Statistics of spinning particles in $2+1$ gravity

K. Gupta
Department of Physics, Syracuse University, Syracuse, New York 13244-1130

A. Stern
Dipartimento di Scienze Fisiche, Istituto Nazionale di Fisica Nucleare, Section of Naples, Mostra d'Oltremare Pad 19, I-80125 Napoli, Italy
*and Department of Physics and Astronomy, University of Alabama, Tuscaloosa, Alabama 35487**

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We examine the dynamics of N spinning particles in $2+1$ gravity, and obtain an effective Lagrangian description for the particles by eliminating the dreibeins and spin connections from the theory. The resulting equations of motion show that locally the particles are free, but globally their conserved momenta and angular momenta are not well defined. Conditions are found under which the effective action is invariant under particle exchanges. Ambiguities exist in passing to the quantum theory and these ambiguities can be exploited to obtain exotic statistics for the particles. We construct exchange operators which give exotic statistics, but which do not in general satisfy the braid group relations when $N>2$.

I. INTRODUCTION

General topological arguments lead to the possibility of having exotic statistics for particles moving in two spatial dimensions [1]. The quantum theory for N such particles may be classified by representations of the braid group B_N [2]. Whether or not exotic statistics are realized in a theory requires a knowledge of the relevant dynamics. For many theories of physical interest, such as those describing the quantum Hall theory [3], this possibility is realized, but it may not always be the case.

In this article we examine the system of spinning particles in $2+1$ gravity. Many novel features [4] have been noted for this system, including the possibility that such particles can be quantized with fractional statistics [5]. In this article we show how an exchange operator can be constructed such that it exhibits exotic statistics. The quantization procedure, however, is not unique, as we can just as well have a system where the particles are quantized as bosons or fermions. At the same time there appears to be nothing which prevents the particle from having an arbitrary fractional spin. Hence at the level of a first-quantized theory, the spin-statistics theorem [6] need not apply.

Gravity in $2+1$ dimensions can be obtained starting from the Chern-Simons action for the $2+1$ Poincaré group ISO(2,1) [7]. Chern-Simons actions are functionals of connection one-forms. If spacetime is topologically trivial, the connections are nondynamical (provided the total action under consideration does not also contain a kinetic-energy term for the potentials). The connection one-forms can then be eliminated from the theory, yielding an effective Lagrangian for any remaining "physical" degrees of freedom of the theory. This procedure, originally outlined by Arovas *et al.* [8], was carried out in detail by Balachandran, Bourdeau, and Jo [9] for particles with internal degrees of freedom coupled to a general Chern-Simons gauge theory. There, the field variables were eliminated from the theory, leaving just a particle theory. We apply the procedure of Ref. [9] to the case where the gauge group is ISO(2,1). For us, the particle sources do not have internal degrees of freedom, but rather are characterized by an arbitrary mass and spin.

The dynamics for relativistic spinning particles in $2+1$ dimensions was obtained starting from a Wess-Zumino particle action in Ref. [10]. We shall consider the system of N spinning particles coupled to the Chern-Simons gravity action. Upon eliminating the fields from the theory, we obtain an effective action for N particles, which, in general, is not invariant under particle exchanges. Upon requiring exchange invariance, certain conditions must be satisfied. For $N=2$, these conditions imply that the particles are identical, i.e., have the same mass and spin. For $N>2$, one obtains, in addition, conditions which do not have a clear physical interpretation. They are analogous to conditions found in Ref. [9]. The conditions are useful for deriving properties of the exchange operator for the quantum theory.

In Sec. II we review the field and particle equations of motion for $2+1$ gravity and show how they are obtained from an action principle. In Sec. III, we substitute the solutions to the field equations into the total action, thereby obtaining the effective particle Lagrangian. Exchange invariance of the two- and three-particle effective Lagrangian is studied in Secs. IV and V, respectively. Quantum exchange operators σ_i are constructed in Sec. VI. They have the property that $\sigma_i^2=1$, and hence only admit bosons or fermions. They are also shown to satisfy the braid relations $\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2$. In Sec. VII we demonstrate that there exist quantization ambiguities, allowing us to construct alternative exchange operators which give the possibility of exotic statistics. However, these exchange operators have the unpleasant feature that they do not in general satisfy the braid relations when $N>2$. Concluding remarks are made in Sec. VIII.

II. REVIEW OF PARTICLE-FIELD DYNAMICS

The Chern-Simons action $S_{\rm CS}=S_{\rm CS}(\omega,e)$ associated with the ISO(2,1) gauge group can be written

$$S_{CS}=\kappa\int_M e_a\wedge R^a\ ,\quad R^a=d\omega^a+\tfrac{1}{2}\epsilon^{abc}\omega_b\wedge\omega_c\ , \tag{2.1}$$

where e^a and ω^a $(a,b,c,\ldots=0,1,2)$ are the dreibein and SO(2,1) spin connection one-forms, respectively, which together comprise the ISO(2,1) connection one-form. κ is the gravitational constant, which in 2+1 dimensions has units of inverse length, and M denotes the space-time manifold. The indices $a,b,c,\ldots$ are raised and lowered with the Minkowski metric tensor $\eta=[\eta_{ab}]=\mathrm{diag}(-1,1,1)$.

Equation (2.1) was shown [7] to be equivalent to the Einstein-Hilbert action for gravity in 2+1 dimensions provided the dreibein fields are everywhere invertible. As well as being invariant under diffeomorphisms of space-time, the action is left unchanged (up to a boundary term) under ISO(2,1) gauge transformation. The latter are given by

$$\omega^a\to(\theta\omega)^a-\tfrac{1}{2}\epsilon^{abc}(d\theta\,\theta^{-1})_{bc}\ , \tag{2.2}$$

$$e^a\to(\theta e)^a-[\theta\, d(\theta^{-1}b)]^a+\epsilon^{adc}b_d(\theta\omega)_c\ . \tag{2.3}$$

Here $\theta=[\theta_{ab}]$ is a space-time-dependent Lorentz matrix, while $b=[b_a]$ is a space-time-dependent Lorentz vector.

The field equations which follow from variations of ω^a and e^a in Eq. (2.1) state that the curvature two-form R^a and torsion two-form

$$T^a=de^a+\tfrac{1}{2}\epsilon^{abc}(e_b\wedge\omega_c+\omega_b\wedge e_c)$$

vanish everywhere on M. The solutions to the sourceless equations are pure gauges and can be expressed according to

$$\omega^a=-\tfrac{1}{2}\epsilon^{abc}(d\theta\,\theta^{-1})_{bc}\ ,\quad e^a=-[\theta\, d(\theta^{-1}b)]^a\ , \tag{2.4}$$

where θ and b were defined earlier.

Upon introducing a point-particle source into the system, R^a and T^a vanish everywhere except along the particle world line. We coordinate the particle world line by $z^\mu=z^\mu(\tau)$, $\mu=0,1,2$, and postulate that

$$\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}R^a_{\ \nu\lambda}(x)=\int d\tau\,\delta^3(x-z(\tau))P^a\dot z^\mu\ , \tag{2.5}$$

$$\frac{\kappa}{2}\epsilon^{\mu\nu\lambda}T^a_{\nu\lambda}(x)=\int d\tau\,\delta^3(x-z(\tau))J^a\dot z^\mu\ , \tag{2.6}$$

where $R^a_{\ \nu\lambda}$ and $T^a_{\nu\lambda}$ are the space-time components of R^a and T^a, respectively, and the dot denotes differentiation with respect to τ. The "charges" in Eqs. (2.5) and (2.6), $P^a=P^a(\tau)$ and $J^a=J^a(\tau)$, are usually interpreted as the momentum and angular-momentum components of the particle. Equations of motion for the particle can be deduced from the Bianchi identities for the fields:

$$dR^a+\epsilon^{abc}\omega_b\wedge R_c=0\ , \tag{2.7}$$

$$dT^a+\epsilon^{abc}(\omega_b\wedge T_c+e_b\wedge R_c)=0\ . \tag{2.8}$$

Upon substituting (2.5) and (2.6) into (2.7) and (2.8), we find

$$\dot P^a+\epsilon^a_{\ bc}\omega^b_\mu(z)P^c\dot z^\mu=0\ , \tag{2.9}$$

$$\dot J^a+\epsilon^a_{\ bc}[\omega^b_\mu(z)J^c+e^b_\mu(z)P^c]\dot z^\mu=0\ , \tag{2.10}$$

where here $\omega^b_\mu(z)$ and $e^b_\mu(z)$ are the components of the one-forms ω^b and e^b, evaluated at the particle position z.

In Refs. [10] it was shown that these equations of motion are also obtained starting from an action principle. For this, in addition to the space-time coordinate $z^\mu(\tau)$, we introduce the dynamical variables $\Lambda=[\Lambda_{ab}(\tau)]$ and $a=[a_b(\tau)]$, the former being a Lorentz matrix and the latter a Lorentz vector. In terms of these variables the momentum P and angular momentum J take the form

$$P^a=\Lambda^a_{\ b}\rho^b\ , \tag{2.11}$$

$$J^a=\Lambda^a_{\ b}\lambda^b+\epsilon^{abc}a_b\Lambda_{cd}\rho^d\ . \tag{2.12}$$

Here ρ^a and λ^a are constant vectors which select a particular adjoint orbit of the Poincaré group. An orbit is the set of all P^a and J^a obtained from (2.11) and (2.12) by transforming

$$\Lambda\to\Theta\Lambda,\quad a\to\Theta a+c\ , \tag{2.13}$$

Θ and c being a Lorentz matrix and Lorentz vector, respectively. Under transformation (2.13), the quantities

$$-m^2\equiv P\cdot P=\rho\cdot\rho\ , \tag{2.14}$$

$$-ms\equiv P\cdot J=\rho\cdot\lambda \tag{2.15}$$

remain invariant and these two quantities classify the adjoint orbits of the Poincaré group.

The particle Lagrangian is [10]

$$L_{\mathrm{part}}=L_0(\Lambda,a)+P_a e^a_\mu(z)\dot z^\mu+J_a\omega^a_\mu(z)\dot z^\mu\ , \tag{2.16}$$

where L_0 gives the dynamics for a free relativistic spinning particle:

$$L_0(\Lambda,a)=P\cdot\dot a+\tfrac{1}{2}\epsilon^{abc}\lambda_a(\Lambda^{-1}\dot\Lambda)_{bc}\ . \tag{2.17}$$

L_{part} is invariant under local Poincaré transformations, (2.3), (2.2), and (2.13) with $\Theta=\Theta(\tau)=\theta(z(\tau))$ and $c=c(\tau)=b(z(\tau))$.

Upon varying a_a and Λ_{ab} in L_{part} and using (2.11) and (2.12), we recover equations of motion (2.9) and (2.10). An additional equation of motion arises from variations of z^μ in L_{part}. After using (2.9) and (2.10) it can be written

$$[P_aT^a_{\mu\nu}(z)+J_aR^a_{\ \mu\nu}(z)]\dot z^\gamma=0\ . \tag{2.18}$$

The field equations (2.5) and (2.6) can be obtained by extremizing the total action with respect to $e^a_\mu(x)$ and $\omega^a_\mu(x)$. For this we define the total action to be

$$S=S_{CS}-\int d\tau\, L_{\mathrm{part}}\ . \tag{2.19}$$

The solutions to the one-particle field equations (2.5) and (2.6) may be expressed as

$$\omega^a=(\theta\Omega)^a-\tfrac{1}{2}\epsilon^{abc}(d\theta\,\theta^{-1})_{bc}\ , \tag{2.20}$$

$$e^a=(\theta E)^a-[\theta\, d(\theta^{-1}b)]^a+\epsilon^{adc}b_d(\theta\Omega)_c\ , \tag{2.21}$$

where $\theta(x)$ and $b(x)$ satisfy $\theta(z(\tau))=\Lambda(\tau)$ and

$b(z(\tau))=a(\tau)$ and

$$\Omega^a=\frac{1}{2\pi\kappa}\rho^a d\phi \quad \text{and} \quad E^a=\frac{1}{2\pi\kappa}\lambda^a d\phi \ , \tag{2.22}$$

$\phi=\phi(x-z(\tau))$ being an angular variable which changes by 2π upon circling the particle world line once.

The above system can be easily generalized to describe N particle sources. For this define $\Lambda^{(\alpha)}$, $a^{(\alpha)}$, and $z^{(\alpha)}$ to be dynamical variables, and $\rho^{(\alpha)}$ and $\lambda^{(\alpha)}$ to be the constant vectors associated with the αth particle, $\alpha=1,2,\ldots,N$. From them we can define the corresponding momenta $P^{(\alpha)a}$ and angular momentum $J^{(\alpha)a}$ analogous to (2.11) and (2.12), along with the αth particle Lagrangian $L^{(\alpha)}_{\text{part}}$, analogous to (2.16). Then L_{part} in (2.19) is replaced by $L=\sum_\alpha L^{(\alpha)}_{\text{part}}$. Now N terms contribute to the right-hand sides of field equations (2.5) and (2.6), having "charges" $P^{(\alpha)}_a$ and $J^{(\alpha)}_a$, $\alpha=1,2,\ldots,N$, respectively.

The general solution to the field equations in Chern-Simons theory with arbitrary gauge group and number of point sources was given in Ref. [9]. In writing the solution the authors define a spacelike region $\Gamma^{(\alpha)}(\tau)$ associated with particle α to be a thin strip along the negative x_2 direction which encloses the αth particle. The choice of the region $\Gamma^{(\alpha)}(\tau)$ corresponds to a gauge fixing. The strips associated with different particles are assumed not to overlap. On each strip an anglelike function $\phi^{(\alpha)}$ (analogous to ϕ) is defined, the value of which increases by 2π as $\Gamma^{(\alpha)}(\tau)$ is traversed from left to right. In applying the solutions of Ref. [8] to the case of the ISO(2,1) gauge group, we set Ω and E of Eqs. (2.20) and (2.21) equal to zero on $M\setminus\bigcup_\alpha\Gamma^{(\alpha)}$ and

$$\Omega^a=\frac{1}{2\pi\kappa}\rho^{(\alpha)a}d\phi^{(\alpha)} \quad \text{and} \quad E^a=\frac{1}{2\pi\kappa}\lambda^{(\alpha)a}d\phi^{(\alpha)} \tag{2.23}$$

on $\Gamma^{(\alpha)}$. Furthermore, $\theta(x)$ and $b(x)$ in Eqs. (2.20) and (2.21) are required to satisfy

$$\theta(z^{(\alpha)}(\tau))=\Lambda^{(\alpha)}(\tau) \quad \text{and} \quad b(z^{(\alpha)}(\tau))=a^{(\alpha)}(\tau) \ . \tag{2.24}$$

III. THE EFFECTIVE ACTION

We now follow the approach of Ref. [9] and substitute the solutions of the field equations back into the total action (2.19), thereby obtaining an effective action for the remaining particle degrees of freedom. Here we neglect self-interactions of the particles.

For the case $z^{(\alpha)}\in M\setminus\bigcup_{\beta\neq\alpha}\Gamma^{(\beta)}$, the effective Lagrangian for the αth particle is

$$L^{(\alpha)}_{\text{eff}}=L_0(\tilde{\Lambda}^{(\alpha)},\tilde{a}^{(\alpha)}) \ , \tag{3.1}$$

where $L_0(\Lambda,a)$ was the free particle Lagrangian given in (2.17). The variables $\tilde{\Lambda}^{(\alpha)}$ and $\tilde{a}^{(\alpha)}$ are defined in terms of $\Lambda^{(\alpha)}$, $a^{(\alpha)}$, θ, and b according to

$$\tilde{\Lambda}^{(\alpha)}=\theta(z^{(\alpha)})^{-1}\Lambda^{(\alpha)}$$

and

$$\tilde{a}^{(\alpha)}=\Theta(z^{(\alpha)})^{-1}[a^{(\alpha)}-b(z^{(\alpha)})] \ . \tag{3.2}$$

These variables are invariant under local Poincaré transformations (2.3), (2.2), and (2.13) with $\Theta=\Theta(\tau)=\theta(z(\tau))$ and $c=c(\tau)=b(z(\tau))$. The conserved momentum $\tilde{P}^{(\alpha)}$ and angular momentum $\tilde{J}^{(\alpha)}$ associated with the αth particle are constructed from the gauge-invariant variables $\tilde{\Lambda}^{(\alpha)}$ and $\tilde{a}^{(\alpha)}$ as follows,

$$\begin{aligned} \tilde{P}^{(\alpha)}&=\tilde{\Lambda}^{(\alpha)}\rho^{(\alpha)} \ , \\ \tilde{J}^{(\alpha)a}&=(\tilde{\Lambda}^{(\alpha)}\lambda^{(\alpha)})^a+\epsilon^{abc}\tilde{a}^{(\alpha)}_b(\tilde{\Lambda}^{(\alpha)}\rho^{(\alpha)})_c \ , \end{aligned} \tag{3.3}$$

and hence are also invariant under local Poincaré transformations.

For the case $z^{(\alpha)}\in\Gamma^{(\beta)}$, the effective particle Lagrangian contains interaction terms

$$L^{(\alpha)}_{\text{eff}}=L_0(\tilde{\Lambda}^{(\alpha)},\tilde{a}^{(\alpha)})+(\tilde{P}^{(\alpha)}\cdot\lambda^{(\beta)}+\tilde{J}^{(\alpha)}\cdot\rho^{(\beta)})\times\frac{\partial_\mu\phi^{(\beta)}(z^{(\alpha)})\dot{z}^{\mu(\alpha)}}{4\pi\kappa} \ . \tag{3.4}$$

In deriving (3.4) we have taken into account the effect of the Chern-Simons term S_{CS}. This gives rise to an overall factor of $\frac{1}{2}$ in the coupling terms in Eq. (3.4). (For a discussion of the factor $\frac{1}{2}$, see Ref. [9].) Again we have assumed that the regions $\Gamma^{(\beta)}$ and $\Gamma^{(\gamma)}$ associated with any two sources β and γ do not overlap. Then Eqs. (3.1) and (3.4) give all cases for the effective action of particle α.

For the case $z^{(\alpha)}\in M\setminus\bigcup_{\beta\neq\alpha}\Gamma^{(\beta)}$, the effective Lagrangian (3.1) corresponds to that of a free relativistic spinning particle. This is so since the particle feels no external curvature or torsion. The same must be true for the case $z^{(\alpha)}\in\Gamma^{(\beta)}$, $\alpha\neq\beta$. For this case as well, the particle feels no curvature or torsion. Hence the Lagrangian (3.4) must be equivalent to a free particle Lagrangian. This is evident after defining yet another set of Poincaré group parameters

$$\begin{aligned} \tilde{\Lambda}^{(\alpha,\beta)}&=\Xi^{(\beta)}(z^{(\alpha)})\tilde{\Lambda}^{(\alpha)} \ , \\ \tilde{a}^{(\alpha,\beta)}&=\Xi^{(\beta)}(z^{(\alpha)})\tilde{a}^{(\alpha)}+h^{(\beta)}(z^{(\alpha)}) \ . \end{aligned} \tag{3.5}$$

The SO(2,1) matrix $\Xi^{(\beta)}(x)$ and vector $h^{(\beta)}(x)$ are functionals of $\phi^{(\beta)}(x)$. They are defined by

$$\begin{aligned} (\Xi^{(\beta)^{-1}}\partial_\mu\Xi^{(\beta)})^{ab}&=-\frac{\partial_\mu\phi^{(\beta)}}{4\pi\kappa}\epsilon^{abc}\rho^{(\beta)}_c \ , \\ \Xi^{(\beta)^{-1}}\partial_\mu h^{(\beta)}&=\frac{\partial_\mu\phi^{(\beta)}}{4\pi\kappa}\lambda^{(\beta)} \ . \end{aligned} \tag{3.6}$$

In terms of these variables the Lagrangian (3.4) can be reexpressed as a free particle Lagrangian:

$$L^{(\alpha)}_{\text{eff}}=L_0(\tilde{\Lambda}^{(\alpha,\beta)},\tilde{a}^{(\alpha,\beta)}) \ . \tag{3.7}$$

When $z^{(\alpha)}\in M\setminus\bigcup_{\beta\neq\alpha}\Gamma^{(\beta)}$, the conserved momentum and angular momentum are given in Eqs. (3.3). Using Lagrangian (3.6), the analogous conserved quantities for the case $z^{(\alpha)}\in\Gamma^{(\beta)}$ are

$$\begin{aligned} \tilde{P}^{(\alpha,\beta)}&=\tilde{\Lambda}^{(\alpha,\beta)}\rho^{(\alpha)} \ , \\ \tilde{J}^{(\alpha,\beta)a}&=(\tilde{\Lambda}^{(\alpha,\beta)}\lambda^{(\alpha)})^a+\epsilon^{abc}\tilde{a}^{(\alpha,\beta)}_b(\tilde{\Lambda}^{(\alpha,\beta)}\rho^{(\alpha)})_c \ . \end{aligned} \tag{3.8}$$

For a system of two particles, let us uniquely determine $\Xi^{(\beta)}(x)$ and $h^{(\beta)}(x)$ by setting $\Xi^{(\beta)}(x)=1_{3\times3}$ and

$h^{(\beta)}(x)=0$ for x at the boundary $\phi^{(\beta)}=0$ of $\Gamma^{(\beta)}$. Then

$$\tilde{P}^{\alpha,\beta)}=\tilde{P}^{(\alpha)} \quad \text{and} \quad \tilde{J}^{(\alpha,\beta)}=\tilde{J}^{(\alpha)} , \tag{3.9}$$

for $z^{(\alpha)}$ evaluated at $\phi^{(\beta)}=0$. On the other hand, the relations (3.9) will not hold for $z^{(\alpha)}$ at the boundary $\phi^{(\beta)}=2\pi$ of $\Gamma^{(\beta)}$. Instead, $\tilde{P}^{(\alpha,\beta)}$ and $\tilde{J}^{(\alpha,\beta)}$ at $\phi^{(\beta)}=2\pi$ will in general differ from $\tilde{P}^{(\alpha)}$ and $\tilde{J}^{(\alpha)}$ by an ISO(2,1) group transformation. The transformation is obtained by acting with a group element which we denote by $t^{(\beta)}$. $t^{(\beta)}$ corresponds to $\Xi^{(\beta)}$ and $h^{(\beta)}$ evaluated at the boundary $\phi^{(\beta)}=2\pi$. The result depends on $\rho^{(\beta)}$ and $\lambda^{(\beta)}$ and defines the ISO(2,1) holonomy element associated with the βth particle. (Actually, it is the square root of the holonomy element if we compute the relevant Wilson loop using the fields in (2.23) [4,11]. This is due to the factor $\frac{1}{2}$ which entered in the coupling term in the effective Lagrangian (3.4).)

The above discussion shows that the conserved momentum and angular momentum are not globally defined in terms of ISO(2,1) group variables. This result is independent of the particular choice made for the solutions and the regions $\Gamma^{(\alpha)}$. It is due to the fact that the manifold Q spanned by the space-time coordinates is not simply connected. (We assume as usual that no two particles have the same space-time coordinates.) The generators of π_1 of Q are associated with the holonomy elements $t^{(1)},t^{(2)},\ldots,t^{(N)}$. For a system of two particles, we treat one particle as a test particle and the other as a source. When $t^{(1)}$ acts on particle 2, it corresponds to rotating particle 2 about particle 1 by 2π. Similarly, when $t^{(2)}$ acts on particle 1, it corresponds to rotating particle 1 about particle 2 by 2π. For a system of two particles these operations commute and hence define an Abelian group. On the other hand, for three or more particles $\pi_1(Q)$ is non-Abelian and defines the braid group. The implications of the fundamental group for the quantum theory are discussed in Sec. VII.

For a system of three or more particles, the regions $\Gamma^{(\beta)}$ and $\Gamma^{(\gamma)}$ associated with two sources β and γ can in principle overlap. When this happens the solutions given in Eqs. (2.20)–(2.23), as well as the effective Lagrangians which followed, are no longer valid. So, as in Ref. [9], we must impose the veto that a third (test) particle does not cross the overlapping region. (This situation is similar to magnetic-monopole theory where a charged particle is not permitted to cross a Dirac string.) The quantum-mechanical consequences of this veto have been studied in general Chern-Simons theory with three or more particles [9]. A set of "braid quantization conditions" was found (analogous to the charge quantization condition of magnetic-monopole theory) which had to be imposed on the quantum states. Although these conditions were seen to be very restrictive, we shall not be concerned with them in what follows in Secs. IV–VI. They will, however, play a role in the construction of anyonic exchange operators in Sec. VII. There we find that the exchange operators generate the braid group only when the braid quantization conditions are satisfied. On the other hand, we argue in Sec. VIII that it may not be necessary to impose the braid quantization conditions.

IV. THE TWO-PARTICLE SYSTEM

We first study the case of two particles (i.e., one source and one test particle), where many of the abovementioned complications do not occur.

The total effective Lagrangian is $L_{\rm eff}=L_{\rm eff}^{(1)}+L_{\rm eff}^{(2)}$. Under what circumstances is $L_{\rm eff}$ invariant under an exchange of particles 1 and 2? The total effective Lagrangian is not in general invariant under the naive replacement

$$(z^{(1)},\tilde{\Lambda}^{(1)},\tilde{a}^{(1)})\rightleftarrows(z^{(2)},\tilde{\Lambda}^{(2)},\tilde{a}^{(2)}) . \tag{4.1}$$

Following Ref. [9], we instead defined an exchange according to

$$z^{(1)}\rightleftarrows z^{(2)} ,$$
$$\tilde{\Lambda}^{(1)}\to\Theta\tilde{\Lambda}^{(2)}\Theta,\quad \tilde{\Lambda}^{(2)}\to\Theta^{-1}\tilde{\Lambda}^{(1)}\Theta^{-1} ,$$
$$\tilde{a}^{(1)}\to\Theta(\tilde{\Lambda}^{(2)}c+\tilde{a}^{(2)})+c$$

and

$$\tilde{a}^{(2)}\to\Theta^{-1}(\tilde{a}^{(1)}-\tilde{\Lambda}^{(1)}\Theta^{-1}c-c) , \tag{4.2}$$

where Θ and c correspond to a constant Lorentz matrix and vector, respectively. Note that under two exchanges $(z^{(\alpha)},\tilde{\Lambda}^{(\alpha)},\tilde{a}^{(\alpha)})\to(z^{(\alpha)},\tilde{\Lambda}^{(\alpha)},\tilde{a}^{(\alpha)})$.

If the following conditions are satisfied:

$$\begin{aligned}&\rho^{(2)}=\Theta\rho^{(1)} ,\\ &\lambda^{(2)a}=(\Theta\lambda^{(1)})^a+\epsilon^{abc}c_b(\Theta\rho^{(1)})_c ,\end{aligned} \tag{4.3}$$

for the set of constant vectors $\lambda^{(\alpha)}$ and $\rho^{(\alpha)}$, $\alpha=1,2$, then some work shows that $L_{\rm eff}^{(1)}\rightleftarrows L_{\rm eff}^{(2)}$ under the exchange (4.2). Consequently, the total effective action is exchange invariant when (4.3) is satisfied.

The conditions (4.3) have a simple interpretation. From them it follows that

$$\rho^{(1)}\cdot\rho^{(1)}=\rho^{(2)}\cdot\rho^{(2)}$$

and (4.4)

$$\rho^{(1)}\cdot\lambda^{(1)}=\rho^{(2)}\cdot\lambda^{(2)} .$$

The former equation implies that particles 1 and 2 have the same mass m, while the latter then implies that they have the same spin s. They are thus identical particles.

It appears that we have recovered the usual exchange symmetry for two identical particles, but there is one important difference. Under the exchange (4.2), the particle "momenta" $\tilde{P}^{(1)},\tilde{P}^{(2)}$ and "angular momenta" $\tilde{J}^{(1)},\tilde{J}^{(2)}$ are not simply interchanged. Rather,

$$\begin{aligned}&\tilde{P}^{(1)}\to\Theta\tilde{P}^{(2)},\quad \tilde{P}^{(2)}\to\Theta^{-1}\tilde{P}^{(1)},\\ &\tilde{J}^{(1)a}\to(\Theta\tilde{J}^{(2)})^a+\epsilon^{abd}c_b(\Theta\tilde{P}^{(2)})_d ,\\ &\tilde{J}^{(2)a}\to(\Theta^{-1}\tilde{J}^{(1)})^a-\Theta_b{}^a\epsilon^{bed}c_e\tilde{P}^{(1)}_d .\end{aligned} \tag{4.5}$$

V. THE THREE-PARTICLE SYSTEM

Now $L_{\rm eff}=L_{\rm eff}^{(1)}+L_{\rm eff}^{(2)}+L_{\rm eff}^{(3)}$. Let us again consider exchanging particles 1 and 2. Following Ref. [9] we supplement the transformations (4.2) with

$$z^{(3)}\rightarrow z^{(3)},\quad \tilde{\Lambda}^{(3)}\rightarrow \Phi\tilde{\Lambda}^{(3)},\quad \tilde{a}^{(3)}\rightarrow \Phi\tilde{a}^{(3)}+f\ , \tag{5.1}$$

Φ and f denoting another constant Lorentz matrix and vector, respectively. In order that $L^{(3)}_{\rm eff}\rightarrow L^{(3)}_{\rm eff}$ under the exchange, we must require that

$$\begin{aligned} &\rho^{(1)}=\Phi\rho^{(2)},\quad \rho^{(2)}=\Phi\rho^{(1)}\ ,\\ &\lambda^{(1)}_a=(\Phi\lambda^{(2)})_a+\epsilon_{abc}f^b(\Phi\rho^{(2)})^c\ ,\\ &\lambda^{(2)}_a=(\Phi\lambda^{(1)})_a+\epsilon_{abc}f^b(\Phi\rho^{(1)})^c\ . \end{aligned} \tag{5.2}$$

As in Sec. IV, we require that $L^{(1)}_{\rm eff}\rightleftarrows L^{(2)}_{\rm eff}$ under the exchange. This again implies conditions (4.3), as well as

$$\begin{aligned} &\rho^{(3)}=\Theta\rho^{(3)}\ ,\\ &\lambda^{(3)}_a=(\Theta\lambda^{(3)})_a+\epsilon_{abd}c^b\rho^{(3)d}\ . \end{aligned} \tag{5.3}$$

Thus if all the conditions (4.3), (5.2), and (5.3) can be simultaneously satisfied for $\rho^{(\alpha)}$ and $\lambda^{(\alpha)}$, $\alpha=1,2,3$, the total effective Lagrangian $L_{\rm eff}$ is invariant under the exchange of particles 1 and 2. As before, the conditions (4.3) imply that particles 1 and 2 have the same mass and spin. In addition, using (3.5) we also find

$$\rho^{(2)}\cdot\rho^{(3)}=\rho^{(1)}\cdot\rho^{(3)}\ , \tag{5.4}$$

$$\rho^{(2)}\cdot\lambda^{(3)}+\rho^{(3)}\cdot\lambda^{(2)}=\rho^{(1)}\cdot\lambda^{(3)}+\rho^{(3)}\cdot\lambda^{(1)}\ . \tag{5.5}$$

We can repeat the above procedure for an exchange of particles 2 and 3. If we then demand that the total effective Lagrangian is invariant this exchange, particles 2 and 3 must have the same mass and spin. Furthermore, in analogy with (5.4) and (5.5), $\rho^{(\alpha)}$ and $\lambda^{(\alpha)}$ must satisfy

$$\rho^{(1)}\cdot\rho^{(2)}=\rho^{(1)}\cdot\rho^{(3)}\ , \tag{5.6}$$

$$\rho^{(1)}\cdot\lambda^{(2)}+\rho^{(2)}\cdot\lambda^{(1)}=\rho^{(1)}\cdot\lambda^{(3)}+\rho^{(3)}\cdot\lambda^{(1)}\ . \tag{5.7}$$

Then for the system to be invariant under all possible exchanges of the three particles, the particles must be identical, and must satisfy (5.4)–(5.7). (The conditions on $\rho^{(\alpha)}$ and $\lambda^{(\alpha)}$ are analogous to Eqs. (7.2) and (7.3) in Ref. [9] for the case of the SU(2) Chern-Simons term.)

We now give solutions to the above conditions for various cases.

a. No spin. Here we set $\lambda^{(\alpha)}=0$, $\alpha=1,2,3$. Since all three particles have the same mass $\rho^{(1)}$, $\rho^{(2)}$, and $\rho^{(3)}$, are related by Lorentz transformations. The most general ansatz for $\rho^{(\alpha)}$ (up to an overall Lorentz transformation) is

$$\rho^{(1)}=m\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \rho^{(2)}=m\begin{bmatrix}\cosh\mu\\ \sinh\mu\\ 0\end{bmatrix},\quad \rho^{(3)}=m\begin{bmatrix}\cosh\mu\\ \sinh\mu\cos\nu\\ \sinh\mu\sin\nu\end{bmatrix}\ . \tag{5.8}$$

To satisfy conditions (5.4) and (5.6) we require that the angles μ and ν are related by

$$\sec\nu=1+\mathrm{sech}\mu\ .$$

In (4.2) and (5.1) an exchange of particles 1 and 2 was defined in terms of two Lorentz matrices Θ and Φ, and two Lorentz vectors c and f. Since here $\lambda^{(\alpha)}=0$, we can set the two Lorentz vectors equal to zero. Let us now rename the two Lorentz matrices $\Theta_{(12)}$ and $\Phi_{(12)}$, respectively. In addition, call $\Theta_{(13)}$ and $\Phi_{(13)}$ the corresponding Lorentz matrices associated with the exchange of particles 1 and 3, and $\Theta_{(23)}$ and $\Phi_{(23)}$ the Lorentz matrices associated with the exchange of particles 2 and 3. Invariance of the effective action under particle exchanges requires that

$$\begin{aligned} &\rho^{(\beta)}=\Theta_{(\alpha\beta)}\rho^{(\alpha)},\quad \rho^{(\gamma)}=\Theta_{(\alpha\beta)}\rho^{(\gamma)}\ ,\\ &\rho^{(\beta)}=\Phi_{(\alpha\beta)}\rho^{(\alpha)}\ ,\quad \rho^{(\alpha)}=\Phi_{(\alpha\beta)}\rho^{(\beta)}\ , \end{aligned} \tag{5.9}$$

for $(\alpha,\beta,\gamma)=(1,2,3)$, $(2,3,1)$ and $(3,1,2)$. Using (5.8) we can solve for $\Theta_{(\alpha\beta)}$ and $\Phi_{(\alpha\beta)}$. For this define L_η to be a boost along the 1 axis and R_Ψ to be a rotation in the 1-2 plane, i.e.,

$$\begin{aligned} L_\eta&=\begin{bmatrix}\cosh\eta & \sinh\eta & \\ \sinh\eta & \cosh\eta & \\ & & 1\end{bmatrix}\ ,\\ R_\Psi&=\begin{bmatrix}1 & & \\ & \cos\Psi & -\sin\Psi\\ & \sin\Psi & \cos\Psi\end{bmatrix}\ . \end{aligned} \tag{5.10}$$

In terms of these matrices the solutions of Eqs. (5.9) for $\Theta_{(\alpha\beta)}$ and $\Phi_{(\alpha\beta)}$ can be written as

$$\begin{aligned} &\Theta_{(23)}=R_\nu,\quad \Phi_{(23)}=L_\mu R_{3\nu}L_\mu R_{\pi-3\nu}L_{-\mu},\\ &\Theta_{(12)}=L_\mu R_{2\nu},\quad \Phi_{(12)}=L_\mu R_\pi\ ,\\ &\Theta_{(13)}=L_\mu R_{-\nu}L_{-\mu},\quad \Phi_{(13)}=R_{-3\nu}L_\mu R_\pi L_\mu R_{3\nu}\ , \end{aligned} \tag{5.11}$$

where we are again assuming the relation $\sec\nu=1+\mathrm{sech}\mu$. From these solutions it is easy to verify the identities

$$[\Phi_{(\alpha\beta)}]^2=1\ , \tag{5.12}$$

$$\Phi_{(23)}\Theta_{(12)}\Theta_{(23)}=\Theta_{(12)}\Theta_{(23)}\Phi_{(12)}\ . \tag{5.13}$$

b. Spin parallel to momenta. Here we take $\lambda^{(\alpha)}$ nonzero and proportional to $\rho^{(\alpha)}$, $\lambda^{(\alpha)}=k^{(\alpha)}\rho^{(\alpha)}$. Since all particles are identical the constant of proportionality $k^{(\alpha)}$ is the same for all α. Then equations (5.5) and (5.7) reduce to (5.4) and (5.6), respectively, and this case is identical to case (a). The solutions (5.9) and (5.11) for case (a) apply to this case as well. Furthermore, the Lorentz matrices $\Theta_{(\alpha\beta)}$ and $\Phi_{(\alpha\beta)}$ satisfy

$$\begin{aligned} &\lambda^{(\beta)}=\Theta_{(\alpha\beta)}\lambda^{(\alpha)},\quad \lambda^{(\gamma)}=\Theta_{(\alpha\beta)}\lambda^{(\gamma)},\\ &\lambda^{(\beta)}=\Phi_{(\alpha\beta)}\lambda^{(\alpha)},\quad \lambda^{(\alpha)}=\Phi_{(\alpha\beta)}\lambda^{(\beta)}\ , \end{aligned} \tag{5.14}$$

for $(\alpha,\beta,\gamma)=(1,2,3)$, $(2,3,1)$, and $(3,1,2)$.

c. Arbitrary spin. Here we initially make no special assumptions for $\lambda^{(\alpha)}$, except that $\lambda^{(\alpha)}\cdot\rho^{(\alpha)}$ is the same, namely, $-ms$, for all α, so that we are dealing with identical particles. Now Eqs. (5.14) are no longer valid since the $\lambda^{(\alpha)}$'s are, in general, not obtainable from one another solely by the Lorentz transformations $\Theta_{(\alpha\beta)}$ and $\Phi_{(\alpha\beta)}$. We must allow for translations, as well as Lorentz transformations. In (4.2) and (5.1) the former were parametrized by c and f for the exchange of particles 1 and 2. Let us rename these parameters $c_{(12)}$ and $f_{(12)}$, respectively, or more generally, let $c_{(\alpha\beta)}$ and $f_{(\alpha\beta)}$ be associated with the exchanges of particles α and β in the set (α,β,γ). Then instead of (5.14), we must require

$$\begin{aligned}
\lambda_a^{(\beta)}&=(\Theta_{(\alpha\beta)}\lambda^{(\alpha)})_a+\epsilon_{abd}c^b_{(\alpha\beta)}\rho^{(\beta)d} ,\\
\lambda_a^{(\gamma)}&=(\Theta_{(\alpha\beta)}\lambda^{(\gamma)})_a+\epsilon_{abd}c^b_{(\alpha\beta)}\rho^{(\gamma)d} ,\\
\lambda_a^{(\alpha)}&=(\Phi_{(\alpha\beta)}\lambda^{(\beta)})_a+\epsilon_{abd}f^b_{(\alpha\beta)}\rho^{(\alpha)d} ,\\
\lambda_a^{(\beta)}&=(\Phi_{(\alpha\beta)}\lambda^{(\alpha)})_a+\epsilon_{abd}f^b_{(\alpha\beta)}\rho^{(\beta)d} ,
\end{aligned} \tag{5.15}$$

for $(\alpha,\beta,\gamma)=(1,2,3)$, (2,3,1), and (3,1,2), in order that the total Lagrangian is exchange invariant. From these relations we can find a set of consistency conditions:

$$\begin{aligned}
(\Theta_{(\alpha\beta)}\lambda^{(\alpha)})\cdot\rho^{(\beta)}&=(\Theta_{(\alpha\beta)}\lambda^{(\gamma)})\cdot\rho^{(\gamma)}\\
&=(\Phi_{(\alpha\beta)}\lambda^{(\beta)})\cdot\rho^{(\alpha)}\\
&=(\Phi_{(\alpha\beta)}\lambda^{(\alpha)})\cdot\rho^{(\beta)}=-ms.
\end{aligned} \tag{5.16}$$

These equations are trivially satisfied for case (b), $\lambda^{(\alpha)}=k\rho^{(\alpha)}$, as they correspond to the solution $c_{(\alpha\beta)}=f_{(\alpha\beta)}=0$. Using (5.9), we can verify that the consistency conditions (5.16) are also satisfied by

$$\lambda^{(1)}=k\rho^{(2)},\quad \lambda^{(2)}=k\rho^{(3)},\quad \lambda^{(3)}=k\rho^{(1)} , \tag{5.17}$$

or

$$\lambda^{(1)}=k\rho^{(3)},\quad \lambda^{(2)}=k\rho^{(1)},\quad \lambda^{(3)}=k\rho^{(2)} . \tag{5.18}$$

For such configurations $c_{(\alpha\beta)}$ and $f_{(\alpha\beta)}$ are not all zero and can be solved for from Eqs. (5.15). We shall not do so explicitly, but rather just note an identity. From the last two equations in (5.15) it follows that the cross products of $\Phi_{(\alpha\beta)}f_{(\alpha\beta)}+f_{(\alpha\beta)}$ with $\rho^{(\alpha)}$ and with $\rho^{(\beta)}$ must simultaneously vanish. Then for $\rho^{(\alpha)}$ not parallel to $\rho^{(\beta)}$, we have

$$\Phi_{(\alpha\beta)}f_{(\alpha\beta)}+f_{(\alpha\beta)}=0 . \tag{5.19}$$

In the next section, this identity will be used in proving that the exchange operator squared is the identity.

VI. QUANTUM THEORY

The quantum algebra for a free massive spinning particle [as described by the Lagrangian L_0 in Eq. (2.17)] corresponds to the Poincaré algebra [10]:

$$\begin{aligned}
&[\mathbf{P}_a,\mathbf{P}_b]=0,\quad [\mathbf{J}_a,\mathbf{J}_b]=i\epsilon_{abc}\mathbf{J}^c ,\\
&[\mathbf{J}_a,\mathbf{P}_b]=i\epsilon_{abc}\mathbf{P}^c .
\end{aligned} \tag{6.1}$$

$\mathbf{P}_a$ and $\mathbf{J}_a$ are the quantum operators associated with the conserved momentum and angular momentum satisfying (2.14) and (2.15).

Let $\{\mathbf{U}(\Lambda,a)\}$ be a unitary irreducible representation of ISO(2,1). Then $\mathbf{U}(\Lambda,a)\mathbf{U}(\Theta,b)=\mathbf{U}(\Lambda\Theta,\Lambda b+a)$. From the commutation relations (6.1),

$$\begin{aligned}
&\mathbf{U}(\Lambda,a)^{-1}\mathbf{P}_a\mathbf{U}(\Lambda,a)=\Lambda_{ab}\mathbf{P}^b ,\\
&\mathbf{U}(\Lambda,a)^{-1}\mathbf{J}_a\mathbf{U}(\Lambda,a)=\Lambda_{ab}\mathbf{J}^b+\epsilon_{abc}a^b(\Lambda\mathbf{P})^c .
\end{aligned} \tag{6.2}$$

The Hilbert space $\{|\xi,\eta,m,s\rangle\}$ can be constructed by utilizing the method of induced representations [12]. For a particle of mass m and spin s we define the state $|0,0,m,s\rangle$ by

$$\begin{aligned}
&\mathbf{P}_a|0,0,m,s\rangle=m\eta_{a0}|0,0,m,s\rangle ,\\
&\mathbf{J}\cdot\mathbf{P}|0,0,m,s\rangle=-ms|0,0,m,s\rangle .
\end{aligned} \tag{6.3}$$

Again η_{ab} is the Minkowski metric tensor. Under the action of $\mathbf{J}_0$ this state changes by a phase

$$\begin{aligned}
\mathbf{U}(R_\psi,0)|0,0,m,s\rangle&=e^{-i\mathbf{J}_0\Psi}|0,0,m,s\rangle\\
&=e^{-is\Psi}|0,0,m,s\rangle .
\end{aligned} \tag{6.4}$$

The remaining states in the Hilbert space may be obtained by first acting on $|0,0,m,s\rangle$ with a boost $\mathbf{U}(L_\eta,0)$, $\eta\neq0$, along the 1 axis and then, with a rotation $\mathbf{U}(R_\xi,0)$ [cf. Eqs. (5.10)],

$$\begin{aligned}
|\xi,\eta,m,s\rangle&\equiv\mathbf{U}(R_\xi,0)|0,\eta,m,s\rangle\\
&=\mathbf{U}(R_\xi,0)\mathbf{U}(L_\eta,0)|0,0,m,s\rangle .
\end{aligned} \tag{6.5}$$

Then

$$P|\xi,\eta,m,s\rangle=m\begin{bmatrix}\cosh\eta\\ \sinh\eta\ \cos\xi\\ \sinh\eta\ \sin\xi\end{bmatrix}|\xi,\eta,m,s\rangle . \tag{6.6}$$

A boost and rotation acts on the state $|\xi,\eta,m,s\rangle$ as

$$\begin{aligned}
&\mathbf{U}(R_\psi,0)|\xi,\eta,m,s\rangle=|\xi+\psi,\eta,m,s\rangle,\quad \text{for } \eta\neq0 ,\\
&\mathbf{U}(L_\chi,0)|\xi,\eta,m,s\rangle=e^{-is\psi'}|\xi',\eta',m,s\rangle ,
\end{aligned} \tag{6.7}$$

where ψ', ξ', and η' are related to χ,ξ, and η by $R_{\xi'}L_{\eta'}R_{\psi'}=L_\chi R_\xi L_\eta$.

For the two-particle system we define two sets of operators $\mathbf{P}_a^{(\alpha)}$ and $\mathbf{J}_a^{(\alpha)}$, $\alpha=1,2$. They are the operator analogues of the conserved momenta and angular momenta of the classical theory. When $z^{(\alpha)}\in M\ \cup_{\beta\neq\alpha}\Gamma^{(\beta)}$, the latter are $\tilde{P}^{(\alpha)}$ and $\tilde{J}^{(\alpha)}$ defined in Eq. (3.3). When $z^{(\alpha)}\in\Gamma^{(\beta)}$, the relevant conserved momenta and angular momenta are $\tilde{P}^{(\alpha,\beta)}$ and $\tilde{J}^{(\alpha,\beta)}$ given in Eqs. (3.8). As stated in Sec. III, the conserved momenta and angular momenta are not globally defined; i.e., $\tilde{P}^{(\alpha,\beta)}\neq\tilde{P}^{(\alpha)}$ and $\tilde{J}^{(\alpha,\beta)}\neq\tilde{J}^{(\alpha)}$ at the boundary $\phi^{(\beta)}=2\pi$ of $\Gamma^{(\beta)}$. Thus the classical values of the conserved momenta and angular momenta are not continuous when the particle crosses the boundary. This, however, does not appear to lead to difficulties in the quantum-mechanical description of the system, because we do not promote the space-time coordinate $z^{(\alpha)}$ to a quantum operator. On the other hand, a quantum mechanical position operator can be defined for the system (cf. Ref. [10]). It has the unusual property that different space-time components do not commute when spin is present.

To obtain the commutation relations for the two-particle system, we replace $\mathbf{P}_a$ and $\mathbf{J}_a$ with $\mathbf{P}_a^{(\alpha)}$ and $\mathbf{J}_a^{(\alpha)}$, respectively, in Eqs. (6.1). In addition,

$$[\mathbf{P}_a^{(\alpha)},\mathbf{P}_b^{(\beta)}]=[\mathbf{J}_a^{(\alpha)},\mathbf{P}_b^{(\beta)}]=[\mathbf{J}_a^{(\alpha)},\mathbf{J}_b^{(\beta)}]=0\quad\text{for } \alpha\neq\beta . \tag{6.8}$$

The Hilbert space consists of the tensor product states $|1\rangle\times|2\rangle$, where $|\alpha\rangle$ denotes the ket$|\xi_\alpha,\eta_\alpha,m_\alpha,s_\alpha\rangle$. The operators $\mathbf{P}_a^{(\alpha)}$ and $\mathbf{J}_a^{(\alpha)}$ act nontrivially only on the αth ket in the tensor product.

Next we define an exchange of particles 1 and 2. The exchange which leaves the effective action invariant is not given by $|1\rangle\times|2\rangle\rightarrow|2\rangle\times|1\rangle$. Instead, it must involve ISO(2,1) transformations on the states. This is because the eigenvalues of $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ are not simply switched under an exchange. When $z^{(1)}\notin\Gamma^{(2)}$ and $z^{(2)}\notin\Gamma^{(1)}$, they transform as $\tilde{P}^{(1)}$ and $\tilde{P}^{(2)}$ in Eq. (4.5). For this case we may define an exchange operator according to

$$\sigma|1\rangle\times|2\rangle=U_{(1)}(\Theta,c)\mathbf{U}_{(2)}(\Theta,c)^{-1}|2\rangle\times|1\rangle\ , \qquad (6.9)$$

where $\{\mathbf{U}_{(\alpha)}(\Theta,c)\}$ is a unitary representation of the ISO(2,1) group element constructed from generators $\mathbf{P}_a^{(\alpha)}$ and $\mathbf{J}_a^{(\alpha)}$. [We could introduce an arbitrary phase in (6.9); however, such a phase could be absorbed in a redefinition of $|1\rangle\times|2\rangle$ or $|2\rangle\times|1\rangle$.] From (6.9) it follows that σ^2 acting on any tensor product state is $\mathbf{1}$.

This result is also valid for the case of $z^{(1)}\in\Gamma^{(1)}$ or $z^{(2)}\in\Gamma^{(1)}$. For example, when $z^{(1)}\in\Gamma^{(2)}$, the quantum operator $\mathbf{P}^{(1)}$ transforms under an exchange as the classically conserved momentum $\tilde{P}^{(1,2)}$ defined in (3.8). This involves the ISO (2,1) element $\mathbf{U}_{(\alpha)}(\Xi^{(2)},h^{(2)})$, as well as $\mathbf{U}_{(\alpha)}(\Theta,c)$. [$\Xi^{(2)}$ and $h^{(2)}$ are defined in (3.7).] More precisely, for $z^{(1)}\in\Gamma^{(2)}$ and $z^{(2)}\notin\Gamma^{1}$, the exchange operator is defined as

$$\sigma|1\rangle\times|2\rangle=\mathbf{U}_{(1)}(\Xi^{(2)},h^{(2)})\mathbf{U}_{(1)}(\Theta,c)\mathbf{U}_{(2)}(\Theta,c)^{-1}\times\mathbf{U}_{(2)}(\Xi^{(2)},h^{(2)})^{-1}|2\rangle\times|1\rangle\ . \qquad (6.10)$$

Once again it follows that $\sigma^2=\mathbf{1}$.

Now let $\alpha=1,2,3$. The Hilbert space is spanned by the tensor product sates $|1\rangle\times|2\rangle\times|3\rangle$. Let the exchange of particles 1 and 2 be obtained from the action of σ_1. This action does not simply switch the first two kets in the tensor product, but from Sec. V, it involves group elements $\mathbf{U}_{(\alpha)}(\Theta_{(12)},c_{(12)})$ and $\mathbf{U}_{(\alpha)}(\Phi_{(12)},f_{(12)})$. The properties of $\Theta_{(\alpha\beta)}$, $\Phi_{(\alpha\beta)}$, $c_{(\alpha\beta)}$, and $f_{(\alpha\beta)}$ were discussed in Sec. V. If we assume $z^{(\beta)}\in M\setminus\bigcup_{\alpha\neq\beta}\Gamma^{(\alpha)}$, for all β, σ_1 can be defined as

$$\sigma_1|1\rangle\times|2\rangle\times|3\rangle=\mathbf{U}_{(1)}(\Theta_{(12)},c_{(12)})\mathbf{U}_{(2)}(\Theta_{(12)},c_{(12)})^{-1}\times\mathbf{U}_{(3)}(\Phi_{(12)},f_{(12)})|2\rangle\times|1\rangle\times|3\rangle\ . \qquad (6.11)$$

Then applying σ_1 twice gives

$$\sigma_1^2=\mathbf{U}_{(3)}(\Phi_{(12)},f_{(12)})^2=\mathbf{U}_{(3)}(1,\Phi_{(12)}f_{(12)}+f_{(12)})=\mathbf{1}\ , \qquad (6.12)$$

where we have used (5.12). The last line follows for cases (a) and (b) in Sec. V, since there $f_{(\alpha\beta)}=0$. It follows for case (c) by the identities (5.19).

Similar results appear for an exchange between particles 2 and 3. We denote the corresponding exchange operator by σ_2:

$$\sigma_2|1\rangle\times|2\rangle\times|3\rangle=\mathbf{U}_{(1)}(\Phi_{(23)},f_{(23)})\times\mathbf{U}_{(2)}(\Theta_{(23)},c_{(23)})\times\mathbf{U}_{(3)}(\Theta_{(23)},c_{(23)})^{-1}\times|1\rangle\times|3\rangle\times|2\rangle\ , \qquad (6.13)$$

which yields

$$\sigma_2^2=\mathbf{U}_{(1)}(\Phi_{(23)},f_{(23)})^2=\mathbf{U}_{(1)}(1,\Phi_{(23)}f_{(23)}+f_{(23)})=\mathbf{1}\ . \qquad (6.14)$$

Using identity (5.13), we can further show that the braid condition

$$\sigma_1\sigma_2\sigma_1=\sigma_2\sigma_1\sigma_2 \qquad (6.15)$$

is satisfied. [For this we also need the relation

$$\Phi_{(23)}(\Theta_{(12)}c_{(23)}+c_{(12)})+f_{(23)}=\Theta_{(12)}(\Theta_{(23)}f_{(12)}+c_{(23)})+c_{(12)}\ . \qquad (6.16)$$

Equation (6.16) is easily satisfied for the case (a) of no spin and case (b), since there, as well, $c_{(\alpha\beta)}=f_{(\alpha\beta)}=0$. On the other hand, the proof of this relation is not obvious for the case (c) of arbitrary spin.]

VII. QUANTIZATION AMBIGUITY AND GRAVITATIONAL ANYONS

In Sec. VI, N-particle wave functions were constructed by taking tensor products of induced representations of ISO(2,1). Exchange operators were defined and only fermionic or bosonic statistics resulted. As stated in Sec. III, the manifold Q spanned by the particle space-time coordinates $z^\mu_{(\alpha)}$ is not simply connected. It is known [13] that quantization ambiguities can occur when the classical configuration space is not simply connected. In this section, we exploit these ambiguities to construct alternative quantum theories for N particles where nontrivial statistics occurs.

The N-particle wave functions of Sec. VI were eigenstates of the momenta operators $\mathbf{P}_a^{(\alpha)}$. The angular momenta $\mathbf{J}_a^{(\alpha)}$ had a well-defined action on the states. But from Sec. III, it was shown that the classical analogs of $\mathbf{P}_a^{(\alpha)}$ and $\mathbf{J}_a^{(\alpha)}$ are not globally defined. By rotating particle α around particle β by 2π, the conserved quantities $\tilde{P}_a^{(\alpha)}$ and $\tilde{J}_a^{(\alpha)}$ transform under the action of an ISO(2,1) group element $t^{(\beta)}$. [$t^{(\beta)}$ was defined from $\Xi^{(\beta)}$ and $h^{(\beta)}$ of Eq. (3.7) evaluated at the boundary $\phi^{(\beta)}=2\pi$. The result depends on $\rho^{(\beta)}$ and $\lambda^{(\beta)}$.] Hence $\tilde{P}_a^{(\alpha)}$ and $\tilde{J}_a^{(\alpha)}$ are not classically observable degrees of freedom. Physical degrees of freedom are obtained only after moding out the action of fundamental group $\pi_1(Q)$. Under the action of $\pi_1(Q)$, $\tilde{P}_a^{(\alpha)}$ and $\tilde{J}_a^{(\alpha)}$ undergo ISO(2,1) transformations. The latter are generated by $t^{(\beta)}$.

For the case of two particles, let $\mathbf{t}^{(\beta)}$ denote the quantum-mechanical analogue of $t^{(\beta)}$. We define its action on the states by

$$\mathbf{t}^{(1)}|1\rangle|2\rangle=\mathbf{U}_{(2)}(t^{(1)})|1\rangle|2\rangle ,$$
$$\mathbf{t}^{(2)}|1\rangle|2\rangle=\mathbf{U}_{(1)}(t^{(2)})|1\rangle|2\rangle . \quad (7.1)$$

Since $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$ act nontrivially on different kets in the tensor product, they commute and thus generate an Abelian group corresponding to $\pi_1(Q)$. $t^{(1)}$ and $t^{(2)}$ are not independent. Using the ISO(2,1) group element defined from Θ and c, we can transform $t^{(1)}$ to $t^{(2)}$. This is since from Eqs. (4.3), Θ and c transform $\rho^{(1)}$ and $\lambda^{(1)}$ to $\rho^{(2)}$ and $\lambda^{(2)}$. Quantum mechanically, this yields the identity

$$\mathbf{U}(t^{(2)})=\mathbf{U}(\Theta,c)\mathbf{U}(t^{(1)})\mathbf{U}(\Theta,c)^{-1} , \quad (7.2)$$

$\mathbf{U}$ being a unitary representation of ISO(2,1).

Exchange operators $\boldsymbol{\sigma}$ for the two-particle system were defined in Eqs. (6.9) and (6.10). Since physical observables are defined after moding out the action of $\pi_1(Q)$, we could equally well define $\boldsymbol{\sigma}\mathbf{t}^{(1)}$ to be the exchange operator for the two-particle system. Using (6.9) and (7.1),

$$\boldsymbol{\sigma}\mathbf{t}^{(1)}|1\rangle|2\rangle=\mathbf{U}_{(1)}(\Theta,c)\mathbf{U}_{(1)}(t^{(1)})\mathbf{U}_{(2)}(\Theta,c)^{-1}|2\rangle|1\rangle . \quad (7.3)$$

Upon taking its square, we find

$$\begin{aligned}(\boldsymbol{\sigma}\mathbf{t}^{(1)})^2|1\rangle|2\rangle&=\mathbf{U}_{(1)}(\Theta,c)\mathbf{U}_{(1)}(t^{(1)})\\&\quad\times\mathbf{U}_{(1)}(\Theta,c)^{-1}\mathbf{U}_{(2)}(t^{(1)})|1\rangle|2\rangle\\&=\mathbf{U}_{(1)}(t^{(2)})\mathbf{U}_{(2)}(t^{(1)})|1\rangle|2\rangle\\&=\mathbf{t}^{(1)}\mathbf{t}^{(2)}|1\rangle|2\rangle ,\end{aligned} \quad (7.4)$$

where we have used (7.1) and (7.2). Now the exchange operator squared is not the identity, but an operator associated with an element of $\pi_1(Q)$. This result persists if we more generally define the exchange operator to be σ times any element of the fundamental group.

It is known [13] that the action of $\pi_1(Q)$ on wave functions written on the universal covering space of Q can be defined in such a way that it commutes with all the observable operators of the theory. The Hilbert space for such theories can then be decomposed into irreducible representations of $\pi_1(Q)$. For Abelian fundamental groups, we can simultaneously diagonalize the generators $\mathbf{t}^{(\alpha)}$ along with a complete set of commuting observables. The eigenvalues of $\mathbf{t}^{(\alpha)}$ are phases which serve to label the irreducible representation. For the system of two particles described above, the phases are associated with the statistics of the particles.

We thus find that the statistics for the two-particle system are ambiguous for two reasons: (1) The exchange operator can be chosen to be $\boldsymbol{\sigma}$ times an operator associated with any element of $\pi_1(Q)$; (2) after defining the exchange operator, its eigenvalue depends on which particular irreducible representation we choose for $\pi_1(Q)$.

The above analysis for two particles can readily be generalized to three or more particles. In this regard, we once again note that the generators $t^{(\alpha)}$ no longer commute and $\pi_1(Q)$ corresponds to the braid group. For three particles, $t^{(1)}$ can act either on particle 2 or 3. Therefore we can construct two different quantum operators $\mathbf{t}^{(1,2)}$ and $\mathbf{t}^{(1,3)}$ associated with the ISO(1,2) group group element $t^{(1)}$:

$$\mathbf{t}^{(1,2)}|1\rangle|2\rangle|3\rangle=\mathbf{U}_{(2)}(t^{(1)})|1\rangle|2\rangle|3\rangle ,$$
$$\mathbf{t}^{(1,3)}|1\rangle|2\rangle|3\rangle=\mathbf{U}_{(3)}(t^{(1)})|1\rangle|2\rangle|3\rangle . \quad (7.5)$$

More generally, define $\mathbf{t}^{(\alpha,\beta)}$ according to

$$\mathbf{t}^{(\alpha,\beta)}|1\rangle|2\rangle|3\rangle=\mathbf{U}_{(\beta)}(t^{(\alpha)})|1\rangle|2\rangle|3\rangle . \quad (7.6)$$

As was the case for two particles, $\mathbf{t}^{(\alpha,\beta)}$ are not all independent. The generalization of identity (7.2) to the case of three particles is

$$\begin{aligned}\mathbf{U}(t^{(\beta)})&=\mathbf{U}(\Theta_{(\alpha,\beta)},c_{(\alpha\beta)})\mathbf{U}(t^{(\alpha)})\mathbf{U}(\Theta_{(\alpha\beta)},c_{(\alpha\beta)})^{-1} ,\\\mathbf{U}(t^{(\gamma)})&=\mathbf{U}(\Theta_{(\alpha\beta)},c_{(\alpha\beta)})\mathbf{U}(t^{(\gamma)})\mathbf{U}(\Theta_{(\alpha\beta)},c_{(\alpha\beta)})^{-1} ,\\\mathbf{U}(t^{(\beta)})&=\mathbf{U}(\Phi_{(\alpha\beta)},f_{(\alpha\beta)})\mathbf{U}(t^{(\alpha)})\mathbf{U}(\Phi_{(\alpha\beta)},f_{(\alpha\beta)})^{-1} ,\\\mathbf{U}(t^{(\alpha)})&=\mathbf{U}(\Phi_{(\alpha\beta)},f_{(\alpha\beta)})\mathbf{U}(t^{(\beta)})\mathbf{U}(\Phi_{(\alpha\beta)},f_{(\alpha\beta)})^{-1} ,\end{aligned} \quad (7.7)$$

for $(\alpha,\beta,\gamma)=(1,2,3)$, (2,3,1), and (3,1,2). Identities (7.7) follow from (5.9) and (5.15).

As an example let us choose $\boldsymbol{\sigma}_1\mathbf{t}^{(1,2)}$ and $\boldsymbol{\sigma}_2\mathbf{t}^{(2,3)}$ to be exchange operators for the three-particle system, where $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ were defined in (6.11) and (6.13). $\boldsymbol{\sigma}_1\mathbf{t}^{(1,2)}$ acts on a three-particle state as

$$\begin{aligned}\boldsymbol{\sigma}_1\mathbf{t}^{(1,2)}|1\rangle|2\rangle|3\rangle&=\mathbf{U}_{(1)}(\Theta_{(12)},c_{(12)})\mathbf{U}_{(2)}(\Theta_{(12)},c_{(12)})^{-1}\\&\quad\times\mathbf{U}_{(3)}(\Phi_{(12)},f_{(12)})\mathbf{U}_{(1)}(t^{(1)})|2\rangle|1\rangle|3\rangle .\end{aligned} \quad (7.8)$$

Upon taking its square we find

$$(\sigma_1\mathbf{t}^{(1,2)})^2=\mathbf{t}^{(1,2)}\mathbf{t}^{(2,1)} , \quad (7.9)$$

where we have used identities (7.7). Similarly,

$$(\boldsymbol{\sigma}_2\mathbf{t}^{(2,3)})^2=\mathbf{t}^{(2,3)}\mathbf{t}^{(3,2)} . \quad (7.10)$$

Thus now, as in the two-particle case, the exchange operator squared is not the identity, but an operator associated with an element of $\pi_1(Q)$. We expect that this result is independent of the particular choice made for the exchange operator.

We again conclude that the statistics is ambiguous for two reasons: (1) The exchange operator can be chosen to be σ times an operator associated with any element of $\pi_1(Q)$. (2) After defining the exchange operator, the result depends on which particular irreducible representation we choose for $\pi_1(Q)$.

Finally, we remark that although $\boldsymbol{\sigma}_\alpha$ defined in Sec. VI satisfy the braid relations [cf. Eq. (6.15)], the exchange operators $\boldsymbol{\sigma}_1\mathbf{t}^{(1,2)}$ and $\boldsymbol{\sigma}_2\mathbf{t}^{(2,3)}$ defined above do not, in general, satisfy the braid relation. Using the identities (7.7), we find

$$\boldsymbol{\sigma}_1\boldsymbol{t}^{(1,2)}\boldsymbol{\sigma}_2\boldsymbol{t}^{(2,3)}\boldsymbol{\sigma}_1\boldsymbol{t}^{(1,2)}=\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{t}^{(1,2)}\boldsymbol{t}^{(2,3)}\boldsymbol{t}^{(1,3)} , \quad (7.11)$$

while

$$\boldsymbol{\sigma}_2\boldsymbol{t}^{(2,3)}\boldsymbol{\sigma}_1\boldsymbol{t}^{(1,2)}\boldsymbol{\sigma}_2\boldsymbol{t}^{(2,3)}=\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{t}^{(1,2)}\boldsymbol{t}^{(1,3)}\boldsymbol{t}^{(2,3)} . \quad (7.12)$$

Now applying (6.15), we conclude that the braid relation for operators $\boldsymbol{\sigma}_1\boldsymbol{t}^{(1,2)}$ and $\boldsymbol{\sigma}_2\boldsymbol{t}^{(2,3)}$ is satisfied only if

$$t^{(2,3)}t^{(1,3)}=t^{(1,3)}t^{(2,3)} . \quad (7.13)$$

Equation (7.13) is precisely the braid quantization condition mentioned in passing in Sec. III and discussed in detail in Ref. [9] [cf. Eq. (5.18) of that reference]. A trivial solution to (7.13) would correspond to all $\rho^{(\alpha)}$'s being equal and all $\lambda^{(\alpha)}$'s equal. In that case we can take all $\Theta_{(\alpha\beta)}=\Phi_{(\alpha\beta)}=1$ and $c_{(\alpha\beta)}=f_{(\alpha\beta)}=0$. It is not apparent whether other solutions exist to (7.13).

Instead of $\sigma_1 t^{(1,2)}$ and $\sigma_2 t^{(2,3)}$, we can define alternative exchange operators which give exotic statics (i.e., their square is not one), but they will not in general satisfy the $N=3$ braid group relation, and hence not generate the braid group. If we require that exchange operators generate the braid group for arbitrary $\rho^{(\alpha)}$ and $\lambda^{(\alpha)}$ consistent with exchange invariance, we are left with operators such as σ_α which do not have exotic statistics.

VIII. CONCLUDING REMARKS

The preceding analysis is easily generalized to $N>3$ particles. Exchange invariance of the effective Lagrangian requires that all particles have the same values of mass and spin; i.e., they are "identical." Furthermore, any two particles α and β of the set must have the same value for $\rho^{(\alpha)}\cdot\rho^{(\beta)}$ and $\rho^{(\alpha)}\cdot\lambda^{(\beta)}+\lambda^{(\alpha)}\cdot\rho^{(\beta)}$. In the quantum theory we again find that squares of all operators σ_α are equal to the identity. Exotic statistical properties for the particles can be obtained by defining the exchange operator to be σ_α times elements of $\pi_1(Q)$. The exchange operators are not standard ones. This is because, as in Eqs. (6.9)–(6.11), σ_α involve Poincaré transformations on the states of the tensor product, which ensures that the eigenvalues of momentum are not naively exchanged after acting with σ_α.

As we found in Sec. VII, the statistics of particles with a fixed spin s are ambiguous in the first-quantized theory. Thus at this level of the theory, we do not obtain a connection between the spin and statistics of particles in $2+1$ gravity. (Recently the spin-statistics theorem was proved without the use of field theory, but just assuming the existence of antiparticles [14].)

Finally we discuss deficiencies of our analysis, which we shall address in a future work.

(i) We have ignored self-interactions of particles. Self-interactions are known to have the effect of inducing a spin to particles in Chern-Simons theory [9–15]. Since spin is already present in our system, self-interactions will possibly redefine the "effective spin" of the particles. It would seem unlikely that the effective spin of the particle is such that the spin-statistics theorem is recovered. On the other hand, a similar effect was shown to occur in a related problem, where spin is induced to a spinless particle with the addition of an SO(2,1) Chern-Simons term to the Einstein gravity action. Deser and McCarthy [16] finds that this induced spin is such that the spin-statistics theorem is valid.

(ii) We have not studied the "braid quantization conditions" of Ref. [9] in detail. They appear to be necessary if one wishes to construct exchange operators which give nontrivial statistics and satisfy the braid group relations. As mentioned in Sec. III, they are analogous to the Dirac charge quantization condition which occurs in the charge-monopole system. The braid quantization conditions were obtained by considering three particles, i.e., a test particle in the presence of two sources, in Chern-Simons theory. They were a consequence of the fact that the relevant fiber bundle of the theory is nontrivial and a global section of the bundle does not exist. For us, if α and β correspond to two sources we had to exclude points $z^{(\alpha)}\in\Gamma^{(\beta)}$ and $z^{(\beta)}\in\Gamma^{(\alpha)}$ from the configuration space, as the solutions (2.20), (2.21), and (2.23) were not valid such points. The authors of Ref. [9] required that the exponential of the "flux" through such points is the identity. They defined this exponential by taking the product of group elements, which became the operators $t^{(\alpha,\beta)}$ upon quantization. The braid quantization conditions such as (7.13) resulted, because the group elements in the product do not in general commute. But such conditions may not in fact be necessary. For instance, no such condition will result if we instead define the exponential of the flux as the exponential of a sum of Lie-algebra elements. Because we can commute and cancel terms in the sum, the exponential of the flux will be identically one in this case.

(iii) In (ii) quantization conditions result for (three particles, i.e.) two sources and a test particle. There we ignored the self-interaction of the test particle. But if we choose not to ignore self-interactions, we can obtain quantization conditions for the two-particle system. In that case we cannot say that we have one source and one test particle, but both particles, labeled by 1 and 2, should be simultaneously treated as sources. Then we must again exclude points $z^{(1)}\in\Gamma^{(2)}$ and $z^{(2)}\in\Gamma^{(1)}$. A quantization condition may then follow for the two-particle (self-interacting) system, which would be analogous to the braid quantization conditions of Ref. [9].

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*Permanent address.

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