Partition function for $N=1, c=3/2$ superconformal model on the supertorus with two punctures

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In view of the Krichever-Novikov (KN) framework, we calculate the partition function for the $N=1, c=3/2$ superconformal model on the supertorus with two distinguished punctures. According to boundary conditions for fermionic fields around two punctures, partition functions are classified into the partition function for the Neveu-Schwarz sector Z_{NS} and the partition function for the Ramond sector Z_R. Considering the two insertions of fields as boundary conditions for the partition function in the KN construction, Z_{NS} (Z_{R}) is identical to the two-point function of vortex operators (spin fields) on the supertorus.

I. INTRODUCTION

Recently Krichever and Novikov (KN) have introduced a new kind of algebra onto a Riemann surface of genus g, as a natural extension of the Virasoro algebra [1]. The KN algebra plays an essential role in the study of conformal field theories over high-genus $(g \geq 1)$ Riemann surfaces. In detail, the KN algebra and its representation should provide a basis framework for an operator description of strings on higher-genus Riemann surfaces, as does the Virasoro algebra for genus 0.

On the other hand, the conformal algebra has supersymmetric generalizations known as N-extended superconformal algebras. We here consider the $N=1$ superconformal algebra, which describes spinning (supersymmetric) strings [2]. The Euclidean world sheets of closed oriented spinning strings are just $N=1$ super Riemann surfaces. Note that the higher-genus $(g \geq 2)$ $N=1$ super Riemann surfaces remain unexplored, while the $N=1$ supertorus is well known [3,4].

In this paper, we wish to calculate the partition function for $N=1, c=3/2$ superconformal models on the supertorus with two punctures. It is emphasized that the KN construction needs Riemann surfaces with two distinguished points (two punctures) to incorporate a global definition of time on Riemann surfaces. For example, on a surface with two punctures it is possible to define "time" so that one of these points lies in the infinite past $(t=-\infty)$ and the other lies in the infinite future $(t=\infty)$. Note that Polyakov amplitudes of the closed oriented bosonic string on Riemann surfaces with genus g and p punctures is identical to the amplitudes for the scattering of p physical on-shell strings, which is calculated by inserting p vertex operators on Riemann surfaces of genus g without punctures [5]. Therefore, with the rest of the paper, we will show whether or not the partition function for $N=1, c=3/2$ superconformal models on the supertorus with two punctures is identical to the two-point function on the supertorus without punctures.

The organization of this paper is as follows. In Sec. II we review the $N=1$ supertorus, and an $N=1$ supertorus with two punctures is discussed in Sec. III. We calculate in Sec. IV the partition function for a doubly periodic boson coupled to the NS sector by introducing two magnetic (vortex) operators at two punctures. In Sec. V we obtain partition function for a doubly periodic boson coupled to the R sector by using the spin fields. Finally we discuss results in Sec. VI.

II. $N=1$ SUPERTORUS

We begin reviewing the results on the uniformization theorem of genus-1 super Riemann surfaces (supertorus). A supertorus is obtained as the quotient of the complex superplane (CSP) with coordinates (z,θ) by a supergroup $G=\mathrm{Osp}(1,2)$ of superconformal transformations of the form

$$z'=\frac{az+b}{cz+d}+\theta\frac{\gamma z+\delta}{(cz+d)^2}\,,\qquad \theta'=\frac{\gamma z+\delta}{cz+d}+\frac{\theta}{cz+d}\left(1+\tfrac{1}{2}\delta\gamma\right) \tag{1}$$

with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,Z)\,.$$

Because a subgroup of G on the supertorus $[=\mathrm{SPL}(2,C)]$ is isomorphic to a fundamental group of a torus, it must be Abelian and has precisely two commuting generators. Furthermore, it can be chosen to preserve the flat supergeometry on the CSP characterized by the complete frame fields and its dual fields [4,6,7]:

$$E^{\theta}=d\theta,\quad E^{\bar\theta}=d\bar\theta,\quad E^{+}=dz+\theta\,d\theta,\quad E^{-}=d\bar z-\bar\theta\,d\bar\theta$$

and

$$E_{\theta}=D_{\theta}^2=\partial_z,\quad E_{\bar\theta}=D_{\bar\theta}^2=-\partial_z\,,\qquad E_{+}=D_{\theta}=\partial_{\theta}+\theta\partial_z,\quad E_{-}=D_{\bar\theta}=\partial_{\bar\theta}-\bar\theta\partial_z\,. \tag{2}$$

The generators of this subgroup can be given by

$$z'=z+1,\quad \theta'=\theta;\quad z'=z+\tau+\theta\delta,\quad \theta'=\theta+\delta \tag{3}$$

for the odd-spin structure [(+,+) boundary conditions] and

$$z'=z+1,\quad \theta'=\theta,\quad z'=z+\tau,\quad \theta'=-\theta \tag{4}$$

for the even-spin structure with (+,−) boundary conditions. The other even-spin structures are obtained by replacing signs (−,−) or (−,+) in the transformations of θ in (4). The even-spin structures are just the superspace version of a torus and lead to no difficulty. However, the periodicity of (3) for the odd-spin structure induces some problems. For example, the requirement of the periodicity (3) on a scalar superfield gives boundary conditions that mix the component fields. Avoiding this difficulty, we introduce the new coordinates (ω,ϕ) related to the (z,θ) as [7]

$$z=\omega+\phi\delta\frac{\omega_I}{\tau_I},\quad \theta=\phi+\delta\frac{\omega_I}{\tau_I}\ , \tag{5}$$

where ω_I and τ_I denote the imaginary part of ω and τ. Rewriting ω and ϕ in terms of $(z,\bar{z},\theta,\bar{\theta},\delta,\bar{\delta},\tau,\bar{\tau})$ we have

$$\begin{aligned}\omega&=z-\frac{z_I}{\tau_I}\theta\delta\left[1+\frac{i\bar{\theta}\bar{\delta}}{2\tau_I}\right],\\ \phi&=\theta-\frac{z_I}{\tau_I}\delta\left[1+\frac{i\bar{\theta}\bar{\delta}}{2\tau_I}\right].\end{aligned} \tag{6}$$

In this system the periodicity (3) reduces to

$$\omega'=\omega+1,\quad \phi'=\phi,\quad \omega'=\omega+\tau,\quad \phi'=\phi\ . \tag{7}$$

On the other hand, the supermodular transformations on the odd supertorus are given by

$$\tau'=\frac{a\tau+b}{c\tau+d},\quad \delta'=\frac{\delta}{(c\tau+d)^{3/2}}\ , \tag{8}$$

when τ (δ) is an even (odd) modular parameter. Physically, τ and δ correspond to zero modes of a two-dimensional graviton and gravitino field. In terms of a new parameter $T\equiv\tau+\theta\delta$, these can be described by

$$T'=\frac{aT+b}{cT+d} \tag{9}$$

which is exactly the same form as the modular transformation law on the torus (see Fig. 1).

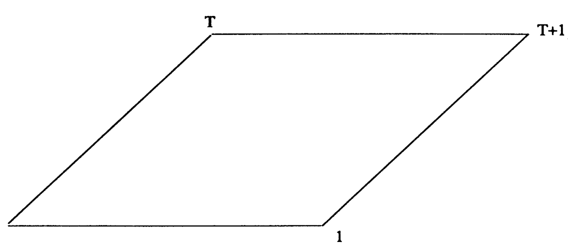

FIG. 1. The supertorus of odd-spin structure is effectively described by the torus with the new modular parameter $T=\tau+\theta\delta$ instead of τ.

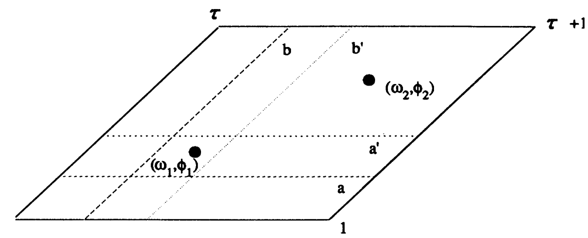

FIG. 2. Four independent cycles a,a',b,b' on a torus with two punctures in the (ω,ϕ) coordinates. The punctures (multipoles) are denoted by two ●'s.

III. $N=1$ SUPERTORUS WITH TWO PUNCTURES

As was discussed, the role of two punctures on super Riemann surfaces plays a crucial role in KN algebra construction. These punctures provide the concept of "time" on the Euclidean world sheets of closed oriented spinning strings. Moreover, note that the structure constants of the KN algebra depend on the chosen puncture as well as the periods of the supertorus. We now label the two punctures as (z_1,θ_1) and (z_2,θ_2). On a doubly punctured supertorus, there are in fact four independent cycles a,a',b,b' as shown in Fig. 2. Correspondingly, there are eight possible spin structures, compared with four spin structures for an ordinary supertorus [8]:

$$\text{NS}\qquad \begin{array}{llllll} a: & + & - & - & + & \\ a': & + & - & - & +\ , & \\ b: & - & + & - & + & \\ b': & - & + & - & + & \end{array} \tag{10}$$

$$\text{R}\qquad \begin{array}{llll} + & - & - & + \\ - & + & + & -\ . \\ - & + & - & + \\ + & - & + & - \end{array} \tag{11}$$

The first four of these spin structures precisely conincide with the four spin structures for a supertorus without punctures, which we have already described in Sec. II. Hence, we here refer to these as the three even- and one odd-spin structures. Furthermore, under parallel transport around either puncture, there is no change of sign. So, we refer to this set of spin structures as the Neveu-Schwarz (NS) sector. The remaining four spin structures appear new. Under parallel transport around either puncture each of these spin structures does result in a change of sign. Hence, we designate this set of spin structures as the Ramond (R) sector. To be specific, we remark that under parallel transport around the cycles a,a',b,b' a spinor field $\psi(z)$ in this sector can change by factors of ±1. Note that for the case of a gravitino field with even-spin structures of the NS sector, there is at least one cycle around which the field changes sign under parallel transport. Such a field cannot have a zero mode (constant mode) and thus there are no odd modular parameters for even-spin structures. In the R sector there is no analogue of the Grassmann-odd modular parameter δ, since there is always at least one cycle around which a spinor changes sign under parallel transport.

IV. PARTITION FUNCTION FOR A DOUBLY PERIODIC BOSON COUPLED TO THE NS SECTOR

The action for the $N=1,c=3/2$ superconformal model on the odd-spin structure of supertorus is given by

$$A_{(++,++)}=\frac{g}{2\pi}\int_{ST}d^4Z\, D_\theta S\, D_{\bar\theta}S\ . \tag{12}$$

In order to obtain a simple situation on the boundary conditions, using (5), we can transform (12) into the action on the ordinary torus

$$\begin{aligned}A_{(++,++)}=\frac{g}{2\pi}\int_T d^2\omega\,\Bigg[&\partial_\omega\Phi_1\partial_{\bar\omega}\Phi_1+\bar\psi_1\partial_\omega\bar\psi_1-\psi_1\partial_{\bar\omega}\psi_1\\&+\frac{\bar\delta\delta}{2\tau_1^2}\bar\psi_1\psi_1-F_1^2-\frac{i\delta}{\tau_I}\psi_1\partial_\omega\Phi_I\\&-\frac{i\bar\delta}{\tau_I}\bar\psi_1\partial_{\bar\omega}\Phi_1\Bigg]\\&+g\frac{\bar\delta\delta}{4\pi\tau_I}\bar\psi_0\psi_0-\frac{g\tau_I F_0^2}{2\pi}\ ,\end{aligned} \tag{13}$$

where we have split $S\ (=S_0+S_1)$ into a zero-mode part S_0 and a non-zero-mode part S_1. The component expansion of superfield S is defined by

$$\begin{aligned}S(W,\bar W)=&\Phi(\omega,\bar\omega)+\phi\psi(\omega,\bar\omega)-\bar\phi\,\bar\psi(\omega,\bar\omega)\\&+\bar\phi\phi F(\omega,\bar\omega)\ .\end{aligned} \tag{14}$$

Here we can eliminate the auxiliary field F by its equation of motion. The partition function for Φ_1 leads to

$$Z_0=\int D\Phi_1 e^{-A_{\Phi_1}}=\left[\frac{g}{2\tau_I}\right]^{1/2}\frac{1}{|\eta(\tau)|^2}\ , \tag{15}$$

where

$$A_{\Phi_1}=\frac{g}{2\pi}\int_T d^2\omega\,\partial_\omega\Phi_1\partial_{\bar\omega}\Phi_1$$

and η is the Dedekind η function. To calculate the fermionic part, let us define the measure for the zero modes as

$$DS_0=C\,dF_0 d\psi_0 d\bar\psi_0,\quad C=\text{const}\ . \tag{16}$$

For convenience, we choose C such that the partition function takes the form

$$Z_1=\int DS_0 D[\psi_1,\bar\psi_1]e^{-A_r}=\frac{\bar\delta\delta}{\tau_I}|\eta(\tau)|^2\ , \tag{17}$$

where

$$A_I=A_{(++,++)}-A_{\Phi_I}\ .$$

Note that there is no contribution from the δ, $\bar\delta$-dependent terms because of the presence of the $\bar\psi_0\psi_0$ term. Also we recall that there is no contribution to the partition function from the doubly periodic free fermion action (Ising model) on the torus, due to the absence of the zero-mode term in action [9]. However, on the supertorus, there is a direct contribution from the fermion action to the partition function because of the presence of the zero-mode term. Now we wish to take into account the classical part of the action since in a finite geometry the boundary conditions generate various constraints. This comes from classical solutions and their winding on the nontrivial homology cycles of the supertorus [10]. Then, for variations along two cycles $(1,\tau)$

$$\begin{aligned}\delta\Phi_{\rm cl}&=\Phi_{\rm cl}(\omega+1)-\Phi_{\rm cl}(\omega)\\&=\int_1(\partial_z\Phi_{\rm cl}dz+\partial_{\bar z}\Phi_{\rm cl}d\bar z)=2\pi m\ ,\end{aligned} \tag{18}$$

$$\begin{aligned}\delta'\Phi_{\rm cl}&=\Phi_{\rm cl}(\omega+\tau)-\Phi_{\rm cl}(\omega)\\&=\int_\tau(\partial_z\Phi_{\rm cl}dz+\partial_{\bar z}\Phi_{\rm cl}d\bar z)=2\pi m'\ ,\end{aligned} \tag{19}$$

the corresponding continuum limit is the frustrated partition function

$$Z_{m,m'}=\exp(-A_{\rm cl})\ , \tag{20}$$

where

$$A_{\rm cl}=\frac{\pi g}{2}\frac{|m'-m\tau|^2}{\tau_I}\ . \tag{21}$$

Further, in order to account for the poles of two punctures at (ω_1,ϕ_1) and (ω_2,ϕ_2), we have to introduce a two-dimensional Coulomb gas formalism. In the Coulomb gas formalism, the two essential operators are the "vertex operator" (O_{e0}) and the "vortex operator" (O_{0n}). Here the subscripts e (n) denote the electric charge (magnetic charge) in Coulomb gas representation. At this stage we need the magnetic (vortex) operators O_{0n}, which create the discontinuity of $2\pi n$ on a cut relating between (ω_1,ϕ_1) and (ω_2,ϕ_2). These operators may take into account appropriately the unique meromorphic one-form with simple poles and residues $\pm n$ at (ω_1,ϕ_1) and (ω_2,ϕ_2) [5]. For this purpose, we introduce another classical field [6]

$$\begin{aligned}S_{\rm cl}=n\Bigg[&\operatorname{Im}\ln\left[\frac{\theta_1(\omega-\omega_1-\phi\phi_1|\tau)}{\theta_1(\omega-\omega_2-\phi\phi_2|\tau)}\right]\\&-\frac{2\pi}{\tau_I}\operatorname{Im}[\omega+\tfrac{1}{2}(\phi_1+\phi_2)\phi]\operatorname{Re}\omega_{12}\\&-2\pi\operatorname{Re}[\phi(\phi_1-\phi_2)+\phi_1\phi_2]\Bigg]\end{aligned} \tag{22}$$

with

$$\omega_{12}=\omega_1-\omega_2-\phi_1\phi_2\ .$$

This is doubly periodic

$$S_{\rm cl}(\omega+1,\phi)=S_{\rm cl}(\omega,\phi)\ ,$$

$$S_{\rm cl}(\omega+\tau,\phi)=S_{\rm cl}(\omega,\phi)$$

and singular at (ω_1,ϕ_1) and (ω_2,ϕ_2). Note that $S_{\rm cl}$ satisfies

$$D_\phi D_{\bar\phi} S_{cl}=0 \quad \text{at } (\omega,\phi)\neq(\omega_1,\phi_1),(\omega_2,\phi_2) .$$

The desired discontinuities around (ω_1,ϕ_1) and (ω_2,ϕ_2) appear as (see Fig. 3)

$$\int_{(\omega_1,\phi_1)\Longrightarrow c_1}(D_\phi S_{cl} d\omega\, d\phi + D_{\bar\phi} S_{cl} d\bar\omega\, d\bar\phi)=2\pi n , \tag{23}$$

$$\int_{(\omega_2,\phi_2)\Longrightarrow c_2}(D_\phi S_{cl} d\omega\, d\phi + D_{\bar\phi} S_{cl} d\bar\omega\, d\bar\phi)=-2\pi n . \tag{24}$$

In order to calculate the corresponding partition function, we should introduce the other field such as $\bar S_{cl}$:

$$\bar S_{cl}=n\left[\ln\left|\frac{\theta_1(\omega-\omega_1-\phi\phi_1|\tau)}{\theta_1(\omega-\omega_2-\phi\phi_2|\tau)}\right| - \frac{2\pi}{\tau_I}\mathrm{Re}[\omega+\tfrac{1}{2}(\phi_1+\phi_2)\phi]\,\mathrm{Re}\omega_{12} - 2\pi\,\mathrm{Re}[\phi(\phi_1-\phi_2)+\phi_1\phi_2)]\right] \tag{25}$$

since this leads to

$$D_\phi D_{\bar\phi}\bar S_{cl}=-\frac{\pi n}{2}[\delta^2(\omega-\omega_1)\delta^2(\phi-\phi_1) - \delta^2(\omega-\omega_2)\delta^2(\phi-\phi_2)] \tag{26}$$

instead of

$$D_\phi D_{\bar\phi} S_{cl}=0 .$$

The partition function due to the presence of two punctures takes the form

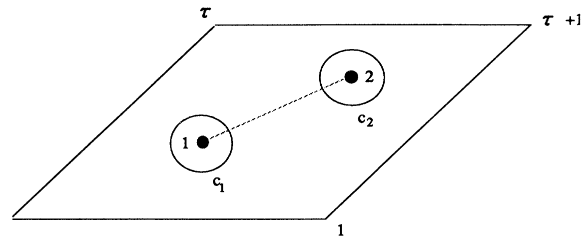

FIG. 3. Magnetic operators create a cut with a discontinuity of $2\pi n$ for the field Φ along c_1 and $-2\pi n$ along c_2.

$$Z_{punc}=\exp\left[\frac{g}{2\pi}\int d^2\phi\int_T d^2\omega\, D_\phi S_{cl} D_{\bar\phi}\bar S_{cl}\right] = \left|\frac{\eta(\tau)}{\theta_1(\omega_{12}|\tau)}\right|^{g/n^2/2}\exp\left[-\frac{g\pi n^2}{2\tau_I}(\mathrm{Re}\omega_{12})^2\right] . \tag{27}$$

Note that in deriving the above result, we prefix the normalization

$$[\tilde Z_0]^{-gn^2/2}\equiv\left|\frac{\eta(\tau)}{\theta_1(0|\tau)}\right|^{gn^2/2} \quad \text{to } Z_{punc}$$

because there exists the singular term $\theta_1(0|\tau)$.

Considering (15), (17), (20), and (27), the partition function on odd-spin structure is given by

$$Z_{odd}=\sum_{m,m'\in Z}\frac{\bar\delta\delta}{\tau_I}\left[\frac{g}{2\tau_I}\right]^{1/2}\left|\frac{\eta(\tau)}{\theta_1(\omega_{12}|\tau)}\right|^{gn^2/2}\exp\left[-\frac{\pi g}{2\tau_I}\{|m'-m\tau|^2+n^2(\mathrm{Re}\omega_{12})^2-2n\,\mathrm{Re}[(m'-m\bar\tau)\omega_{12}]\}\right] \tag{28}$$

$$\equiv\sum_{m,m'\in Z} Z_1 Z''_{m,m'} \tag{29}$$

$$\equiv\sum_{m,m'\in Z} Z_1 Z_{punc} Z'_{m,m'} . \tag{30}$$

One can show that

$$Z''_{m,m'}(\omega_{12}+1)=Z''_{m,m'-n}(\omega_{12}) , \tag{31}$$

$$Z''_{m,m'}(\omega_{12}+\tau)=Z''_{m+n,m'}(\omega_{12}) . \tag{32}$$

Note that $n\notin Z$, $Z''_{m,m'}$ is not periodic, since shifting ω_{12} by $1,\tau$, or $1+\tau$ is equivalent to adding a new frustration line wrapping around the torus.

In order to obtain the full partition function for the $N=1,c=3/2$ superconformal model on the odd supertorus, we have to consider the coupling between Φ and ψ, which is purely induced by the boundary conditions [11]. Considering the partition function of the SU(2), $k=2$ Wess-Zumino-Witten model on an odd-spin structure of torus, the partition function $N=1,c=3/2$ superconformal model for the coupling Φ and ψ is given by

$$Z'_{\Phi-\psi}\left[\frac{g}{2}\right]=\sum_{r,s=0,1}\hat Z_2(r,s)\sum_{\substack{m=r[2]\\ m'=s[2]}}Z'_{m,m'}\left[\frac{g}{2}\right] , \tag{33}$$

where $\hat Z_2(r,s)$ denotes the partition function for an Ising-type model with twisted boundary conditions $e^{i\pi r}$ ($e^{i\pi s}$) on the spin variables σ. Also $Z'_{m,m'}(g/2)$ is already defined in (28). $\hat Z_2(r,s)$ enjoy modular covariance properties as

$$T(\tau\to\tau+1):\hat Z_2(r,s)=\hat Z_2(r,r+s) , \tag{34}$$

$$S\left[\tau\to-\frac{1}{\tau}\right]:\hat Z_2(r,s)=\hat Z_2(-s,r) . \tag{35}$$

Further the soliton sector $Z'_{m,m'}$ changes under modular transformations as

$$T: Z'_{m,m'}(\tau+1)=Z'_{m,m+m'}(\tau)\ , \tag{36}$$

$$S: Z'_{m,m'}\left[-\frac{1}{\tau}\right]=Z'_{-m',m}(\tau)\ . \tag{37}$$

Therefore, it is easily shown that the partition function $Z'_{\Phi-\psi}$ in (33) is a modular-invariant quantity. In our case, the corresponding partition function $Z_{\Phi-\psi}$ can be obtained by replacing $Z'_{m,m'}$ with $Z''_{m,m'}$:

$$Z_{\Phi-\psi}=\sum_{r,s=0,1}\hat{Z}_2(r,s)\sum_{\substack{m=r[2]\\ m'=s[2]}}Z''_{m,m'}\left[\frac{g}{2}\right]\ . \tag{38}$$

We obtain the partition function of a doubly periodic boson coupled to the NS sectors for a fermionic field on the odd-spin structure of a supertorus expressed in terms of the (ω,ϕ) coordinates:

$$\begin{aligned} Z_{\rm NS}=Z_{\rm odd}+Z_{\Phi-\psi}=&\sum_{m,m'\in Z}Z_1Z''_{m,m'}\left[\frac{g}{2}\right]+Z_2\left[\sum_{m,m'\in e}-\sum_{m\in e,m'\in o}+\sum_{m\in o,m'\in e}+\sum_{m,m'\in o}\right]Z''_{m,m'}\left[\frac{g}{2}\right]\\ &+Z_3\left[\sum_{m,m'\in e}+\sum_{m\in e,m'\in o}+\sum_{m\in o,m'\in e}-\sum_{m,m'\in o}\right]Z''_{m,m'}\left[\frac{g}{2}\right]\\ &+Z_4\left[\sum_{m,m'\in e}+\sum_{m\in e,m'\in o}-\sum_{m\in o,m'\in e}+\sum_{m,m'\in o}\right]Z''_{m,m'}\left[\frac{g}{2}\right]\ , \end{aligned} \tag{39}$$

where

$$Z_\nu=\frac{1}{2}\left|\frac{\theta_\nu(0|\tau)}{\eta(\tau)}\right|,\quad \nu=2,3,4\ . \tag{40}$$

Z_ν corresponds to the partition function for an Ising Majorana fermion on the torus with $(++,--)$, $(--,--)$ and $(--,++)$ boundary conditions, respectively. Here e (o) denotes even (odd) integers.

Finally, let us transform the (ω,ϕ) coordinates into the (z,θ) coordinates on the $(++,++)$ odd-spin structure. The relation between ω_{12} and Z_{12} is given by

$$\omega_{12}=Z_{12}-\frac{(\mathrm{Im}z_1)(\theta_1+\theta_2)\delta}{\tau_I}\left[1+\frac{i\bar{\theta}_1\bar{\delta}}{2\tau_I}\right]+\frac{(\mathrm{Im}z_2)(\theta_1+\theta_2)\delta}{\tau_I}\left[1+\frac{i\bar{\theta}_2\bar{\delta}}{2\tau_I}\right]\ . \tag{41}$$

Because of the presence of the $\bar{\delta}\delta$ term in (28), we can substitute τ into $T_{12}\equiv\tau+(\theta_1+\theta_2)\delta$ in the $(++,++)$ sector. In this case, the relation between $\theta_1(\omega_{12}|\tau)$ and $\theta_1(Z_{12}|T_{12})$ is given by

$$\theta_1(\omega_{12}|\tau)=\theta_1(Z_{12}|T_{12})\left[1+(\theta_1+\theta_2)\delta\left[-\frac{1}{4\pi i}\frac{\theta_1''(Z_{12}|T_{12})}{\theta_1(Z_{12}|T_{12})}+\left[-\frac{\mathrm{Im}z_1}{\mathrm{Im}T_1}+\frac{\mathrm{Im}z_2}{\mathrm{Im}T_2}\right]\frac{\theta_1'(Z_{12}|T_{12})}{\theta_1(Z_{12}|T_{12})}\right]\right]\ , \tag{42}$$

where $\theta_1(Z_{12}|T_{12})$ is a superelliptic function (supertheta function) on the supertorus [4,6,7], $T_1\equiv\tau+\theta_1\delta$ and $T_2\equiv\tau+\theta_2\delta$.

As is shown in (38) through (29), however, $Z_{\Phi-\psi}$ does not contain the prefactor $\bar{\delta}\delta$ which has the nilpotent property $(\delta^2=\bar{\delta}^2=0)$. Recall that this coupling term comes from the boundary conditions of a doubly periodic boson on the odd $(++,++)$ spin structure and a twisted fermionic field on three even-spin structures $((++,--),(--,++),(--,--))$. In this case, the new coordinates (ω,ϕ) are identical to the (z,θ) coordinates, because the odd-modular parameter δ does not appear on the even-spin structures. Therefore, it is easy to rewrite $Z_{\Phi-\psi}$ in terms of the (z,θ) coordinates.

After a calculation, we can rewrite $Z_{\rm NS}$ in terms of the (z,θ) coordinates as

$$\begin{aligned} Z_{\rm NS}=&\left[\left[\frac{g}{2}\right]^{1/2}\frac{\delta\bar{\delta}}{(\mathrm{Im}T_{12})^{3/2}}\sum_{m,m'\in Z}\right]\\ &\times\left|\frac{\eta(T_{12})}{\theta_1(Z_{12}|T_{12})}\right|^{gn/2}\exp\left[-\frac{\pi g}{2\,\mathrm{Im}T_{12}}\{-|m'-mT_{12}|^2+n^2(\mathrm{Re}Z_{12})^2-2n\,\mathrm{Re}[(m'-mT_{12})Z_{12}]\}\right]\\ &+\left[\left[\frac{g}{2\tau_I}\right]^{1/2}\frac{1}{|\eta(\tau)|^2}\sum_{r,s=0,1}\hat{Z}_2(r,s)\sum_{m=r[2],m'=s[2]}\right]\\ &\times\left|\frac{\eta(\tau)}{\theta_1(Z_{12}|\tau)}\right|^{gn/2}\exp\left[-\frac{\pi g}{2\tau_I}\{-|m'-m\tau|^2+n^2(\mathrm{Re}Z_{12})^2-2n\,\mathrm{Re}[(m'-m\tau)Z_{12}]\}\right]\ . \end{aligned} \tag{43}$$

We wish to notice that the partition function $Z_{\rm NS}$ is actually identical to the total correlation function (two-point function) $\langle O_{0n}(z_1,\theta_1)O_{0-n}(z_2,\theta_2)\rangle$ of vortex operators on the supertorus without punctures.

Here we wish to note that the correlation function of Kanno, Nishimura, and Tamekiyo [12] is just the quantum correlation function (propagator of a scalar superfield) on the odd-spin structure of supertorus (refer to Eq. (A9) in [6]). In constructing a general two-point function of electromagnetic operators (particularly electric vertex operators) on the supertorus, we have to use the quantum propagator as is shown in Eq. (38) of Ref. [6]. However, our approach to calculating the partition function is to take into account the meromorphic one-form with simple poles at two punctures given in the Krichever-Novikov construction by using vortex operators instead of vertex operators. During our calculations, we did not use any vertex operators. If we use the electromagnetic operators, instead of magnetic operators, to take into account the poles, two-point quantum propagator of Kanno, Nishimura, and Tamekiyo is necessary to complete the desired two-point function.

V. PARTITION FUNCTION FOR A DOUBLY PERIODIC BOSON COUPLED TO THE R SECTOR

So far, we have considered only coupling the soliton sector of a bosonic field with the NS sector. Two punctures in the Ramond sector can be described by bases for the fermionic field which have branch points at ω_1 and ω_2, compared with multipoles at ω_1 and ω_2 in the case of the NS sector. Two of the simple branch points can be created by introducing the spin field $\sigma(\omega,\overline{\omega})$ at ω_1 and ω_2 [10,13]. Indeed, in order to make these bases single valued we have to introduce a branch cut relating between ω_1 and ω_2 as shown in Fig. 4. As is shown in (11), under parallel transport around a cycle which crosses the cut (e.g., a',b'), the spinor field changes sign with respect to parallel transport around a corresponding cycle which does not cross the cut (e.g., a,b, respectively). In our approach, introducing spin fields $\sigma(\omega,\overline{\omega})$ at ω_1 and ω_2 and a branch cut relating between ω_1 and ω_2 actually takes into account the required sign changes of the fermion in Ramond sector around the cycles a,a',b,b'. For the spinor field which does not cross a branch cut there is no change of sign, while along the cycle which crosses a branch cut there is change of sign for the field $\psi(z)$. According to $N=1$ superconformal algebra, furthermore, one can construct the vacuum for Ramond state from that for the NS state ($|0\rangle$) by the holomorphic spin field $\sigma^{\pm}(\omega)$ as [13]

$$|\sigma^{\pm}\rangle=\sigma^{\pm}(0)|0\rangle\ , \tag{44}$$

where $\sigma^{\pm}(\omega)$ and $\overline{\sigma}^{\,\pm}(\overline{\omega})$ are defined by the relations as

$$\sigma(\omega,\overline{\omega})=\sigma^{+}\overline{\sigma}^{\,-}+\sigma^{-}\overline{\sigma}^{\,+}\ ,$$
$$\mu(\omega,\overline{\omega})=\sigma^{-}\overline{\sigma}^{\,-}+\sigma^{+}\overline{\sigma}^{\,+}\ . \tag{45}$$

In this sense, it is not unnatural to introduce the spin field $\sigma(\omega,\overline{\omega})$ in order to account for the Ramond sector. The role of spin fields on the complex z plane was discussed in

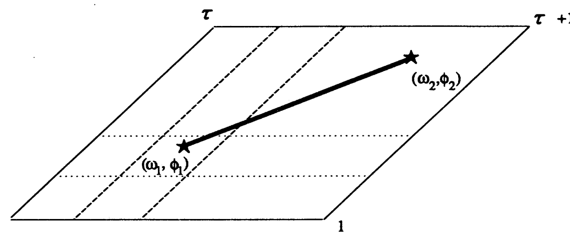

FIG. 4. A branch cut between two punctures on the torus in the (ω,ϕ) coordinates. The punctures (branch points) are represented by two ★'s.

Ref. [14]. By the similar way as in Sec. IV, we have to consider a soliton sector with (m,m') characteristic to construct the partition function. We introduce a real classical field such that

$$\partial_z\Phi_{\rm cl}=A(1,2)\Omega(\omega,\omega_1,\omega_2)\ . \tag{46}$$

Here Ω_1 has the form

$$\Omega_1(\omega,\omega_1,\omega_2)=\frac{\theta_1(\omega-\frac{1}{2}(\omega_1+\omega_2)|\tau)}{[\theta_1(\omega-\omega_1|\tau)\theta_1(\omega-\omega_2|\tau)]^{1/2}}\ , \tag{47}$$

which satisfies the correct local monodromy properties around ω_1 and ω_2.

Now let us fix $A(1,2)$ by the condition in (18) and (19). Then we easily find

$$A(1,2)=i\pi\frac{m\overline{I}\,'_1-m'\overline{I}_1}{\mathrm{Im}(\overline{I}_1I'_1)}\ , \tag{48}$$

with

$$I_1(I'_1)=\int_{1(\tau)}\Omega_1(\omega,\omega_1,\omega_2)d\omega\ . \tag{49}$$

The corresponding classical action reads

$$A_{\rm cl}=\frac{\pi g}{2}\frac{|m'I_1-mI'_1|^2}{[\mathrm{Im}(\overline{I}_1I'_1)]^2}\int_T|\Omega_1|^2d^2\omega\ . \tag{50}$$

Actually, to calculate $\int_T|\Omega_1|^2d^2\omega$, we can consider this in the simply connected region $\widetilde{T}$ whose boundary is the parallelogram representing the torus, relating by a thin neck to a contour surrounding the cut in Fig. 5. In $\widetilde{T}$, there exists a function such that $df=\Omega_1d\omega$. Using Green's theorem and integrating in a symmetric way along 1, $1+\tau$ (τ, $1+\tau$), we find

$$\begin{aligned}\int_T|\Omega_1|^2d^2\omega&=\frac{1}{2i}\int_{\widetilde{T}}\Omega_1d\omega\wedge\overline{\Omega}_1d\overline{\omega}\\&=\frac{1}{2i}\left[\int_1\overline{\Omega}_1\int_\tau\Omega_1-\int_1\Omega_1\int_\tau\overline{\Omega}_1\right]\\&=\mathrm{Im}(\overline{I}_1I'_1)\ .\end{aligned} \tag{51}$$

Note that the contribution of the contour surrounding the cut is disappearing because the corresponding integral of Ω_1 vanishes.

As $\omega_1\to\omega_2$, one finds

$$\Omega_1\to1,\quad I_1\to1,\quad I'_1\to\tau\ ,$$
$$\overline{\Omega}_1\to1,\quad \overline{I}_1\to1,\quad \overline{I}\,'_1\to\overline{\tau}\ . \tag{52}$$

Therefore, A_{cl} naturally recovers the result in (21). The corresponding partition function takes the form

$$Z''_{m,m'}\left[\frac{g}{2}\right] \equiv \left[\frac{g}{2\tau_I}\right]^{1/2} \frac{1}{|\eta(\tau)|^2} \times \exp\left[-\frac{\pi g}{2}\frac{|mI'_1 - m'I_1|^2}{\mathrm{Im}(\bar{I}_1 I'_1)}\right] . \quad (53)$$

According to Ref. [9], the spin-spin correlation function is given by

$$\langle \sigma(\omega_1,\bar{\omega}_1)\sigma(\omega_2,\bar{\omega}_2)\rangle_{\nu=2,3,4} = \frac{|\theta'_1(0|\tau)|^{1/4}|\theta_\nu(\frac{1}{2}(\omega_1-\omega_2)|\tau)|}{|\theta_\nu(0|\tau)||\theta_1(\omega_1-\omega_2|\tau)|^{1/4}} . \quad (54)$$

In the case of $\nu=1$, one notes that $\langle\sigma\sigma\rangle_1$ is singular due to the term $|\theta_1(0|\tau)|$. One point to remark is that $\langle\sigma(1)\sigma(2)\rangle_\nu$ is not periodic by itself. This is because spin operators are not local in terms of Majorana fermions ψ. Actually, translating $\omega_1-\omega_2$ by 1, τ or $\tau+1$ amounts to creating a frustration line winding around the supertorus and changes the sign of ψ along 1, τ, or 1 and τ.

In order to account for the periodicity of the total correlation function and make $\langle\sigma\sigma\rangle_{\nu=1}$ finite, we prefix

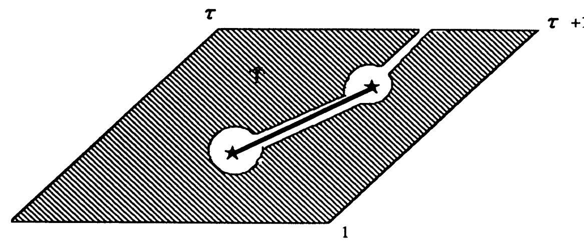

FIG. 5. The simply connected region $\tilde{T}$ whose boundary is the usual parallelogram of torus related by a thin neck to a contour surrounding the branch cut.

$\frac{1}{2}\tilde{Z}_0$, $Z_{\nu=2,3,4}$ to $\langle\sigma(1)\sigma(2)\rangle_{\nu=1,2,3,4}$ respectively:

$$Z_\nu\langle\sigma(\omega_1,\bar{\omega}_1)\sigma(\omega_2,\bar{\omega}_2)\rangle_\nu = \frac{1}{2}\frac{|\theta_\nu(0|\tau)|}{|\eta|}\langle\sigma\sigma\rangle_\nu = \frac{(2\pi)^{1/4}}{2|\eta|^{1/4}}\frac{|\theta_\nu(\frac{1}{2}(\omega_1-\omega_2)|\tau)|}{|\theta_1(\omega_1-\omega_2|\tau)|^{1/4}} . \quad (55)$$

As $\omega_1\to\omega_2$, we obtain the correct operator-product expansions:

$$\tilde{Z}_1\langle\sigma\sigma\rangle_1 \cong \tfrac{1}{2}|\omega_1-\omega_2|^{3/4}\pi|\eta|^2 , \quad (56)$$

$$Z_{\nu\neq 1}\langle\sigma\sigma\rangle_{\nu\neq 1} \cong \frac{Z_\nu}{|\omega_1-\omega_2|^{1/4}}\left|1+\frac{(\omega_1-\omega_2)^2}{48}\left[\frac{3\theta'''_\nu(0|\tau)}{\theta_\nu(0|\tau)}-\frac{\theta'''_1(0|\tau)}{\theta'_1(0|\tau)}\right]\right|^2 . \quad (57)$$

Finally, we construct the partition function for a doubly periodic boson coupled to the Ramond sector as

$$Z_R = \frac{\tilde{Z}_1}{2}\langle\sigma\sigma\rangle_1 \sum_{m,m'\in Z} Z''_{m,m'}\left[\frac{g}{2}\right] + \sum_{r,s=0,1}\tilde{Z}_2(r,s)\sum_{\substack{m=r[2]\\ m'=s[2]}} Z''_{m,m'}\left[\frac{g}{2}\right] , \quad (58)$$

where the partition functions of spin fields with the branch cut $[\tilde{Z}_2(r,s)]$ are given by

$$\begin{aligned}
\tilde{Z}_2(0,0) &= Z_2\langle\sigma\sigma\rangle_2 + Z_3\langle\sigma\sigma\rangle_3 + Z_4\langle\sigma\sigma\rangle_4 ,\\
\tilde{Z}_2(0,1) &= -Z_2\langle\sigma\sigma\rangle_2 + Z_3\langle\sigma\sigma\rangle_3 + Z_4\langle\sigma\sigma\rangle_4 ,\\
\tilde{Z}_2(1,1) &= Z_2\langle\sigma\sigma\rangle_2 - Z_3\langle\sigma\sigma\rangle_3 + Z_4\langle\sigma\sigma\rangle_4 ,\\
\tilde{Z}_2(1,0) &= Z_2\langle\sigma\sigma\rangle_2 + Z_3\langle\sigma\sigma\rangle_3 - Z_4\langle\sigma\sigma\rangle_4 .
\end{aligned} \quad (59)$$

Note that the partition function Z_R is identical to the two-point function of twisted fields on the supertorus without punctures.

VI. DISCUSSIONS

Now let us suppose that one may rewrite Z_R in terms of the (z,θ) coordinates. We remark that in the Ramond sector the bosonic field Φ and the fermionic fields $\psi(z)$ cannot be combined in a superfield multiplet because the fermionic field should satisfy the special boundary conditions along the cycles a,a',b,b' in (11). As a result, the presence of spin fields σ which account appropriately for the boundary conditions of the fermion prevent us from expressing Z_R in terms of the (z,θ) coordinates. However, one can easily rewrite Z_R in terms of the z coordinate by the substitution $\omega\to z$, since in the Ramond sector the new coordinate ω is identical to the z coordinate.

We note also that the Ramond sector of the partition function for the $N=1, c=3/2$ superconformal model on the odd-spin structure of the torus with two punctures has the same form as in (58). This point shows us a difference between Z_{NS} and Z_R. Furthermore, in the case of the partition function for the same model on the odd-spin structure of a torus (not supertorus), the combination of odd modular parameters $(\bar{\delta}\delta)$ never appears. That is, Z_{odd} in Z_{NS} disappears on the torus and all formulas expressed in terms of the (ω,ϕ) coordinates are not distinguished from that expressed in terms of the (z,θ) coordinates.

Our strategy was to construct the partition function of the $N=1, c=3/2$ superconformal model on the odd-spin structure of the supertorus. Even though our interest stays only at the odd $(++,++)$ spin structure, we consider coupling a doubly periodic bosonic field Φ on the odd-spin structure to all sectors of the spinor fields $\psi(z)$ from the boundary conditions of Φ and ψ. It thus assumes that the total partition function for the $N=1, c=3/2$ superconformal model on the odd-spin structure of supertorus with two punctures may be given by

$$Z = Z_{\rm NS} + Z_{\rm R} \ . \tag{60}$$

However, it is problematic to add $Z_{\rm NS}$ and $Z_{\rm R}$, since the construction of $Z_{\rm NS}$ is quite different from that of $Z_{\rm R}$. For example, when one either inserts vortex operators or spin operators at given points, one gets either one or the other part of the partition function. Also if we take the above assumption for granted, it remains to interpret the total partition function in (60) as a two-point function of one operator which gives both vortex and spin operators on the supertorus without punctures. Note that the first part of Z ($Z_{\rm NS}$) is identical to the two-point function of vortex operators, while the second ($Z_{\rm R}$) is identical to that of spin fields. As explained in Sec. V, to describe the coupling of a doubly periodic boson to the Ramond sector, we need interesting physical quantities such as spin fields. These are not described by operators of the vertex or vortex (electromagnetic operators), but instead by twist operators. It should be emphasized that the vortex operators create a branch cut with the amplitude discontinuity of $2\pi n$ for the bosonic field Φ, while the twist operators create a branch cut with the change of sign for the fermionic field ψ. When one writes a correlation function, it depends whether one has spin operators inserted or vortex operators, and one gets different results. Therefore, it is problematic to add them ($Z_{\rm NS}$ and $Z_{\rm R}$) and look for one operator that gives both.

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