d^2\mathbf{r}\, r^2\rho_\omega\ . \tag{4.15}$$

Substituting the $\omega=0$ formulas from (4.7), (4.10), and (4.11) for the appropriate expressions in (4.13)–(4.15) gives, after a change of integration variable for $\mathbf{r}$ to $\mathbf{R}$,

$$E_\omega = (1+\tfrac{1}{4}\omega^2T^2)E + \frac{m\omega^2}{8}\int d^2\mathbf{R}\, R^2\rho(T,\mathbf{R}) - \frac{m\omega^2 T}{4}\int d^2\mathbf{R}\,\mathbf{R}\cdot\mathbf{j}(T,\mathbf{R}) - \frac{m\omega}{2}\int d^2\mathbf{R}\,\mathbf{R}\times\mathbf{j}(T,\mathbf{R})\ , \tag{4.16}$$

$$P^i_\omega = \frac{1}{\cos\frac{\omega}{2}t} P^j R^{ji}\left(\frac{\omega}{2}t\right) - \frac{1}{2}m\omega\epsilon^{ij}\int d^2\mathbf{R}\, R^j\rho(T,\mathbf{R}), \tag{4.17}$$

$$J_\omega = J\ . \tag{4.18}$$

Again the angular momentum coincides with its $\omega=0$ value, while the integrals in the formulas for E_ω and $\mathbf{P}_\omega$ can be expressed in terms of the $\omega=0$ constants of motion from (1.13), (1.14), (1.19) and (1.20):

$$E_\omega = E + \frac{\omega^2}{4}K - \frac{\omega}{2}J\ , \tag{4.19}$$

$$P^i_\omega = P^i + \frac{\omega}{2}\epsilon^{ij}B^j\ . \tag{4.20}$$

Thus when the $\omega=0$ solution is the boosted static solution, we get, from (2.8), (2.9), (2.16), and (2.17),

$$E_\omega = \frac{1}{2}M\mathbf{v}_a^2 + \frac{1}{8}M\omega^2\langle\mathbf{r}\rangle^2 - \frac{\omega}{2}\langle\mathbf{r}\rangle\times M\mathbf{v}_a - \frac{\omega}{2}\epsilon(\kappa)N + \frac{M}{2}\left[a^2+\frac{\omega^2}{4}\right](\Delta\mathbf{r})^2\ , \tag{4.21}$$

$$P^i_\omega = Mv^i_a - \frac{1}{2}M\omega\epsilon^{ij}\langle r^j\rangle\ , \tag{4.22}$$

$$J_\omega = \langle\mathbf{r}\rangle\times M\mathbf{v}_a + \epsilon(\kappa)N\ . \tag{4.23}$$

Note that

$$E_\omega + \omega J_\omega = \frac{P_\omega^2}{2M} + \frac{\omega}{2}\epsilon(\kappa)N + \frac{M}{2}\left[a^2+\frac{\omega^2}{4}\right](\Delta\mathbf{r})^2\ . \tag{4.24}$$

This shows that apart from the internal energy, proportional to $(\Delta\mathbf{r})^2$, the kinematics of our vortex soliton is that of a point-particle, with mass M, spin $\epsilon(\kappa)N$ and unit g factor, moving in an external magnetic field.

For the static, unboosted, radially symmetric solution (2.20), we have

$$E_\omega = \tfrac{1}{8}M\omega^2\langle r^2\rangle - \frac{\omega}{2}\epsilon(\kappa)N = \tfrac{1}{8}M\omega^2 r_0^2\left[\frac{\pi/n}{\sin\pi/n}\right] - 2\pi\omega|\kappa|\ . \tag{4.25}$$

which diverges for $n=1$. As seen previously, the single vortex-soliton has infinite interaction energy.

B. Semiclassical quantization

Just like the harmonic-oscillator solution, the one in the external magnetic field is periodic. Note that

$$\frac{1}{\cos\frac{\omega}{2}t}R^{ij}\left(\frac{\omega}{2}t\right) = \delta^{ij} - e^{ij}\tan\frac{\omega}{2}t\ ;$$

therefore, (4.10) shows that coordinates $T,\mathbf{R}$ are periodic with period $2\pi/\omega$ [8]. The Jacobian factor in (4.7), viz. $1/\cos\frac{1}{2}\omega t$, really enters with an absolute value [compare (3.10)]; hence the period of $\psi_\omega(t,\mathbf{r})$ is $2\pi/\omega$ and the semiclassical quantization is, as in (3.17);

$$E_\omega = \omega\left[\mathcal{N} - \frac{1}{2\pi}\int_0^{2\pi/\omega} dt\, L_\omega\right] \tag{4.26}$$

When the Lagrangian is evaluated on a solution we have, as in (3.18a),

$$L_\omega = \int d^2\mathbf{r}\left[A^0_\omega - \frac{1}{2m|\kappa|}\rho_\omega\right]\rho_\omega\ . \tag{4.27a}$$

which is evaluated in terms of the $\omega=0$ solution to be

$$L_\omega = (1+\tfrac{1}{4}\omega^2T^2)\int d^2\mathbf{R}\left[A^0(T,\mathbf{R}) - \frac{1}{2m|\kappa|}\rho(T,\mathbf{R})\right]\rho(T,\mathbf{R}) - \frac{\omega}{2}\int d^2\mathbf{R}\left[\mathbf{R}\times\mathbf{A}(T,\mathbf{R}) + \frac{\omega}{2}T\mathbf{R}\cdot\mathbf{A}(T,\mathbf{R}) + \frac{N}{2\pi\kappa}\right]\rho(T,\mathbf{R})\ , \tag{4.27b}$$

where the change of integration variables from r^i to $R^i = R^{ij}(\omega t/2)r^j/\cos\omega t/2$ has been made. The $\omega=0$ solution is taken to be the boosted static solution, so (4.27b) becomes evaluated from (2.12) and (2.13), after another change of variables from $\mathbf{R}$ to $\mathbf{r}=(\mathbf{R}-\mathbf{v}T)/(1-aT)$, as

$$L_\omega = \frac{1+\frac{1}{4}\omega^2T^2}{(1-aT)^2}\int d^2\mathbf{r}\left[A^0(\mathbf{r}) - \frac{1}{2m|\kappa|}\rho(\mathbf{r})\right]\rho(\mathbf{r}) + \frac{1+\frac{1}{4}\omega^2T^2}{1-aT}\int d^2\mathbf{r}(\mathbf{v}-a\mathbf{r})\cdot\mathbf{A}(\mathbf{r}) - \frac{\omega}{2}\int d^2\mathbf{r}\left[\left[\mathbf{r}+\frac{\mathbf{v}T}{1-aT}\right]\times\mathbf{A}(\mathbf{r}) + \frac{\omega}{2}T\left[\mathbf{r}+\frac{\mathbf{v}T}{1-aT}\right]\cdot\mathbf{A}(\mathbf{r}) + \frac{N}{2\pi\kappa}\right]\rho(\mathbf{r}) \tag{4.27c}$$

The first integral vanishes according to (2.6a). In the integrals over $\mathbf{A}$, both $\int d^2\mathbf{r}\,\mathbf{A}(\mathbf{r})\rho(\mathbf{r})$ and $\int d^2\mathbf{r}\,\mathbf{r}\cdot\mathbf{A}(\mathbf{r})\rho(\mathbf{r})$ vanishes as before, while $\int d^2\mathbf{r}\,\mathbf{r}\times\mathbf{A}(\mathbf{r})\rho(\mathbf{r})$ is given by (1.9a) as

$$-\frac{1}{2\pi\kappa}\int d^2\mathbf{r}\,d^2\mathbf{r}'\frac{\mathbf{r}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\rho(\mathbf{r})\rho(\mathbf{r}')\ .$$

Since

$$\mathbf{r}\cdot(\mathbf{r}-\mathbf{r}')=\tfrac{1}{2}(r^2-r'^2)+\tfrac{1}{2}|\mathbf{r}-\mathbf{r}'|^2\ ,$$

the first term leads to an antisymmetric integrand and a vanishing integral, while the second term gives

$$-\frac{1}{4\pi\kappa}\int d^2\mathbf{r}\,d^2\mathbf{r}'\rho(\mathbf{r})\rho(\mathbf{r}')=-\frac{N^2}{4\pi\kappa}\ ,$$

thus contributing $\omega N^2/8\pi\kappa$ to (4.27c). The last integral in (4.27c) gives $-\omega N^2/4\pi\kappa$, for a final result of

$$L_\omega=-\frac{\omega N^2}{8\pi\kappa}\ . \tag{4.27d}$$

Therefore the quantization condition (4.26) becomes [9]

$$E_\omega=\omega\left[\mathcal{N}+\frac{N^2}{8\pi\kappa}\right]\ . \tag{4.28}$$

C. Discussion

The external magnetic field $\mathcal{B}$ is coupled minimally to the vortex-soliton system in (4.1), (4.2) and (4.6). However, there exists a "natural" nonminimal coupling, which is worth discussing.

Recall that the contact interaction terms in (4.2) arises in a spinorial formulation [1] as a non-minimal magnetic interaction $[\epsilon(\kappa)/2m]B\rho_\omega$; this equals $-(1/2m|\kappa|)\rho_\omega^2$ by virtue of the Chern-Simons relation (1.3a). Therefore a spinorial formulation for the external field problem would lead to the nonminimal term

$$\frac{\epsilon(\kappa)}{2m}(B+e\mathcal{B})\rho_\omega=-\frac{1}{2m|\kappa|}\rho_\omega^2+\frac{1}{2}\epsilon(\kappa)\omega\rho_\omega\ .$$

Thus, a "natural" modification of the previous theory is to supplement the Hamiltonian (4.2) by $\frac{1}{2}\epsilon(\kappa)\omega\int d^2\mathbf{r}\,\rho_\omega$; i.e., replace (4.2) by

$$H'_\omega=\int d^2\mathbf{r}\left[\frac{1}{2m}|\mathbf{D}_\omega\psi'_\omega|^2-\frac{1}{2m|\kappa|}\rho'^2_\omega+\frac{1}{2}\epsilon(\kappa)\omega\rho'_\omega\right]. \tag{4.29}$$

(The primes indicate that a modified model is under discussion.) This leads to the equation of motion

$$i\partial_t\psi'_\omega=\frac{\delta H'_\omega}{\delta\psi'^*}=-\frac{1}{2m}\mathbf{D}^2_\omega\psi'_\omega+\left[A'^0_\omega-\frac{1}{m|\kappa|}\rho'_\omega+\frac{1}{2}\epsilon(\kappa)\omega\right]\psi'_\omega \tag{4.30}$$

instead of (4.6).

But now we see that the altered dynamics produces a trivial modification: ψ'_ω is given by $\exp[-i\frac{1}{2}\epsilon(\kappa)\omega t]\psi_\omega$, where ψ_ω solves (4.6).

The altered dynamics does produce a change in the energy:

$$E'_\omega=E_\omega+\tfrac{1}{2}\epsilon(\kappa)\omega N\ . \tag{4.31}$$

But one can check that the semiclassical quantization condition for E'_ω is still (4.28).

A related point is that the Hamiltonian H'_ω may also be written, apart from boundary terms, as

$$H'_\omega=\frac{1}{2m}\int d^2\mathbf{r}|(D_{\omega1}-i\epsilon(\kappa)D_{\omega2})\psi'_\omega|^2\ ; \tag{4.32}$$

compare (4.13).

This suggests looking for a self-dual solution to the external field problem. Indeed, such solutions have already appeared in the literature [10], both for the attractive nonlinearity as in the model here considered, and also for the related model with repulsive nonlinearity. The repulsive model is of course different from the attractive model, which we have discussed in the main body of this paper. However, the two are connected by the spinorial formulation [1]. Recall that the spinorial formulation involves a two-component spinor χ, with decoupled dynamics for the two components. The nonlinearity arises as $B\chi^\dagger\sigma^3\chi=-(1/\kappa)(\chi^\dagger\chi)(\chi^\dagger\sigma^3\chi)$, and is attractive for one component and repulsive for the other. We do not describe further these topics, since they are discussed by others [10].

V. CONCLUSIONS

Coordinate transformations on the planar, gauged, nonlinear Schrödinger equation insert an external harmonic force or an external magnetic field into the dynamics. Correspondingly, solutions of the equation without external fields are transformed into periodic solutions with external fields. In particular vortex-soliton configurations bind to the external fields, but single vortices carry infinite binding energy.

The periodic bound-state solutions can be quantized semiclassically by the Bohr-Sommerfield procedure. The resulting energy spectra, Eqs. (3.19) and (4.28), are in qualitative agreement with known solutions to the quantum mechanical problem [4,7,11]. Although our quantization is performed on transforms of Galileo and conformally boosted solutions to equations without external fields, the boosting does not affect the final result, which could just as well be obtained by starting with vortices that are static in the absence of external fields.

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