PHYSICAL REVIEW D VOLUME 44, NUMBER 1 1 JULY 1991

# Relativistic treatment of fermion-antifermion bound states

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(Received 4 March 1991)

We discuss the relativistic treatment of fermion-antifermion bound states by an effective-Hamiltonian method which imitates their description in terms of nonrelativistic potential models: the effective interaction potential, to be used in a Schrödinger equation which incorporates relativistic kinematics, is derived from the underlying quantum field theory. This approach is equivalent to the instantaneous approximation to the Bethe-Salpeter equation called the Salpeter equation but comes closer to physical intuition than the latter one.

## I. INTRODUCTION

Beyond doubt, the appropriate framework for a relativistic description of quantum systems is quantum field theory. However, the treatment of bound states within this framework becomes a rather cumbersome and unrewarding task. Therefore, we would like to argue here in favor of an effective-Hamiltonian method. This approach is reminiscent of the investigation of bound states in terms of nonrelativistic potential models. It allows one, however, to remain from the very beginning on fully relativistic grounds whereas the nonrelativistic formalism incorporates only the static limit of the theory and its first relativistic corrections, summarized in the form of the well-known Breit-Fermi Hamiltonian.

The idea behind the effective-Hamiltonian method is very simple. The interaction between the bound-state constituents is described by an effective potential. By considering the elastic scattering of the particles which build up the bound state, this potential is derived from the quantum field theory which describes, in fact, their basic interaction. With this effective potential at hand, the Hamiltonian controlling the bound-state system is constructed.

This paper is organized as follows. In Sec. II we briefly recall some generalities concerning the description of scattering processes in quantum theory, which we need in Sec. III in order to derive an effective interaction potential from quantum field theory. In Sec. IV we give the relevant transition amplitudes for the physically most interesting case, that is, a fermion-antifermion system.

Since the nonrelativistic limit has been the object of investigation already for some decades, we focus in Sec. V our interest to the opposite extreme, the massless and thus ultrarelativistic case. There the expressions obtained for the scattering amplitudes before simplify considerably. Section VI is devoted to the eigenvalue problem of our effective Hamiltonian.

The mass difference between corresponding spin-singlet and spin-triplet mesons serves as a tool to demonstrate in Sec. VII the wide range of applicability of the effective-Hamiltonian method as just one illustrative example. Section VIII summarizes the proposed procedure and points out its relation to the quantum-field-theoretic description of bound states. In Sec. II we retain Planck's constant $\hbar$, whereas for the remainder of the paper we use natural units such that $\hbar=c=1$.

## II. SCATTERING THEORY

Let us recall some general facts about the description of scattering processes in quantum theory [1–3].

We assume that the full Hamiltonian $\mathsf{H}$ governing the dynamics of the quantum-mechanical system under consideration can be split up into a free (or, more generally, unperturbed) Hamiltonian $\mathsf{H}_0$ and an interaction potential (or perturbation) $V$,

$$\mathsf{H}(t)=\mathsf{H}_0+\mathsf{V}(t) \ . \tag{1}$$

In the interaction picture the evolution in time of the Hilbert-space state vectors $|\psi(t)\rangle$ is controlled by a (unitary) time-evolution operator $\mathsf{U}(t,t_0)$,

$$|\psi(t)\rangle=\mathsf{U}(t,t_0)|\psi(t_0)\rangle \ , \tag{2}$$

which satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\mathsf{U}(t,t_0)=\mathsf{V}(t)\mathsf{U}(t,t_0), \quad \mathsf{U}(t_0,t_0)=1 \ , \tag{3}$$

or the equivalent integral equation

$$\mathsf{U}(t,t_0)=1-\frac{i}{\hbar}\int_{t_0}^{t}dt'\mathsf{V}(t')\mathsf{U}(t',t_0) \ . \tag{4}$$

In scattering theory, the physical situation of interest is to specify initial states $|i\rangle$ at time $t=-\infty$ and final states $|f\rangle$ at time $t=+\infty$ by unperturbed states $|a\rangle$, i.e., eigenstates of the free Hamiltonian $\mathsf{H}_0$ corresponding to some energy eigenvalue $E$:

$$\mathsf{H}_0|a\rangle=E|a\rangle \ . \tag{5}$$

 

With the help of $\mathsf{U}(t,t_0)$ the corresponding states for finite times, in particular for $t=0$, are defined by

$$|a,\text{in}\rangle \equiv \mathsf{U}(0,-\infty)|a\rangle \ , \tag{6}$$

$$|a,\text{out}\rangle \equiv \mathsf{U}(0,+\infty)|a\rangle \ . \tag{7}$$

Because of the integral equation (4) for $\mathsf{U}(t,t_0)$, the states $|a,\text{in}\rangle$, $|a,\text{out}\rangle$ satisfy the (implicit) Lippmann-Schwinger equations

$$|a,\text{in}\rangle = |a\rangle + \lim_{\epsilon\to+0} \frac{1}{E-\mathsf{H}_0+i\hbar\epsilon}\mathsf{V}_S|a,\text{in}\rangle \ , \tag{8}$$

$$|a,\text{out}\rangle = |a\rangle + \lim_{\epsilon\to+0} \frac{1}{E-\mathsf{H}_0-i\hbar\epsilon}\mathsf{V}_S|a,\text{out}\rangle \ , \tag{9}$$

where $\mathsf{V}_S$ denotes the potential in the Schrödinger picture. Operators in the Schrödinger picture, $\mathsf{O}_S$, are obtained from their counterparts in the interaction picture, $\mathsf{O}(t)$, by means of the unitary transformation

$$\mathsf{O}_S = \exp\left[-\frac{i}{\hbar}\mathsf{H}_0 t\right]\mathsf{O}(t)\exp\left[\frac{i}{\hbar}\mathsf{H}_0 t\right], \quad \mathsf{H}_{0,S}=\mathsf{H}_0 \ . \tag{10}$$

From relations (8) and (9) it can immediately be deduced that the states $|a,\text{in}\rangle, |a,\text{out}\rangle$ are eigenstates of the full Hamiltonian in the Schrödinger picture, $\mathsf{H}_S$, with the same energy eigenvalue $E$: i.e.,

$$\mathsf{H}_S|a,\text{in}\rangle = E|a,\text{in}\rangle \ , \tag{11}$$

$$\mathsf{H}_S|a,\text{out}\rangle = E|a,\text{out}\rangle \ . \tag{12}$$

The explicit solutions of the Lippmann-Schwinger equations read

$$|a,\text{in}\rangle = \left[1+\lim_{\epsilon\to+0}\frac{1}{E-\mathsf{H}_S+i\hbar\epsilon}\mathsf{V}_S\right]|a\rangle \ , \tag{13}$$

$$|a,\text{out}\rangle = \left[1+\lim_{\epsilon\to+0}\frac{1}{E-\mathsf{H}_S-i\hbar\epsilon}\mathsf{V}_S\right]|a\rangle \ . \tag{14}$$

The $S$-matrix element $S_{fi}$ for the transition $i\to f$ is defined as the projection of the initial state $|i,\text{in}\rangle$ onto the final state $|f,\text{out}\rangle$:

$$S_{fi} \equiv (f|S|i) := \langle f,\text{out}|i,\text{in}\rangle \ , \tag{15}$$

where, using Eqs. (6) and (7), the $S$ operator is related to the time-evolution operator in the interaction picture by

$$\mathsf{S} = \mathsf{U}(+\infty,-\infty) \ . \tag{16}$$

Adopting the Lippmann-Schwinger equation (8) or (9) and a specific representation of the $\delta$ function, viz. the relation

$$\lim_{t\to+\infty}\left[\lim_{\epsilon\to+0}\frac{e^{ixt}}{x-i\epsilon}\right] = -\lim_{t\to-\infty}\left[\lim_{\epsilon\to+0}\frac{e^{ixt}}{x+i\epsilon}\right] = 2\pi i\delta(x) \ , \tag{17}$$

the $S$-matrix can be rewritten like

$$S_{fi} = \delta_{fi} - 2\pi i\delta(E_f-E_i)R_{fi} \ , \tag{18}$$

where the transition amplitude $R_{fi}$ is defined by

$$R_{fi} \equiv (f|\mathsf{R}|i) := (f|\mathsf{V}_S|i,\text{in}\rangle = \langle f,\text{out}|\mathsf{V}_S|i\rangle \tag{19}$$

and, according to (13) or (14), the $R$ operator is given by

$$\mathsf{R} = \mathsf{V}_S + \lim_{\epsilon\to+0}\mathsf{V}_S\frac{1}{E-\mathsf{H}_S+i\hbar\epsilon}\mathsf{V}_S \ . \tag{20}$$

With the help of the Lippmann-Schwinger equation for $|i,\text{in}\rangle$, Eq. (8), the transition amplitude $R_{fi}$ may be cast into the form

$$R_{fi} = (f|\mathsf{V}_S|i) + \lim_{\epsilon\to+0}\sum_n \frac{(f|\mathsf{V}_S|n)R_{ni}}{E_i-E_n+i\hbar\epsilon} \ . \tag{21}$$

The Born approximation $R_{fi}^B$ consists of retaining just the first term in either of the above expressions (20) or (21):

$$R_{fi}^B \equiv (f|\mathsf{V}_S|i) \ . \tag{22}$$

As a special case, we shall consider as initial and final states two-particle states of definite momentum, i.e., the eigenstates

$$|i) = |\mathbf{p}_1,\mathbf{p}_2), \quad (f| = (\mathbf{q}_1,\mathbf{q}_2| \tag{23}$$

of the one-particle momentum operators. These states may be built up from tensor products of corresponding single-particle momentum eigenstates $|\mathbf{p})$ with normalization

$$(\mathbf{q}|\mathbf{p}) = \delta^{(3)}(\mathbf{q}-\mathbf{p}) \tag{24}$$

and wave function

$$(\mathbf{x}|\mathbf{p}) = \frac{1}{(2\pi)^{3/2}}e^{i\mathbf{p}\cdot\mathbf{x}} \ . \tag{25}$$

Assuming that the potential $\mathsf{V}_S$ depends only on the relative distance of the two particles, i.e., only on the difference

$$\mathbf{x} \equiv \mathbf{x}_1 - \mathbf{x}_2 \tag{26}$$

of their coordinate operators $\mathbf{x}_1,\mathbf{x}_2$, and abbreviating the involved momentum transfer by

$$\mathbf{k} \equiv \mathbf{p}_1 - \mathbf{q}_1 \ , \tag{27}$$

the free two-particle matrix element of the potential operator $V_S(\mathbf{x})$ takes the form

$$(\mathbf{q}_1,\mathbf{q}_2|V_S(\mathbf{x})|\mathbf{p}_1,\mathbf{p}_2) = \frac{1}{(2\pi)^3}\delta^{(3)}(\mathbf{q}_1+\mathbf{q}_2-\mathbf{p}_1-\mathbf{p}_2) \times \int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}V_S(\mathbf{x}) \ . \tag{28}$$

## III. EFFECTIVE POTENTIAL FROM QUANTUM FIELD THEORY

In quantum field theory, in the absence of external fields, the total momentum is conserved due to translational invariance. Factorizing off the overall momentum conservation from the scattering amplitude $R_{fi}$, the reduced $T$-matrix element $T_{fi}$ is defined by the decomposition

$$S_{fi}=\delta_{fi}+i(2\pi)^4\delta^{(4)}(P_f-P_i)T_{fi} \tag{29}$$

of the $S$ matrix, where $P_i$ and $P_f$ denote the total momentum of initial and final state, respectively. Consequently, the $T$-matrix element $T_{fi}$ from (29) is related to the transition amplitude $R_{fi}$ from (18) by

$$R_{fi}=-(2\pi)^3\delta^{(3)}(\mathbf{P}_f-\mathbf{P}_i)T_{fi} \ . \tag{30}$$

The interaction of two particles which in fact are described by some quantum field theory can be approximated by an effective potential—at least as long as the Born approximation (22) makes sense. The simplest quantum-field-theoretical analogue of the potential scattering considered previously is the elastic scattering

$$\mathcal{P}_1(\mathbf{p}_1)+\mathcal{P}_2(\mathbf{p}_2)\rightarrow\mathcal{P}_1(\mathbf{q}_1)+\mathcal{P}_2(\mathbf{q}_2) \tag{31}$$

of the two particles $\mathcal{P}_1,\mathcal{P}_2$ with masses $m_1,m_2$. The total and relative momenta of the particles in initial and final state are given by

$$\mathbf{P}_i\equiv\mathbf{p}_1+\mathbf{p}_2, \quad \mathbf{p}\equiv\eta_2\mathbf{p}_1-\eta_1\mathbf{p}_2 \ , \tag{32}$$

$$\mathbf{P}_f\equiv\mathbf{q}_1+\mathbf{q}_2, \quad \mathbf{q}\equiv\eta_2\mathbf{q}_1-\eta_1\mathbf{q}_2 \ , \tag{33}$$

with

$$\eta_1+\eta_2=1 \ . \tag{34}$$

From Eqs. (22), (28), and (30), the Born approximation $T_{fi}^B$ to the $T$-matrix element for the above elastic two-particle scattering is given by

$$T_{fi}^B=-\frac{1}{(2\pi)^6}\int d^3x\, e^{i\mathbf{k}\cdot\mathbf{x}}V_S(\mathbf{x}) \ . \tag{35}$$

Accordingly, the potential $V_S(\mathbf{x})$ can be extracted from the $T$-matrix element $T_{fi}$ by Fourier transformation with respect to the momentum transfer $\mathbf{k}$:

$$V_S(\mathbf{x})=-(2\pi)^3\int d^3k\, e^{-i\mathbf{k}\cdot\mathbf{x}}T_{fi}^B \ . \tag{36}$$

The effective interaction potential $V_S(\mathbf{x})$ is thus the Fourier transform of the Born approximation to the reduced $T$-matrix element $T_{fi}$ for the elastic scattering of the involved particles [4,5].

## IV. THE TRANSITION AMPLITUDE

In the preceding section we found a prescription for the derivation of the (perturbatively accessible part of the) effective potential acting between two particles described by a quantum field theory. Let us now apply this prescription to a fermion-antifermion system. Accordingly, we consider the elastic scattering

$$f(\mathbf{p}_1,\tau_1)+\bar{f}(\mathbf{p}_2,\tau_2)\rightarrow f(\mathbf{q}_1,\tau_3)+\bar{f}(\mathbf{q}_2,\tau_4) \tag{37}$$

of the involved fermion $f$ and antifermion $\bar{f}$ (with masses $m_1$ and $m_2$, respectively).

Expressed in terms of Dirac spinors $u(\mathbf{p},\tau)$ and $v(\mathbf{p},\tau)$, the general form of the $T$-matrix element for a process of the type (37) reads

$$T=\frac{1}{(2\pi)^6}\frac{m_1m_2}{(E_1E_2E_3E_4)^{1/2}}\bar{u}(\mathbf{q}_1,\tau_3)\Gamma_1u(\mathbf{p}_1,\tau_1) \times\bar{v}(\mathbf{p}_2,\tau_2)\Gamma_2v(\mathbf{q}_2,\tau_4)K \ , \tag{38}$$

where $\Gamma_i$, $i=1,2$, represent some Dirac matrices. $K$ denotes an (unspecified) interaction kernel, which is usually assumed to depend only on the momentum transfer $k\equiv p_1-q_1$. The Fourier transform of just this kernel yields the static interaction potential. There are many indications that the dominant spin structure for quark-antiquark interaction, originating from quantum chromodynamics, is vector $\Gamma_1\otimes\Gamma_2=\gamma_\mu\otimes\gamma^\mu$ plus scalar $\Gamma_1\otimes\Gamma_2=1\otimes1$. (For a very recent review see, e.g., Ref. [5].)

Explicitly, in the Dirac representation the Dirac spinors are given by

$$u(\mathbf{p}_i,\tau_i)=\left[\frac{S_i}{2m_i}\right]^{1/2}\begin{bmatrix}1\\ \dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_i}{S_i}\end{bmatrix}\chi_{\tau_i} \ , \tag{39}$$

$$v(\mathbf{p}_i,\tau_i)=\left[\frac{S_i}{2m_i}\right]^{1/2}\begin{bmatrix}\dfrac{\boldsymbol{\sigma}\cdot\mathbf{p}_i}{S_i}\\ 1\end{bmatrix}\chi^c_{\tau_i} \ ,$$

$$\chi^c_{\tau_i}\equiv-i\sigma_2\chi^*_{\tau_i} \ ,$$

where $\chi_\tau$ is the two-component spinor corresponding to spin projection $\tau$, and we defined

$$E_1\equiv(\mathbf{p}_1^2+m_1^2)^{1/2}, \quad E_2\equiv(\mathbf{p}_2^2+m_2^2)^{1/2} \ ,$$
$$E_3\equiv(\mathbf{q}_1^2+m_1^2)^{1/2}, \quad E_4\equiv(\mathbf{q}_2^2+m_2^2)^{1/2} \ , \tag{40}$$

and

$$S_1\equiv E_1+m_1, \quad S_2\equiv E_2+m_2 \ ,$$
$$S_3\equiv E_3+m_1, \quad S_4\equiv E_4+m_2 \ . \tag{41}$$

Inserting this into Eq. (38), we immediately obtain, in the center-of-momentum system,

$$\mathbf{p}\equiv\mathbf{p}_1=-\mathbf{p}_2, \quad \mathbf{q}\equiv\mathbf{q}_1=-\mathbf{q}_2, \tag{42}$$

for the $T$-matrix element of vectorial spin structure [6],

$$T_V = N_1 N_2 N_3 N_4 \left[ \delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} + \frac{1}{S_1 S_3}\delta_{\tau_2\tau_4}(\mathbf{p}\cdot\mathbf{q}\delta_{\tau_1\tau_3} - i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_1) + \frac{1}{S_2 S_4}\delta_{\tau_1\tau_3}(\mathbf{p}\cdot\mathbf{q}\delta_{\tau_2\tau_4} - i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_2) \right.$$
$$+ \frac{1}{S_1 S_2}[\mathbf{p}^2\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - \mathbf{p}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + (\mathbf{p}\cdot\boldsymbol{\sigma}_1)(\mathbf{p}\cdot\boldsymbol{\sigma}_2)] + \frac{1}{S_3 S_4}[\mathbf{q}^2\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - \mathbf{q}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + (\mathbf{q}\cdot\boldsymbol{\sigma}_1)(\mathbf{q}\cdot\boldsymbol{\sigma}_2)]$$
$$+ \frac{1}{S_1 S_4}[\mathbf{p}\cdot\mathbf{q}\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_+ + (\mathbf{p}\cdot\mathbf{q})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) - (\mathbf{q}\cdot\boldsymbol{\sigma}_1)(\mathbf{p}\cdot\boldsymbol{\sigma}_2)]$$
$$+ \frac{1}{S_2 S_3}[\mathbf{p}\cdot\mathbf{q}\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_+ + (\mathbf{p}\cdot\mathbf{q})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) - (\mathbf{p}\cdot\boldsymbol{\sigma}_1)(\mathbf{q}\cdot\boldsymbol{\sigma}_2)]$$
$$\left. + \frac{1}{S_1 S_2 S_3 S_4}[(\mathbf{p}\cdot\mathbf{q})^2\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - i(\mathbf{p}\cdot\mathbf{q})(\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_+) - (\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_1)(\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_2)] \right] K_V \tag{43}$$

and, for the $T$-matrix element of scalar spin structure [6],

$$T_S = -N_1 N_2 N_3 N_4 \left[ \delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - \frac{1}{S_1 S_3}\delta_{\tau_2\tau_4}(\mathbf{p}\cdot\mathbf{q}\delta_{\tau_1\tau_3} - i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_1) - \frac{1}{S_2 S_4}\delta_{\tau_1\tau_3}(\mathbf{p}\cdot\mathbf{q}\delta_{\tau_2\tau_4} - i\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_2) \right.$$
$$\left. + \frac{1}{S_1 S_2 S_3 S_4}[(\mathbf{p}\cdot\mathbf{q})^2\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - i(\mathbf{p}\cdot\mathbf{q})(\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_+) - (\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_1)(\mathbf{p}\times\mathbf{q}\cdot\boldsymbol{\sigma}_2)] \right] K_S . \tag{44}$$

Here we introduced the shorthand notation

$$\boldsymbol{\sigma}_1 \equiv \chi^\dagger_{\tau_3}\boldsymbol{\sigma}\chi_{\tau_1}, \quad \boldsymbol{\sigma}_2 \equiv \chi^\dagger_{\tau_4}\boldsymbol{\sigma}\chi_{\tau_2}, \quad \boldsymbol{\sigma}_+ \equiv \boldsymbol{\sigma}_1\delta_{\tau_2\tau_4} + \boldsymbol{\sigma}_2\delta_{\tau_1\tau_3} . \tag{45}$$

The normalization factors $N_i$ are given by

$$N_i \equiv \frac{1}{(2\pi)^{3/2}}\left[\frac{S_i}{2E_i}\right]^{1/2} . \tag{46}$$

From this, the interaction Hamiltonian may be found from the Fourier transform with respect to the momentum transfer $\mathbf{k}\equiv\mathbf{p}-\mathbf{q}$. In the nonrelativistic expansion up to next-to-lowest order one recovers in this way, of course, the well-known Breit-Fermi Hamiltonian.

## V. THE MASSLESS CASE

Let us now investigate in some more detail the scattering of *massless* fermions, i.e., $m_i = 0$. In this case the factors $S_i$ in (43) and (44) reduce to

$$S_1 = S_2 = E_1 = E_2 = \sqrt{\mathbf{p}^2}, \quad S_3 = S_4 = E_3 = E_4 = \sqrt{\mathbf{q}^2}, \tag{47}$$

whereas the normalization factors $N_i$ are no longer $\mathbf{p}$ dependent:

$$N_i = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}} . \tag{48}$$

Defining the unit vectors

$$\hat{\mathbf{p}} \equiv \frac{\mathbf{p}}{\sqrt{\mathbf{p}^2}}, \quad \hat{\mathbf{q}} \equiv \frac{\mathbf{q}}{\sqrt{\mathbf{q}^2}} \tag{49}$$

and abbreviating their difference by

$$\mathbf{k} \equiv \hat{\mathbf{p}} - \hat{\mathbf{q}}, \quad \mathbf{k}^2 = 2(1 - \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) , \tag{50}$$

the $T$-matrix elements (43) and (44) simplify to

$$T_V = \frac{1}{(2\pi)^6}\frac{1}{4}[(3+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(1+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} - i(3+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_+) - \mathbf{k}^2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + (\mathbf{k}\cdot\boldsymbol{\sigma}_1)(\mathbf{k}\cdot\boldsymbol{\sigma}_2) - (\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_1)(\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_2)]K_V \tag{51}$$

and

$$T_S = -\frac{1}{(2\pi)^6}\frac{1}{4}\left[\frac{1}{4}(\mathbf{k}^2)^2\delta_{\tau_1\tau_3}\delta_{\tau_2\tau_4} + \frac{i}{2}\mathbf{k}^2(\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_+) - (\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_1)(\hat{\mathbf{p}}\times\hat{\mathbf{q}}\cdot\boldsymbol{\sigma}_2)\right]K_S . \tag{52}$$

## VI. THE BOUND-STATE ENERGY

The primary aim of any investigation of bound states are the energy eigenvalues $E$ of the system—which give the mass spectrum of the composite particles—and the corresponding state vectors $|\psi\rangle$.

We consider as two-particle states simultaneous eigenstates of the two-particle Hamiltonian H,

$$\mathsf{H}|\psi(\mathbf{K})\rangle = E_{\mathbf{K}}|\psi(\mathbf{K})\rangle , \tag{53}$$

as well as of the total momentum $\mathbf{p}_1+\mathbf{p}_2$,

$$(\mathbf{p}_1+\mathbf{p}_2)|\psi(\mathbf{K})\rangle=\mathbf{K}|\psi(\mathbf{K})\rangle \ . \tag{54}$$

$E_{\mathbf{K}}$ and $\mathbf{K}$ denote energy and momentum of the two-particle system. The normalization of these states then has to read

$$\langle\psi(\mathbf{L})|\psi(\mathbf{K})\rangle=\delta^{(3)}(\mathbf{L}-\mathbf{K}) \ . \tag{55}$$

The energy eigenvalue $E_{\mathbf{K}}$ is thus obtained from

$$E_{\mathbf{K}}\delta^{(3)}(\mathbf{L}-\mathbf{K})=\langle\psi(\mathbf{L})|\mathsf{H}|\psi(\mathbf{K})\rangle \ . \tag{56}$$

We assume that the Hamiltonian $\mathsf{H}$ which governs the dynamics of the two-particle system under consideration is of the form (1), $\mathsf{H}=\mathsf{H}_0+\mathsf{V}$, with $\mathsf{H}_0$ describing the free motion of the particles and an interaction potential $\mathsf{V}$ which does not depend on the center-of-momentum coordinate of the two particles, that is, $\mathsf{V}=V(\mathbf{x})$. This latter feature ensures that the Hamiltonian commutes with the total momentum, $[\mathsf{H},\mathbf{p}_1+\mathbf{p}_2]=0$, which is an unavoidable prerequisite in order to be able to define the simultaneous eigenstates considered above. In general, the potential $\mathsf{V}$ will consist of a part $\mathsf{V}_{\mathrm{p}}$ which can be grasped by perturbation theory along the lines sketched in Sec. III, and a nonperturbative part $\mathsf{V}_{\mathrm{np}}$ which is beyond reach of perturbation theory. The two-particle Hamilton operator we are dealing with thus reads

$$\mathsf{H}=H_{0,1}(\mathbf{p}_1)+H_{0,2}(\mathbf{p}_2)+V_{\mathrm{p}}(\mathbf{x})+V_{\mathrm{np}}(\mathbf{x}) \ . \tag{57}$$

We introduce the two-particle wave functions

$$\begin{aligned}(\mathbf{x}_1,\mathbf{x}_2|\psi(\mathbf{K})\rangle&\equiv\psi(\mathbf{x}_1,\mathbf{x}_2;\mathbf{K}) \ ,\\ (\mathbf{p}_1,\mathbf{p}_2|\psi(\mathbf{K})\rangle&\equiv\tilde{\psi}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{K}) \ ,\end{aligned} \tag{58}$$

which are related by Fourier transformation:

$$\begin{aligned}\psi(\mathbf{x}_1,\mathbf{x}_2;\mathbf{K})&=\frac{1}{(2\pi)^3}\int d^3p_1\,d^3p_2\exp[i(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2)]\tilde{\psi}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{K}) \ ,\\ \tilde{\psi}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{K})&=\frac{1}{(2\pi)^3}\int d^3x_1\,d^3x_2\exp[-i(\mathbf{p}_1\cdot\mathbf{x}_1+\mathbf{p}_2\cdot\mathbf{x}_2)]\psi(\mathbf{x}_1,\mathbf{x}_2;\mathbf{K}).\end{aligned} \tag{59}$$

The center-of-momentum and relative coordinates and momenta are defined in the usual manner:

$$\mathbf{P}\equiv\mathbf{p}_1+\mathbf{p}_2, \quad \mathbf{p}\equiv\eta_2\mathbf{p}_1-\eta_1\mathbf{p}_2 \ , \tag{60}$$

$$\mathbf{X}\equiv\eta_1\mathbf{x}_1+\eta_2\mathbf{x}_2, \quad \mathbf{x}\equiv\mathbf{x}_1-\mathbf{x}_2 \ , \tag{61}$$

with

$$\eta_1+\eta_2=1 \ . \tag{62}$$

By analogy with the nonrelativistic case one may specify the factors $\eta_i$ by the mass fractions

$$\eta_i\equiv\frac{m_i}{m_1+m_2} \ . \tag{63}$$

As a consequence of (54) the wave functions (59) factorize like

$$\begin{aligned}\psi(\mathbf{x}_1,\mathbf{x}_2;\mathbf{K})&=\frac{1}{(2\pi)^{3/2}}\exp(i\mathbf{K}\cdot\mathbf{X})\psi(\mathbf{x}) \ ,\\ \tilde{\psi}(\mathbf{p}_1,\mathbf{p}_2;\mathbf{K})&=\delta^{(3)}(\mathbf{P}-\mathbf{K})\tilde{\psi}(\mathbf{p}) \ .\end{aligned} \tag{64}$$

In the center-of-momentum system of the two particles, $\mathbf{K}=0$, using (30), the expectation value of the Hamiltonian (57) is then given by

$$\begin{aligned}\mathbf{E}_0=&\int d^3p\,|\tilde{\psi}(\mathbf{p})|^2[H_{0,1}(\mathbf{p})+H_{0,2}(-\mathbf{p})]\\ &-(2\pi)^3\int d^3p\,d^3q\,\tilde{\psi}^*(\mathbf{q})T_{fi}^B\tilde{\psi}(\mathbf{p})\\ &+\int d^3x\,|\psi(\mathbf{x})|^2V_{\mathrm{np}}(\mathbf{x}) \ .\end{aligned} \tag{65}$$

According to the prescription of Sec. III, the perturbative part $\mathsf{V}_{\mathrm{p}}$ of the potential is related to the $T$-matrix element $T_{fi}$ for two-particle scattering by

$$(\mathbf{q}_1,\mathbf{q}_2|\mathsf{V}_{\mathrm{p}}|\mathbf{p}_1,\mathbf{p}_2)=-(2\pi)^3\delta^{(3)}(\mathbf{P}_f-\mathbf{P}_i)T_{fi}^B.$$

As has been demonstrated in Sec. IV for the case of fermion-antifermion systems, $T$-matrix elements are more easily derived in momentum space. Therefore we give in (65) the expectation value of $\mathsf{V}_{\mathrm{p}}$ in momentum-space representation. The nonperturbative part $\mathsf{V}_{\mathrm{np}}$ of the potential cannot be derived from a scattering process but has to be obtained from somewhere else. The simplest possibility for this is to guess its form in configuration space, which is the reason why we give in (65) its expectation value in coordinate-space representation.

## VII. SINGLET-TRIPLET MASS DIFFERENCES OF MESONS

In the quark model, hadrons are regarded as bound states of quarks. Mesons, in particular, are considered as being built up from a quark-antiquark pair forming a so-called "quarkonium" state.

Just for the purpose of illustration we will apply here the ideas developed so far to the mass differences between mesons which differ only in the total spin of the constituting quark-antiquark pair. Since the quarks, like any fundamental fermions, carry spin $\frac{1}{2}$, the total spin $S$ of their bound state will be either $S=0$ or $S=1$, which corresponds to a spin-singlet or spin-triplet state, respectively. As already mentioned, for simplicity we will treat the quarks as massless particles.

Experimentally, the differences of the squared masses of corresponding spin-singlet and spin-triplet quarkonium states which contain at least one light quark have been found to be constant to a surprisingly high degree of accuracy. For instance, from Table I one may read off

TABLE I. Differences of the squared masses of spin-singlet and spin-triplet partners [7].

| Spin triplet | Spin singlet | $M^2_{S=1}-M^2_{S=0}(\mathrm{GeV}^2)$ |
|---|---|---|
| $\rho$ | $\pi$ | 0.57 |
| $K^*$ | $K$ | 0.55 |
| $D^*$ | $D$ | 0.55 |
| $D_s^*$ | $D_s$ | 0.58 |
| $B^*$ | $B$ | 0.56 |

$$M^2_{S=1}-M^2_{S=0}\simeq 0.56\ \mathrm{GeV}^2\ . \tag{66}$$

The free relativistic Hamiltonian

$$H_{0,i}(\pm\mathbf{p})=(\mathbf{p}^2+m_i^2)^{1/2}$$

entering in (65) reduces in the massless case to $H_{0,i}(\pm\mathbf{p})=\sqrt{\mathbf{p}^2}$. We only consider ground states, with vanishing orbital angular momentum, which entails spherical symmetry, i.e.,

$$\begin{aligned}&\psi(\mathbf{x})=\psi(r),\quad r\equiv\sqrt{\mathbf{x}^2}\ ,\\ &\tilde{\psi}(\mathbf{p})=\tilde{\psi}(p),\quad p\equiv\sqrt{\mathbf{p}^2}\ .\end{aligned} \tag{67}$$

The Fourier transform (36) of the vector product $\mathbf{p}\times\mathbf{q}$ is the orbital angular momentum. Accordingly, all terms in the $T$-matrix elements (43) and (44) involving this expression do not contribute to the energy expectation value (65) for ground states.

In gauge theories, the perturbative part of the interaction is very likely to originate from gauge-boson exchange. Consequently, the corresponding potential $V_{\mathrm{p}}(\mathbf{x})$ is of vector type. For one-vector-boson exchange the interaction kernel $K_V$ in (43) reads

$$K_V=-\frac{\kappa}{(p_1-q_1)^2}=\frac{\kappa}{(\mathbf{p}^2\mathbf{q}^2)^{1/2}\mathbf{k}^2}\ . \tag{68}$$

In quantum electrodynamics, for one-photon exchange, the parameter $\kappa$ equals $Q_1Q_2e^2$, where $Q_i$ are the electric charges of the involved particles in units of the electron charge $e$. In quantum chromodynamics, for one-gluon exchange between quarks in a color-singlet state, the parameter $\kappa$ equals $\frac{4}{3}g_s^2$, where $g_s$ is the strong coupling constant and the factor $\frac{4}{3}$ arises from color.

For an arbitrary *scalar* function $f(\mathbf{p},\mathbf{q})$, due to spherical symmetry the relation

$$\int d^3p\,d^3q\,k_ik_jf(\mathbf{p},\mathbf{q})=\tfrac{1}{3}\delta_{ij}\int d^3p\,d^3q\,\mathbf{k}^2f(\mathbf{p},\mathbf{q}) \tag{69}$$

holds. The requirement of spherical symmetry is satisfied for any wave function corresponding to a state with vanishing orbital angular momentum and, of course, also for the interaction kernel $K$ in Eq. (38). Consequently, decomposing the product $k_ik_j$ showing up in (51) like

$$k_ik_j=\left[k_ik_j-\tfrac{1}{3}\mathbf{k}^2\delta_{ij}\right]+\tfrac{1}{3}\mathbf{k}^2\delta_{ij} \tag{70}$$

into a traceless part (the term in parentheses) and a trace part, one finds that only the trace part contributes to the energy expectation value.

Furthermore, we assume that the nonperturbative part $V_{\mathrm{np}}(\mathbf{x})$ of the interaction is described by a central potential $V_{\mathrm{np}}(\mathbf{x})=V_{\mathrm{np}}(r)$. There are good reasons to believe that for quark-antiquark bound states this potential is of scalar type and that its shape is not very different from linear rise [5], $V_{\mathrm{np}}(r)=ar$.

Under the above assumptions the bound-state energy (65) is given by

$$\begin{aligned}E_0=&2\int d^3p\,|\tilde{\psi}(p)|^2p\\ &-\frac{1}{(2\pi)^3}\frac{\kappa}{4}\left|\int_0^\infty dp\,p\tilde{\psi}(p)\right|^2[\Omega^2-\tfrac{2}{3}(4\pi)^2\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle]\\ &+a\int d^3x\,|\psi(r)|^2r\ ,\end{aligned} \tag{71}$$

with

$$\Omega^2\equiv\int d\Omega_p\,d\Omega_q\,\frac{1}{\mathbf{k}^2}(3+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})(1+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})\ . \tag{72}$$

The spin expectation value $\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle$ depends on the total spin $S$ of the two-fermion state $|\psi\rangle$:

$$\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle=\begin{cases}-3 & \text{for spin singlets,}\quad S=0,\\ +1 & \text{for spin triplets,}\quad S=1.\end{cases} \tag{73}$$

We adopt a standard variational technique [8] by evaluating the energy expectation value (71) with the help of some trial states $|\psi(\lambda)\rangle$ characterized by a variational parameter $\lambda$,

$$E(\lambda)=\langle\psi(\lambda)|\mathrm{H}|\psi(\lambda)\rangle\ , \tag{74}$$

and by minimizing the resulting expression with respect to $\lambda$:

$$E\simeq E(\lambda_{\min}),\quad \left.\frac{dE(\lambda)}{d\lambda}\right|_{\lambda_{\min}}=0\ . \tag{75}$$

In principle, any variational method can, of course, only provide an upper bound on the energy level under consideration. By definition, the energy of the ground state will always be less than or equal to the value obtained by the variational method. It has, however, been shown that in practice, at least within a nonrelativistic treatment, for a potential which is a superposition of a Coulomb part [which arises from the one-vector-boson exchange (68) in the instantaneous limit] and a linear part, the evaluation of the energy expectation value (74) by Gaussian trial functions yields a good approximation (with an error of a few percent) to the numerically obtained exact energy [9].

We use as trial functions the Gaussian wave functions

$$\begin{aligned}&\psi(\mathbf{x})=\frac{\lambda^{3/2}}{\pi^{3/4}}\exp\left[-\frac{\lambda^2r^2}{2}\right]\\ &\tilde{\psi}(\mathbf{p})=\frac{1}{\lambda^{3/2}\pi^{3/4}}\exp\left[-\frac{p^2}{2\lambda^2}\right]\ ,\end{aligned} \tag{76}$$

as well as the Hydrogen-like wave functions

$$\begin{aligned}&\psi(\mathbf{x})=\left[\frac{\lambda^3}{\pi}\right]^{1/2}\exp(-\lambda r)\\ &\tilde{\psi}(\mathbf{p})=\frac{\sqrt{8\lambda^5}}{\pi}\frac{1}{(p^2+\lambda^2)^2}\ ,\end{aligned} \tag{77}$$

both of them with normalization

$$\int d^3x\,|\psi(\mathbf{x})|^2=\int d^3p\,|\tilde{\psi}(\mathbf{p})|^2=1\ . \tag{78}$$

Already from dimensional considerations the general structure of $E(\lambda)$ as a function of $\lambda$ is

$$E(\lambda)=A\lambda+B\frac{a}{\lambda}\ . \tag{79}$$

The constants $A$ and $B$ are given for Gaussian trial functions by

$$A=\frac{4}{\sqrt{\pi}}-\frac{\kappa}{32\pi^4\sqrt{\pi}}[\Omega^2-\tfrac{2}{3}(4\pi)^2\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle],\quad B=\frac{2}{\sqrt{\pi}} \tag{80}$$

and for hydrogenlike trial functions by

$$A=\frac{16}{3\pi}-\frac{\kappa}{16\pi^5}[\Omega^2-\tfrac{2}{3}(4\pi)^2\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle],\quad B=\tfrac{3}{2}\ . \tag{81}$$

Minimization of $E(\lambda)$ in the form (79) yields

$$\lambda_{\min}=\left[\frac{Ba}{A}\right]^{1/2},\quad E(\lambda_{\min})=\sqrt{ABa}\ . \tag{82}$$

The energy in the rest system of the bound state is, of course, nothing else but the mass of the composite particle. Using (80) or (81) in the expression (82) for the minimum $E(\lambda_{\min})$ of the energy yields for the mass-squared difference we are looking for

$$M^2_{S=1}-M^2_{S=0}\simeq\begin{cases}\dfrac{32}{3\pi^3}\kappa a & \text{for Gaussian trial functions,}\\[2ex] \dfrac{16}{\pi^3}\kappa a & \text{for hydrogenlike trial functions.}\end{cases} \tag{83}$$

These expressions have to be compared with the result [5] obtained in the nonrelativistic case on the grounds of the instantaneous-limit approximation to the interaction kernel (68):

$$M^2_{S=1}-M^2_{S=0}\simeq\frac{2}{3\pi}\kappa a\ . \tag{84}$$

Obviously, all predictions for the mass-squared differences are independent of the mass of the particles which constitute the bound state. However, in the nonrelativistic case this mass independence follows from the neglect of terms of higher order in the inverse masses of the components [10] and the assumption that light constituents will be mainly affected by the linear part of the potential. In contrast with that, in the ultrarelativistic case this mass independence is enforced by the assumption of vanishing masses of the bound-state constituents. Since in this case there is no other dimensional parameter than the slope $a$ of the linear potential, any quantity of dimension mass squared has to be proportional to this slope.

## VIII. SUMMARY

In the present work we made the case for a relativistic treatment of bound states which might be regarded as the relativistic generalization of the approach based on nonrelativistic potential models. It consists of two main steps.

(1) Compute the effective interaction potential between two particles (at least to the extent you can trust in perturbation theory) from the transition amplitude for the elastic scattering of the involved particles. An example for this is provided by the $T$-matrix elements for a fermion-antifermion system given in Eqs. (43) and (44).

(2) Use this potential in a multiparticle Schrödinger equation with a relativistically correct Hamiltonian in order to determine the energy eigenvalues and corresponding eigenstate vectors of the bound state under consideration. This has been done for the two-particle case in Eq. (65).

The obvious advantage of this approach is its physical transparency.

In the application of this recipe we restricted ourselves to the case of massless components of the bound state, in order to be able to give an explicit expression for the bound-state energy. In general, it will not always be possible to obtain analytic results. One has to stick to numerical methods.

An analysis similar to the present one has been performed recently by Gara and co-workers [11,12] on the basis of the Salpeter equation [13] which is obtained from the Bethe-Salpeter equation [14] upon eliminating any dependence on timelike variables in a suitable manner. (See, for instance, also Refs. [4] and [5].) After some standard and plausible approximations, such as the restriction to positive-energy solutions, this approach coincides with the effective-Hamiltonian method advocated for in the present work. The reduced Salpeter equation found in this way [Eq. (12) in Ref. [11] or Eq. (5) in Ref. [12]] is equivalent to our result (65) for the bound-state energy. In fact, it is nothing else but the (momentum-space representation of the) Schrödinger equation (53) with the Hamiltonian (57), after dropping the center-of-momentum motion of the whole system. The interaction functions entering in this equation of motion [Eqs. (13a) and (13b) in Ref. [11] or Eqs. (6a) and (6b) in Ref. [12]] are identical to our $T$-matrix elements (43) and (44).

## ACKNOWLEDGMENTS

We would like to thank H. Grosse and H. Narnhofer for numerous helpful discussions.

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