Hoop conjecture for black-hole horizon formation

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The hoop conjecture was proposed by Thorne in 1972 as a loosely formulated, necessary and sufficient condition for the formation of a horizon in nonspherical gravitational collapse. In this paper we discuss some issues that arise in formulating various precise interpretations of the conjecture, and review some of the small amount of literature that has appeared on this subject. We further specialize the conjecture to static, vacuum, axisymmetric spacetimes and show that it is satisfied in three particular cases of such spacetimes. We also prove a theorem for oblate geometries giving a sufficient condition for the conjecture to be satisfied.

I. INTRODUCTION AND SUMMARY

It is well known that spherical gravitational collapse produces (i) an event horizon, which seals off the collapsing matter and prevents it from causally influencing the external universe, and (ii) a singularity, at which classical general relativity breaks down. It is the simultaneous occurrence of (i) and (ii) which ensures that external observers can never witness any such breakdown. In his 1972 review of nonspherical gravitational collapse, Thorne showed that horizons do not form in idealized planar and cylindrical collapse, in contrast with the spherical case [1]. This suggests the following question: Do horizons always form in the collapse to a singularity of realistic configurations of matter? This "cosmic censorship" [2] question has been called by Hawking "the most important unsolved problem in classical general relativity" [3].

Thus it is of interest to have sufficient and/or necessary conditions for the formation of horizons. On the basis of his observations above, Thorne made the following conjecture [1,4], which later became known as the hoop conjecture (HC): *Black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is* $\mathcal{C} \lesssim 4\pi M$. Roughly speaking, the required condition is that a hoop of circumference $\mathcal{C}$ can be slipped over the region and rotated through 360°. This criterion is sufficiently loosely formulated to allow many different precise interpretations; in particular different definitions of mass, horizon, and of circumference are all possible. It is thus essentially a suggestion as to the form a rigorous result might take, while encapsulating the crucial physical idea that the collapsing matter must be strongly compacted in all three spatial dimensions in order for a horizon to be formed. The relation $\mathcal{C} \lesssim 4\pi M$ is intended as an order of magnitude guideline, so that different numeric constants (not necessarily 4π) may be required for the separate "if" and "only if" parts of the conjecture.

What evidence is there in favor of such a conjecture? Some reasons for believing it are that (i) no compelling counterexamples are known; (ii) it is in accord with the behavior of spherical, cylindrical, and planar collapse [1]; (iii) it is in accord with the results of several numerical relativity calculations of nonrotating, axially symmetric collapse [5–8], and (iv) several examples of static solutions of Einstein equations are known to be consistent with it [9–15]. We proceed to present some of this evidence and briefly summarize past research in this area.

The HC in momentarily static, nonsingular, axisymmetric spacetimes was investigated by Redmount [9]. By imposing the weak energy condition in the matter-containing region he succeeded in deriving geometric constraints that the matter-vacuum boundary must satisfy in any such spacetime. Although these constraints were unfortunately not directly related to HC criteria, he found no counterevidence to the HC in the particular geometries that he examined (the Γ metric [10] and the Bach-Weyl ring metric [11]), and his work inspired future research [19,12]. In [12] and [13], Bonnor considered static spheres and charged perfect fluid with charge q greater than their mass m. He showed that if a suitable energy condition is satisfied then $\mathcal{C} > 2\pi m$, where $\mathcal{C}$ is the circumference of the matter surface, thus supporting the spirit if not the letter of the conjecture. Chamorro, Gregory, and Stewart [14] showed that solutions of the Einstein equations representing static finite thin discs satisfy the HC, without the imposition of an energy condition being necessary, and subsequently Lamberti and Hamity [15] gave a similar demonstration for thin discs of counterrotating particles.

Investigations of horizon formation in evolving dynamic spacetimes have been mostly numeric. Early numerical calculations by Nakamura and Sato of the collapse of nonrotating, axisymmetric fluid bodies indicated that apparent horizons could be formed in highly nonspherical situations, but that for certain initial conditions apparent horizons are *not* formed during the collapse [5]. Subsequently, Nakamura, Shapiro, and Teukolsky analytically solved the initial-value problem for momentarily static dust spheroids, both prolate and oblate [7]. They numerically analyzed the resulting three-dimensional manifolds for various values of the eccentricity to determine the minimum circumference of surfaces surrounding the spheroids, and to check whether or not apparent horizons occur. More recently, Shapiro and Teukolsky [8] numerically modeled the full dynamical collapse of such spheroids, again calculating minimum circumferences and testing the horizon formation. In these calculations, both momentarily static and collapsing, apparent horizons were found to be present whenever the minimum circumference $\mathcal{C}$ was less than 4π times the mass, and not otherwise, to within 10%. This is strong evidence for both the necessary and the sufficient condition aspects of the HC.

In a more general setting, several investigations have been made concerning the question "What is a sufficient degree of compactification of matter to ensure horizon formation?" [16–18]. In 1983 a striking development took place with the proof by Schoen and Yau [19] of the following very powerful and general theorem, motivated by previous work on the HC.

Theorem 1 (Schoen and Yau). *Suppose Σ is any spacelike hypersurface in spacetime, and Ω is a bounded region in Σ on which $\rho-(J_\mu J^\mu)^{1/2}\geq\Lambda>0$, for some $\Lambda>0$. Here $\rho\equiv T_{\mu\nu}n^\mu n^\nu$ and $J_\mu\equiv T_{\mu\nu}n^\nu+\rho n_\mu$ are the energy density and flux, and n^μ is the unit normal to Σ. If $\mathcal{R}(\Omega)>\pi\sqrt{3/2\Lambda}$, where $\mathcal{R}(\Omega)$ is a suitably defined measure* [25] *of the radius of Ω, then Σ contains an apparent horizon.*

Thus by assuming a lower bound for the mass density, they derive an upper bound for the matter radius for spacetimes without black holes.

Given such a definitive result, one may well ask what motivation is there for continuing to study the HC as originally formulated. The following are some reasons.

First, the HC purports to give a sufficient condition for horizons *not* to form, unlike the theorem of Schoen and Yau above, which would be relevant to cosmic censorship. Recent work by Bzion, Malec, and O'Murchadha (Eq. (67) of Ref. [20]), in which they derive such a condition for spherically symmetric systems, indicate that there is hope to prove this portion of the HC generally.

Second, from a physical point of view, one would like to have a criterion applicable to measurements made by an observer external to the collapsing body. External observers can measure circumferences but not radii, and mass but not density. Such criteria, if provable, might be a consequence of the vacuum field equations holding outside the body, and thus may be independent of the equations of state, singularities, etc., of the interior.

The most important reason, however, for continued interest in the HC is the recent strong numerical evidence in its favor described above [7,8]. In light of this new evidence, it now seems likely that some version of the HC is true. However, to date very little attention has been paid to the issue of finding precise formulations of the HC in a general setting; most investigations have been restricted to static spacetimes. Accordingly, one of the two principal purposes of this paper is to identify issues that one must confront in trying to find and prove such a precise and general formulation, and to propose some routes that one might follow. This we undertake in Sec. II, where we consider various ways of defining mass, horizon, and circumference. We identify two likely candidate formulations, one *global* in spirit based on the asymptotic Arnowitt-Deser-Misner (ADM) definition of mass [21], and one *local* in spirit incorporating some suitable "quasilocal" definition of mass. We also distinguish between two approaches towards finding a proof: an *interior* approach which tries to make deductions from energy conditions, etc., holding inside the matter region, and an *exterior* approach which instead works from the vacuum field equations which hold outside.

The second principal purpose of this paper is to present some new evidence in favor of the HC in static, vacuum, axisymmetric spacetimes, evidence which indicates that the exterior approach has a good chance of succeeding. In particular, in Sec. III we show that several particular such spacetimes are consistent with the HC, and we derive sufficient conditions for oblate geometries to satisfy the conjecture. A similar result is also obtained for prolate geometries. Although these results leave much room for improvement and do not by any means constitute a proof of the HC in these spacetimes, they do at least demonstrate that there is hope to make progress along the directions we outline.

II. MAKING THE HOOP CONJECTURE PRECISE: ISSUES AND TOOLS

A. Apparent horizon versus event horizon

Although ideally one would like to obtain a result involving absolute event horizons, in practice it is usually difficult to determine whether or not one exists. This is the case if one is working with data specified on an initial spacelike hypersurface, as the existence of an absolute event horizon depends on the structure of the spacetime in the far future. It is easier to prove the existence of an apparent horizon, or what is a slightly stronger condition, that a closed trapped surface exists (see [22] for definitions of these types of surface). They are closely related as the boundary of a region containing closed trapped surfaces is always an apparent horizon [22], and it is expected that generically trapped surfaces lie inside apparent horizons, with exceptions forming a set of "measure zero" [1].

We now consider the question of to what extent the two types of horizon are equivalent. It is not known whether trapped surfaces are always accompanied by absolute event horizons. Israel [23] presents evidence in this direction, and suggests an approach towards a direct proof. If one assumes cosmic censorship [2], then Penrose's singularity theorem [24] implies that the result is true under quite general circumstances. The following is essentially the same result in a different guise: In spacetimes which are future asymptotically predictable, absolute event horizons must be present when one has a marginally trapped surface, or an outer trapped surface which is the boundary of a three-dimensional region ([31], p. 310). (Asymptotic predictability is a technical condition related to the absence of naked singularities [22].) However the recent work of Nakamura, Shapiro, and Teukolsky [7] referred to above casts doubt on the unqualified validity of this cosmic-censorship hypothesis. In the converse direction, it is not true that the formation of an absolute event horizon implies that there is an apparent horizon nearby. However, if an absolute event horizon forms, it is believed at least that an apparent horizon will at some stage be formed, as it is generally accepted that black holes eventually settle down into a stationary state, and for stationary black-hole spacetimes both types of horizon exist and coincide [22].

In conclusion, it seems that outer trapped surfaces or apparent horizons are probably an acceptable substitute for absolute event horizons when dealing with sufficient conditions for horizon formation, but they might not be so when dealing with necessary conditions.

B. The hoop concept

In this section we consider how to define an appropriate measure of the size of a collapsing body that could be used to diagnose horizon formation in the spirit of the HC. Suppose that we have an asymptotically flat spacelike slice of spacetime Σ, and a connected region Ω in Σ such that the stress-energy tensor vanishes outside Ω. As discussed in the Introduction, there exists a definition of radius $\mathcal{R}(\Omega)$ due to Schoen and Yau [19] which is useful in this context. This $\mathcal{R}(\Omega)$ is essentially the size of the largest torus that fits inside Ω [25]. However, in this paper we are interested in measures that depend only on the surface $S=\partial\Omega$ of Ω, and not on its interior. Some quantities that could be considered are the area of the surface $\mathcal{A}(S)$, and its circumference $\mathcal{C}(S)$. We discuss these in turn.

One might expect that the dimensionless quantity $\mathcal{A}(S)/m^2$ (where m is the ADM mass of Σ [21]) would be useful as a measure of degree of compactification. However there exists a class of static spacetimes (Eq. (5.12) of Ref. [9]) in which $\mathcal{A}(S)/m^2$ can be arbitrarily small. Hence this quantity cannot provide a sufficient condition for the formation of horizons, and we are led to consider alternative measures (see below). On the other hand, an area does appear in the following necessary condition for the formation of trapped surfaces, namely that closed trapped surface of area $\mathcal{A}$ must satisfy

$$\mathcal{A}\le 16\pi m^2 \ . \tag{2.1}$$

This result is true for surfaces S of spherical topology in spacetimes which are asymptotically flat at past null infinity, and for which there exists a nonsingular null hypersurface connecting S to past null infinity [26]. It is also a simple consequence of some suitable form of the cosmic-censorship assumption such as asymptotic predictability [22], together with the black-hole uniqueness and no-hair theorems; see Eq. (3.20) of Ref. [7]. Criteria involving area may also be useful in conjunction with the Hawking definition of mass, see Sec. II C.

The most appropriate external measure, however, seems to be circumference, in accordance with the HC as it is usually interpreted [7]. How does one define the circumference of a surface in general? For axisymmetric surfaces S, one can simply take the maximum of the lengths of those closed curves which are the analogs of lines of longitude and latitude. [Call this quantity $\mathcal{C}_1(S)$.] This is the definition for which we have the most evidence that the HC is true [7–9], and it is this definition which we shall adopt in our discussion of vacuum axisymmetric spacetimes in Sec. III. In this section, however, we consider how to define circumference more generally.

The most straightforward definition one might try is

$$\mathcal{C}_2(S)=2\max\{d_S(x,y)\,|\,x,y\in S\} \ , \tag{2.2}$$

where $d_S(x,y)$ is the length of the shortest smooth curve in S joining x to y. However this definition is not suitable: although it gives the correct answer for spherical surfaces, we show in Appendix A that for axisymmetric surfaces it does not reduce to the definition discussed above. Moreover we show that there exist horizon-free spacetimes containing surfaces S for which $\mathcal{C}_2(S)/m$ can be arbitrarily small.

A more promising notion of circumference is the following. Given a surface S, fix a point $\mathcal{P}$ in S and consider families of closed curves $D(\mathcal{P},\lambda)$, for $0\le\lambda\le 1$, which contain $\mathcal{P}$ and such that all points other than $\mathcal{P}$ on S lie on exactly one of the curves $D(\mathcal{P},\lambda)$. Such families of curves start at $\mathcal{P}$, sweep around the surface and then return to $\mathcal{P}$. Now take the length of the longest curve in the family, minimize over all such families,and then maximize over all points $\mathcal{P}$. This yields the definition

$$\mathcal{C}_3(S)=\max_{\mathcal{P}}\ \min_{\{D(\mathcal{P},\lambda)\}}\ \max_{\lambda} L[D(\mathcal{P},\lambda)] \ . \tag{2.3}$$

While this does not always equal $\mathcal{C}_1(S)$ for axisymmetric surfaces, as shown in Fig. 1, it may be appropriate and useful in formulating the HC. Another definition of circumference which may be a generalization of $\mathcal{C}_1$ is the following. Let K be a two-dimensional plane whose extrinsic curvature tensor in the spacelike slice Σ is traceless. Define $\mathcal{C}_K(S)$ to be the maximum of the lengths of the closed curves which make up $S\cap K$, and put $\mathcal{C}_4(S)=\max_K \mathcal{C}_K(S)$. This definition depends on the extrinsic and intrinsic geometries of S.

Suppose now that we have chosen a suitable definition of circumference $\mathcal{C}(S)$. If we consider various closed surfaces S surrounding a matter-containing region Ω, sometimes the minimum circumference $\inf_S \mathcal{C}(S)$ is not achieved by the matter surface $\partial\Omega$, but by some surface outside this [7]. Hence we should actually use the measure

$$\mathcal{C}_e(\Omega)\equiv\inf_S \mathcal{C}(S) \ , \tag{2.4}$$

where the minimum is taken over all closed surfaces S on

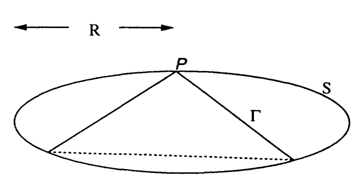

FIG. 1. For a pancake shaped surface S, the circumference function $\mathcal{C}_3(S)$ is given by the length of the curve Γ. The dashed portion of the curve is on the underside of the surface. Notice that this length is less than $\mathcal{C}_1(S)$ which is $2\pi R$.

some spacelike hypersurface Σ that contain Ω in their interior.

Now in a general spacetime with no special symmetries, there is no preferred family of spacelike slices Σ to which to apply the HC criterion. One might hope to establish some criterion which would be applicable to *all* spacelike hypersurfaces. However this is not possible, since the three-geometry as perceived on piecewise almost-null hypersurfaces will be distorted due to Lorentz contraction. For example, let f be a suitably smoothed periodic sawtooth function of compact support, whose amplitude A and period λ satisfy $0<1-2A/\lambda \ll 1$ and $\lambda \ll 1$. Then in a local Lorentz frame with nearly Loretnzian coordinates $\{t,x,y,z\}$, such a hypersurface Σ will be given by $t=f(x)$. Upon averaging over length scales large compared to λ, the effective three-metric of Σ will be given by ${}^{(3)}ds^2=\varepsilon^2 dx^2+dy^2+dz^2$, for some constant $\varepsilon \ll 1$. If T is a world tube of the form $x^2/a^2+y^2/b^2+z^2/b^2 \le 1$ with $b \ll a$, then on such a hypersurface Σ the circumference of T will be $\mathcal{C}_e(T\cap\Sigma)\approx 2b$ instead of the expected value of $\approx 2a$. This clearly violates the spirit of the type of measure we are trying to construct.

Thus it is necessary to restrict the class of hypersurfaces in some way, for example, by placing constraints on their extrinsic curvature. In particular one could demand that only maximal slices of spacetime be considered (i.e., that the extrinsic curvature be traceless); this would not be too restrictive as given any closed smooth spacelike two-surface of spherical topology one can always find a spacelike hypersurface that contains the two-surface and is maximal inside it [27]. These considerations lead to an expression of the HC of the following form: If T is the timelike world tube of a collapsing body, a horizon is formed if $\mathcal{C}_e(T\cap\Sigma)\le 4\pi m$ for some spacelike hypersurface Σ satisfying a suitable extrinsic flatness condition.

An alternate approach is the following. Define for each point p inside a world tube T

$$\mathcal{C}(T,p)\equiv \sup_{\Sigma} \mathcal{C}_e(\Sigma\cap T) , \tag{2.5}$$

where the maximum is taken over all asymptotically flat spacelike hypersurfaces Σ containing p. Then a possible criterion could be $\mathcal{C}(T,p)\le 4\pi m$ for some $p\in T$. In Appendix B we show that in the case of static spacetimes $\mathcal{C}(T,p)=\mathcal{C}_e(\Sigma_o\cap T)$ for all points p, where Σ_o is any hypersurface which is everywhere orthogonal to the timelike killing vector field. Thus $\mathcal{C}(T,p)$ reduces to the usual definition of circumference in this case.

C. Definitions of mass and the effects of gravitational radiation

It has been customary to interpret the mass that appears in the HC inequality as the ADM mass [21] of a slice of spacetime. While this seems reasonable in static spacetimes where the HC has usually been considered, there are problems associated with this interpretation in more general spacetimes, and also sometimes even in static ones. For example, if the region external to the collapsing matter is not vacuum, there will be contributions to the asymptotic mass from stress-energy outside the matter. This applies in particular to electrovacuum spacetimes. A more serious problem is the fact that in any realistic gravitational collapse spacetime there will be gravitational waves extending outside the matter source, which in principle could have a mass as large as that of the matter. In these situations the asymptotic mass is not a good measure of the actual mass undergoing collapse.

One approach to the problem of gravitational radiation is suggested by the calculations of Shapiro and Teukolsky discussed above [8]. Gravitational radiation is emitted by the collapsing spheroids in these calculations, but only at a level of less than one percent of the total mass, so that they are still in accord with the HC. It might be possible to prove a result that deals with realistic initial configurations of matter and establishes as a byproduct that such configurations never emit so much gravitational radiation as to become a serious problem for the HC.

However, we suggest that the best way to avoid the problems associated with the use of asymptotic mass is to use instead some "quasilocal" definition of mass: Although gravitational energy is traditionally considered to be nonlocalizable, recently several such quasilocal definitions (i.e., definitions based on surface integrals rather than volume integrals) have been suggested [28,29]. Also in the special case of stationary spacetimes there is a useful formulation of local mass due to Komar [30,31]. Using some such appropriate interpretation of "mass inside Ω" $m(\Omega)$, the following could be a suitable quasilocal reformulation of the HC. If Ω is any region in a suitably chosen hypersurface Σ (cf. previous section), then a trapped surface will be formed whenever $\mathcal{C}(\partial\Omega)$ is less than $4\pi m(\Omega)$. A stronger result would be to assert that $\partial\Omega$ itself would be a trapped surface, but this is less likely to be true. In any such formulation, the definition of mass cannot depend solely on the stress-energy tensor, as it is known that imploding nearly spherical shells of gravitational radiation can form black holes [32].

In favor of the above interpretation of the HC as a local statement is the following fact: some examples of spacetimes which were found to violate the HC bound of $4\pi m$ by a factor of 2 [12,33] are in accord with a version of the HC which uses the local Komar mass. We sketch a proof of this result: The spacetimes consist of static spheres of charged perfect fluid joined onto an exterior Reissner-Nordström region, where the join occurs at a value of the Schwarzschild radial coordinate $r=r_0>r_+\equiv m+\sqrt{m^2-q^2}$. Here m and q have their usual meanings, and $m>q$. If S_a is the surface given by $r=a$ for $a>r_0$, then the Komar mass of the volume interior to this surface is $m(S_a)=m-q^2/a$. We find that

$$\inf_a \frac{\mathcal{C}(S_a)}{4\pi m(S_a)}=g\left[\frac{q}{m}\right] , \tag{2.6}$$

where

$$g(x)=\begin{cases} 2x^2 \quad \text{for } \dfrac{\sqrt{3}}{2}\le x\le 1 , \\ \dfrac{(1+\sqrt{1-x^2})^2}{2(1-x^2+\sqrt{1-x^2})} \quad \text{for } 0\le x\le \dfrac{\sqrt{3}}{2} ; \end{cases} \tag{2.7}$$

see Fig. 2. In particular it can be seen that

$$\mathcal{C}(S_a) \geq 4\pi m(S_a) \quad \forall\ a > r_+ \ \text{ and } q < m \tag{2.8}$$

as claimed.

There are also some indications in favor of a quasilocal version of the HC incorporating the Hawking [28] definition of mass. In spherically symmetric spacetimes, the Hawking mass $m(S)$ of a spherical surface S coincides with the "mass $m(r)$ inside Schwarzschild radius r" [34], which plays a central role in the standard theory of spherical systems [35]. Moreover, for this $m(r)$ a local version of the HC is known to be valid: spheres of Schwarzschild radius r where $r < 2m(r)$ must be trapped. In addition, if S is any closed two-surface in a static, possibly nonspherical spacetime, which is everywhere orthogonal to the timelike killing field, then the Hawking mass $m(S)$ satisfies [36]

$$m(S)^2 \leq \mathcal{A}(S)/16\pi \ . \tag{2.9}$$

Hence if as seems plausible $\mathcal{A}(S)$ is bounded above by some multiple of $\mathcal{C}(S)^2$, we then obtain a result of the form

$$m(S) \leq k\,\mathcal{C}(S) \tag{2.10}$$

for some constant k.

We now address the following question: under what conditions will the use of asymptotic mass be appropriate? If we grant that some local form of the HC is valid, and Ω_0 is a region such that

$$m(\Omega) \simeq m_\infty \quad \forall\ \Omega \supset \Omega_0 \ , \tag{2.11}$$

where m_∞ is the asymptotic mass, it then follows that a horizon will be formed whenever

$$m_\infty \gtrsim \inf\{\mathcal{C}(\partial\Omega) \mid \Omega \supset \Omega_0\} \tag{2.12}$$

so that we recover a formulation of the conjecture which incorporates the definition (2.4) of circumference used above. If we constrain the initial-value hypersurface in such a way as to ensure that gravitational radiation is unimportant on that hypersurface, then for reasonable definitions of mass Eq. (2.11) will be satisfied when Ω_0 is taken to be the region where the stress-energy tensor is nonzero. Thus we obtain the normal global version of the HC (but with a gravitational-radiation caveat) as a consequence of the local version.

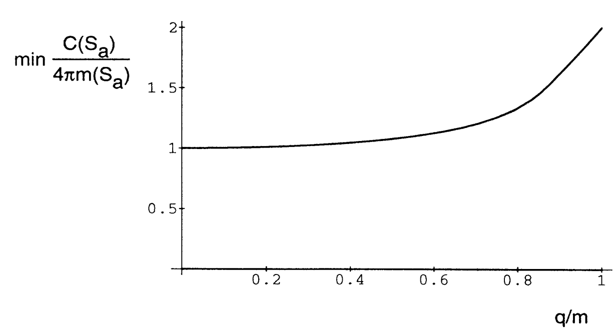

FIG. 2. The minimum circumference to mass ratio [Eq. (4)] for spherical surfaces enclosing static spheres of charged perfect fluid, as a function of the fluid's charge to mass ratio. The circumference to mass ratio is always greater than 4π in accordance with the hoop conjecture. The use of mass enclosed instead of mass at infinity is important in this regard; see text.

In investigating the HC there are two different approaches which can be adopted. One is an interior approach, which focuses on the interior of the matter-containing region, and by imposing energy conditions, etc., there tries to derive constants on the spacetime geometry. This was the procedure followed by Redmount, and by Schoen and Yau. The other is an exterior approach, which instead works with the vacuum field equations that hold outside the matter region. It may be that the HC property is a consequence of these exterior field equations, and independent of the details of interior solutions. Some indication that this might be the case comes from the fact that surfaces of minimal circumference are sometimes located in the vacuum *outside* the matter region [cf. the discussion preceding Eq. (2.4) above]. The exterior approach is also easier to analyze. To do so it is not necessary to consider any matching of interior matter solutions to exterior vacuum solutions; as Redmount has suggested [9] one can consider purely vacuum spacetimes.

Accordingly, we next consider how the conjecture applies to static vacuum spacetimes with singular sources. As trapped surfaces cannot occur in static regions of spacetime and gravitational radiation is absent, the HC reduces in this case to the following statement: there are no surfaces S enclosing the source region which satisfy $\mathcal{C}(S) < 4\pi m_\infty$, where m_∞ is the asymptotic mass. The same conclusion can be reached if one considers event horizons instead of trapped surfaces, since if an event horizon can be found then by Israel's black-hole uniqueness theorem [37] the spacetime must be a portion of the static region of the Schwarzschild spacetime for which no surfaces satisfying the above inequality exist.

III. EVIDENCE IN SUPPORT OF THE EXTERIOR APPROACH: STATIC, AXISYMMETRIC SPACETIMES

In this section we present evidence in support of our view that the exterior approach to proving the HC has a good chance of succeeding, at least in static, vacuum spacetimes. For such spacetimes the appropriate version of the conjecture is as follows (cf. the end of Sec. II C).

The HC for vacuum static spacetimes. *The circumferences of all surfaces surrounding the singular source region should be greater than or of the order of* $4\pi m$, *where* m *is the asymptotic mass.*

We shall investigate the validity of this conjecture in axisymmetric, static spacetimes.

We start in Sec. III A by briefly describing the Weyl formalism [38] for solving the static, axisymmetric, vacuum Einstein field equations, and we then show that in the case of three particular Weyl spacetimes or families of spacetimes, the conjecture holds true. In Sec. III B we specialize to oblate geometries and show that under certain circumstances, for a spacetime to satisfy the conjecture it is sufficient to have $\mathscr{C}(S) \geq 4\pi m$ for those surfaces S which are level surfaces of the lapse function. This result considerably simplifies any analyses of the HC in these spacetimes, and we use it to derive a simple sufficient condition for it to hold true. Finally in Sec. III C we turn to prolate geometries and show that if S is a convex level surface of the lapse function, then $\mathscr{C}(S) \geq 4\pi m$ whenever the value of the lapse on the surface is sufficiently large.

A. General Weyl spacetimes

We start by recalling the Weyl description of static, vacuum, axisymmetric spacetimes [38]. In these spacetimes there exist coordinates $\{t,\rho,z,\phi\}$ such that the line element takes the form

$$ds^2 = -e^{2\psi}dt^2 + e^{2(\gamma-\psi)}(d\rho^2+dz^2)+\rho^2 e^{-2\psi}d\varphi^2 , \quad (3.1)$$

where $\psi=\psi(\rho,z)$, and $\gamma=\gamma(\rho,z)$. The vacuum Einstein field equations reduce to

$${}^{(F)}\nabla^2\psi \equiv \rho^{-1}(\rho\psi_{,\rho})_{,\rho}+\psi_{,zz}=0 , \quad (3.2)$$

$$\gamma_{,\rho}=\rho(\psi_{,\rho}^2-\psi_{,z}^2) , \text{ and } \gamma_{,z}=2\rho\psi_{,\rho}\psi_{,z} . \quad (3.3)$$

Here the Laplacian ${}^{(F)}\nabla^2$ is calculated with respect to a flat, nonphysical three-metric, which is given by ${}^{(F)}ds^2=d\rho^2+dz^2+\rho^2 d\varphi^2$. The appropriate boundary conditions for these equations are that $\gamma=0$ at $z=0$ (except at singularities), and that $\psi \to 0$ at least as fast as $1/r$ as $r\to\infty$, where $r=\sqrt{\rho^2+z^2}$ [39]. Equations (3.3) may be solved by quadrature, so that solutions of Eq. (3.2) determine the geometry. The asymptotic mass of the three-geometry is [31]

$$m=\frac{1}{4\pi}\int_S \frac{{}^{(3)}\partial e^{\psi}}{\partial n}{}^{(3)}d^2S , \quad (3.4)$$

where S is any surface enclosing the matter region, the prefix (3) means that the calculation is to be carried out in the physical curved three-geometry, and $\partial/\partial n$ denotes the derivative in the direction of the outward pointing normal to S. From the metric (3.1) with $dt=0$ one obtains that

$$m=\frac{1}{4\pi}\int_S \frac{{}^{(F)}\partial\psi}{\partial n}{}^{(F)}d^2S , \quad (3.5)$$

i.e., the asymptotic mass is that of the corresponding Newtonian problem.

Consider now surfaces enclosing the singular source region in these spacetimes. We shall restrict attention to axisymmetric surfaces S as it seems plausible that those surfaces for which the circumference function $\mathscr{C}(S)$ is a minimum do not break the symmetry of the surrounding spacetime. Any such surface S of spherical topology is determined by a curve D in the ρz half plane by rotation about the z axis; see Fig. 3. We will from here on always use D and S to denote a curve and a surface related in this way. In terms of D the asymptotic mass of the three-geometry is

$$m=\frac{1}{2}\int_D \rho\frac{\partial\psi}{\partial n}dl . \quad (3.6)$$

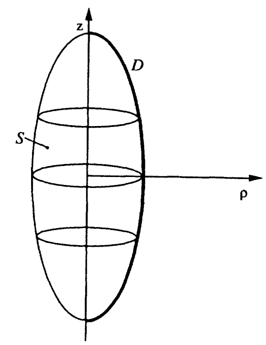

FIG. 3. An axially symmetric surface S in Euclidean three-space with a cylindrical coordinate system determines and is determined by a unique curve D in the ρz half plane.

The maximum lengths of azimuthal and polar curves on S, i.e., those curves which are the analogues of lines of latitude and longitude, are

$$L_e=\sup_D 2\pi\rho e^{-\psi} , \quad (3.7)$$

and

$$L_p=2\int_D e^{\gamma-\psi}dl . \quad (3.8)$$

The definition of circumference that we use is (cf. Sec. II B)

$$\mathscr{C}(S)=\max(L_e,L_p) . \quad (3.9)$$

In the above equations dl denotes the element of proper length with respect to the flat nonphysical geometry on the ρz half plane determined by the metric ${}^{(2)}ds^2=d\rho^2+dz^2$, and $\sup_D f$ means $\sup\{f(x)|x\in D\}$. We will call a surface oblate if $L_e(S)>L_p(s)$ and prolate otherwise, and similarly we will call a Weyl spacetime oblate (prolate) if the surfaces on which ψ is constant are oblate (prolate). Thus in these spacetimes consideration of the HC naturally divides into two cases. While it is not always true that $L_p\geq 4\pi m$, or that $L_e\geq 4\pi m$, it may well be that $\mathscr{C}(S)=\max(L_e,L_p)\geq 4\pi m$ always.

For several solutions of the vacuum field equations (3.2) and (3.3) it is straightforward to show that indeed $\mathscr{C}(S)$ is greater than $4\pi m$ for all axisymmetric surfaces S. The simplest example is the Curzon solution given by

$$\psi=-\frac{m}{r} , \quad \gamma=-\frac{m^2\sin^2\theta}{2r^2} , \quad (3.10)$$

where the coordinates (r,θ) are defined by $z=r\cos\theta$, $\rho=r\sin\theta$. From Eq. (3.7) we obtain, for any axisymmetric surface S,

$$\mathscr{C}(S)\geq L_e=\sup_D 2\pi r\sin\theta e^{m/r} \geq \inf_r 2\pi r e^{m/r}=2\pi m e , \quad (3.11)$$

so that $\mathscr{C}(S)>4\pi m$ always.

As a second example we take a fictitious source for Laplace's equation consisting of a line of uniform linear mass density $\Gamma/2$ on the symmetry axis from $z=-a$ to

$z=a$. Define coordinates $\{u,v\}$ by $z+i\rho=a\cosh(u+iv)$, for $0\le u<\infty$ and $0\le v\le\pi$. Then the metric is given by [10]

$$\psi=\Gamma\ln\tanh\frac{u}{2}\ ,\qquad \gamma=-\frac{\Gamma^2}{2}\ln\left[1+\frac{\sin^2 v}{\sinh^2 u}\right]\ , \tag{3.12}$$

and the mass is $m=\Gamma a$. A calculation of the Riemann invariant $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$ reveals that the coordinate singularity at $u=0$ is a physical singularity except when $\Gamma=1$, in which case it is the Schwarzschild horizon. Numerical studies by Redmount ([9], p. 711) have indicated that $\mathcal{C}(S_u)\ge 4\pi m$ for all the ellipsoidal $u=$const surfaces in these spacetimes. In the case where $0<1-\Gamma\ll 1$ corresponding to a prolate but almost spherical geometry, this can be confirmed by explicit calculation: we find that

$$L_{p,\min}(\Gamma)=4\pi m[1-\varepsilon\ln\varepsilon+O(\varepsilon)]\ , \tag{3.13}$$

where $\varepsilon=1-\Gamma$ and $L_{p,\min}$ is the smallest of the polar circumferences $L_p(S_u)$. This is illustrated in Figs. 4 and 5.

In the case $\Gamma>1$, corresponding to an oblate geometry, it is possible to show that the conjecture is satisfied for all closed surfaces by using the same technique as in the above Curzon case. If S is an arbitrary axisymmetric surface, then from Eq. (3.7),

$$\begin{aligned}\mathcal{C}(S)\ge L_e&=\sup_D 2\pi\rho e^{-\psi}\\ &=\sup_D 2\pi a\sinh u\sin v\tanh^{-\Gamma}(u/2)\\ &\ge\inf_{u>0}2\pi a\sinh u\tanh^{-\Gamma}(u/2)\\ &=2\pi a(\Gamma+1)\left[\frac{\Gamma-1}{\Gamma+1}\right]^{(1-\Gamma)/2}\ ,\end{aligned} \tag{3.14}$$

so that, using $m=\Gamma a$,

$$\frac{\mathcal{C}(S)}{4\pi m}\ge h(\Gamma)\equiv\frac{\Gamma+1}{2\Gamma}\left[\frac{\Gamma-1}{\Gamma+1}\right]^{(1-\Gamma)/2}\ . \tag{3.15}$$

The function h satisfies $h(\Gamma)\ge 1$ for all $\Gamma>1$ as shown in Fig. 5.

Finally consider the class of spacetimes given by

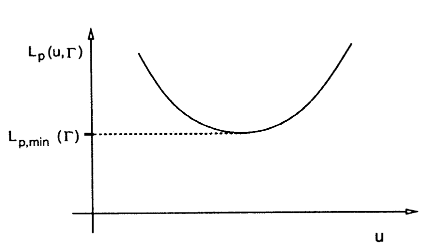

FIG. 4. A graph of the lengths of polar curves on the ellipsoidal surfaces of constant u in the Γ-metric spacetimes. Notice that this length (and consequently also the circumference $\mathcal{C}_1$ of the surfaces) diverges as u tends towards zero, even though the corresponding surfaces are nested inside each other and apparently getting smaller and smaller.

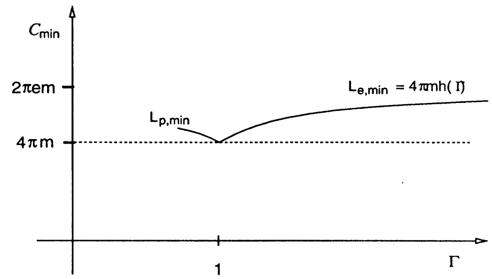

FIG. 5. A graph of the minimum obtainable circumference to mass ratio $\mathcal{C}_1(S)/m$ in the Γ-metric spacetimes as a function of the "oblateness" parameter Γ. For $\Gamma<1$ the longest curves on the surfaces of constant u are polar, while for $\Gamma>1$ they are equatorial. Again the hoop conjecture is validated.

$$\psi(r,\theta)=-\frac{m}{r}+\frac{\alpha}{r^{n+1}}P_n(\cos\theta)\ , \tag{3.16}$$

where α is any real number and P_n is the nth Legendre polynomial. Define θ_n to be such that $\cos\theta_n$ is the smallest positive root of the equation $P_n(x)=0$. Then we obtain

$$\begin{aligned}\frac{\mathcal{C}(S)}{4\pi m}&\ge\sup_D\frac{r}{2m}\sin\theta\exp\left[\frac{m}{r}-\frac{\alpha}{r^{n+1}}P_n(\cos\theta)\right]\\ &\ge\inf_r\frac{re^{m/r}}{2m}\sin\theta_n\ge\frac{1}{2}\alpha_n e\ ,\end{aligned} \tag{3.17}$$

where $\alpha_n=\sin\theta_n$. The sequence (α_n) starts out $(1,\sqrt{2/3},\sqrt{2/5},0.88,0.71,\ldots)$ and is bounded below, so that $\mathcal{C}(S)\gtrsim 4\pi m$ for all axisymmetric surfaces S.

B. Oblate Weyl spacetimes

In investigating the validity of the HC in general Weyl spacetimes, the most natural course of action is to consider the spacetime geometry as fixed and to calculate the circumferences of various surfaces in this spacetime. However we have found the following approach to be more useful. We choose a closed surface S in the unphysical flat background geometry, and specify the value of the potential ψ on S. Then ψ on the region exterior to S will be determined as the solution of the following boundary-value problem: ${}^{(F)}\nabla^2\psi=0$ outside S, $\psi_{|S}=$as specified, and $\psi\to 0$ as $r\to\infty$. Thus the pair $(S,\psi_{|S})$ determines a solution of the vacuum field equations, and all exterior solutions may be obtained in this way. If we specify S and the corresponding curve D by a smooth function $\rho=R(z)$, for $z_0\le z\le z_1$, then the normal derivative $\partial\psi/\partial n_{|D}$ is determined from R and $\psi_{|D}$ by solving the boundary value problem. The value of the function $\gamma_{|D}$ at a point p on D is determined by integrating $\nabla\gamma$ along that portion of D joining the z axis to p, where $\nabla\gamma$ is given in terms of $\psi_{|D}$ and $\partial\psi/\partial n_{|D}$ by Eqs. (3.3). The mass of the spacetime and the circumference of S are then given by Eqs. (3.6) to (3.8), i.e., are functionals of R and $\psi_{|D}$, and the hoop conjecture will be true in these Weyl spacetimes if

$$\mathscr{C}(R,\psi_{|D})\geq 4\pi m(R,\psi_{|D}) \tag{3.18}$$

always.

We now show that for oblate surfaces it is sometimes sufficient to consider the special case $\psi_{|D}=$ const. If, in any Weyl spacetime, we define S_λ for $\lambda>0$ to be the surface $\{p\,|\,\psi(p)=-\lambda\}$, then specifically we have the following theorem.

Theorem 1. *In a Weyl spacetime suppose that $\lambda_0>0$ is such that (i) $\nabla\psi\neq 0$ everywhere on $\{p\,|\,\psi(p)\geq-\lambda_0\}$, and (ii) the coordinate function ρ has only one local maximum on each curve D_λ corresponding to S_λ* for $\lambda\leq\lambda_0$. *Then $L_e(S)\geq 4\pi m$ for all axisymmetric surfaces S of spherical topology outside of and enclosing S_{λ_0} if and only if $L_e(S_\lambda)\geq 4\pi m$ for all S_λ outside S_{λ_0}.*

Note that conditions (i) and (ii) will be satisfied for surfaces S_{λ_0} sufficiently far from the singular source, i.e., for sufficiently small λ_0. The precise distance required will depend on the sizes of the dipole and higher-order multipole mass moments of the fictitious Newtonian source compared to that of the monopole moment.

Proof. We first show that each surface S_λ for $\lambda\leq\lambda_0$ is of spherical topology. If any of these surfaces is not connected, put $\bar\lambda=\inf\{\lambda\,|\,S_\lambda$ has more than one connected component$\}$, which is strictly positive as $\psi\approx-m/r$ as $r\to\infty$. Then it can be seen that there is a point p in $S_{\bar\lambda}$ with $\nabla\psi(p)=0$, which contradicts assumption (i). Hence each S_λ is connected, compact, orientable two-dimensional manifold, and so is topologically a sphere with n handles. An argument similar to that just given shows that the genus n must be zero. Also, the outward normal derivative $\partial\psi/\partial n$ is positive everywhere on each S_λ by condition (i), and so that region exterior to S_λ is the union of the S_σ for $\sigma<\lambda$.

We now prove the "if" part of the theorem as the "only if" part is obvious. Suppose that we are given a surface S as in the statement of the theorem. Define $\lambda_1=\inf_S(-\psi),\lambda_2=\sup_S(-\psi)$ (see Fig. 6), and for λ between λ_1 and λ_2 let p_λ be the unique point on D_λ furthest from the z axis. Then we have that $\eta(p_\lambda)=\sup\{\eta(p)|p\in S_\lambda\}$, where η is the function $\rho e^{-\psi}/(2m)$. Hence

$$\eta(p_\lambda)\geq 1\ ,\quad \lambda_1\leq\lambda\leq\lambda_2\ , \tag{3.19}$$

by Eq. (3.7). Let Γ be the smooth curve generated by the p_λ's and let S_i, S_e denote the regions interior to and exterior to S. We claim that $p_2\equiv p_{\lambda_2}$ lies in $S_i\cup S$. Otherwise there would exist a neighborhood of p_2 in S_e, and as $\nabla\psi(p_2)\neq 0$, there would exist a point $q\in S_e$ with $\psi(q)<\psi(p_2)=\inf_S\psi=\inf_{S_e}\psi$ (using the fact that ψ is harmonic) which is a contradiction. Similarly $p_1\equiv p_{\lambda_1}\in S_e\cup S$, and so it follows that the curve Γ cuts S at some point, say p_σ. It follows from Eqs. (3.7) and (3.19) that

$$\frac{L_e(S)}{4\pi m}=\sup_S\eta\geq\eta(p_\sigma)\geq 1\ . \tag{3.20}$$

This concludes the proof.

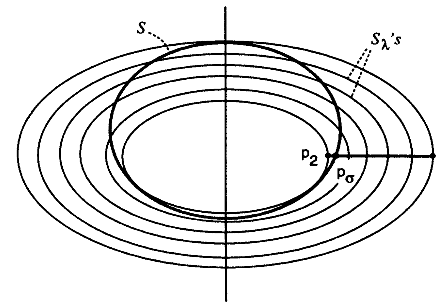

FIG. 6. An illustration of the situation in theorem 1. The ellipses represent level surfaces of the lapse function $e^{2\psi}$.

We now turn to an analysis of the inequality (3.18) under the simplifying assumption that $\psi_{|D}$ is constant. First we introduce the following generalized notion of capacity which is a geometric measure of the size of a region. For Ω any connected open region in a Riemannian manifold Σ, let u be the unique function satisfying $\nabla^2u=0$ on the complement Ω_e of Ω in Σ, $u_{|\partial\Omega}=-1$, and $u\to 0$ at infinity. Then the capacity of Ω is

$$R_0(\Omega)\equiv\frac{1}{4\pi}\int_{\partial\Omega}n^a\nabla_a u\ , \tag{3.21}$$

where n^a is the outward pointing unit normal to $\partial\Omega$. It can be shown [40] that

$$R_0(\Omega)=\inf\left\{\int_{\Omega_e}g^{ab}\nabla_a u\nabla_b u\,\middle|\,u\in C^\infty(\Omega_e)\ ,\right.$$
$$\left.u_{|\partial\Omega}=-1\ ,\quad u\to 0\ \text{ at }\infty\right\}, \tag{3.22}$$

and it follows that the capacity function R_0 is monotone in the sense that

$$\Omega\subset\Omega'\Longrightarrow R_0(\Omega)\leq R_0(\Omega')\ . \tag{3.23}$$

For Ω a connected region, we define $R_0(\partial\Omega)=R_0(\Omega)$.

Suppose now that we are given a curve D and we specify $\psi_{|D}$. Then from Eqs. (3.2) and (3.5) the mass is given by $m(D,\psi_{|D})=-\psi_{|D}R_0(S)$, as m is a linear function of the number $\psi_{|D}$. Hence to obtain $m>0$ we must take $\psi_{|D}=-\lambda$, where $\lambda>0$. Here the capacity is calculated with respect to ${}^{(F)}ds^2$, the flat three-dimensional metric. [In terms of the physical curved three-metric we have $m=(1-e^{-\nu})R_0(S_\nu)$ for any $\nu>0$.] We now let ψ_0 be the solution of the boundary value problem with the boundary condition $\psi_{0|D}=-1$, with the mass being m_0 and γ_0 being the corresponding function obtained from Eqs. (3.3). Then $\psi=\lambda\psi_0$ and $\gamma=\lambda^2\gamma_0$, and Eqs. (3.6) and (3.7) yield that

$$L_e=\sup_D 2\pi\rho e^\lambda \tag{3.24}$$

and

$$m=\lambda m_0=\lambda R_0(S)\ . \tag{3.25}$$

Now if we use the monotonicity property (3.23) together with the fact that the capacity R_0 of a sphere is just its ordinary radius, we obtain that $R_0(S) \le r_m \equiv \sup_D r$, where $r = \sqrt{\rho^2 + z^2}$. Also if we define $\rho_m = \sup_D \rho$, then, from Eqs. (3.24) and (3.25),

$$\frac{L_e}{4\pi m} = \frac{\rho_m}{2R_0(S)} \frac{e^\lambda}{\lambda} \ge \left[\frac{e}{2}\right] \frac{\rho_m}{r_m} . \tag{3.26}$$

Thus for curves D for which $\rho_m \approx r_m$, L_e is approximately equal to m and the HC for these surfaces is approximately satisfied. Only in cases where $\rho_m \ll r_m$ can it be violated, and we expect that in such cases $L_p > 4\pi m$ instead, so that by Eq. (3.9) the inequality $\mathcal{C}(S) > 4\pi m$ is still maintained.

More generally, if we define the "eccentricity" of a curve D via

$$\epsilon(D) \equiv \frac{\rho_m}{R_0(D)} , \tag{3.27}$$

and then we see from the inequality (3.26) that $L_e(S) \ge 4\pi m$ if and only if $\epsilon(D) \ge 2\lambda e^{-\lambda}$. Combining this with theorem 1 we obtain the following result.

Theorem 2. *Suppose in a Weyl spacetime (i)* $\nabla\psi \neq 0$ *everywhere, and (ii) the level surfaces of* ψ *are convex with respect to the fictitious flat three-geometry. Then* $L_e(S) \ge 4\pi m$ *for all axisymmetric surfaces S of spherical topology enclosing the singularity if and only if* $\epsilon(D_\lambda) \ge 2\lambda e^{-\lambda}$ *for every* $\lambda > 0$.

The condition on the eccentricity will be satisfied in particular when $\rho_m > (2/e) r_m$ for all curves D on which ψ is constant. As an example we apply this theorem to the "Γ metrics" discussed earlier [cf. Eq. (3.12)] and rederive the result that the hoop conjecture is satisfied for $\Gamma > 1$. Using $\rho_m = a \sinh u$, $R_0(D) = \Gamma a/\lambda$, and $\lambda = -\Gamma \ln \tanh(u/2)$ we obtain that

$$\inf_\lambda \epsilon(D_\lambda) \frac{e^\lambda}{2\lambda} = \inf_\lambda \frac{\exp[\lambda(1 - 1/\Gamma)]}{\Gamma(1 - e^{-2\lambda/\Gamma})} = \frac{\Gamma+1}{2\Gamma} \left[\frac{\Gamma-1}{\Gamma+1}\right]^{(1-\Gamma)/2} \ge 1 \quad \forall\ \Gamma > 1 , \tag{3.28}$$

where the last expression is the same as that obtained previously in Eq. (3.15).

C. Prolate Weyl spacetimes

We now turn to a consideration of the inequality (3.18) in the case when the surface S is prolate. We again restrict attention to equipotential surfaces, although in this case we have not been able to show that it is sufficient to consider these surfaces only. Our result is as follows.

Theorem 3. *In a Weyl spacetime, let D be a curve in the* ρz *half plane with* $\psi_{|D} = -\lambda$. *Suppose that (i) D is convex, and (ii)* $\lambda \le \lambda_{\text{crit}}(D)$, *where*

$$\lambda_{\text{crit}}^{-1} = 2\int_0^{\rho_m} \left[\frac{\partial \psi_0}{\partial n}\right]^2 \rho\, d\rho$$

and $\psi_0 \equiv \lambda^{-1}\psi$. *Then*

$$\frac{L_p(S)}{4\pi m} \ge \frac{e}{\pi} \left[1 + \frac{\pi^2}{4}\right]^{-1/2} \approx \frac{1}{2} .$$

Thus the spirit, if not the letter, of the hoop conjecture is satisfied for sufficiently small λ in the special case of convex D. To prove this result, we have from Eq. (3.8) that

$$L_p = 2\int_0^{L(D)} d\xi\, e^{\lambda + \lambda^2 \gamma_0(\xi)} . \tag{3.29}$$

Here $L(D)$ is the length of D with respect to ${}^{(2)}ds^2$, and ξ is the proper length parameter. Now from Eqs. (3.3) we calculate that $\gamma_0' = -\psi_{0,n}^2 \rho\rho'$, where the prime denotes differentiation with respect to ξ, and the n denotes normal derivative. If $\mu \equiv -\gamma_0$, then it follows that $\mu_m \equiv \sup_D \mu = \int_0^{\rho_m} \psi_{0,n}^2 \rho\, d\rho$, and so by (ii) above $1/\lambda \ge 2\mu_m \ge -2\gamma_0(\xi)$ for all ξ. Hence

$$e^{\lambda + \lambda^2 \gamma_0} > e^{\lambda/2} , \tag{3.30}$$

and combining this with Eqs. (3.25) and (3.29) yields that

$$\frac{L_p}{4\pi m} \ge \frac{e^{\lambda/2}}{\lambda} \frac{L(D)}{2\pi m_0} \ge \frac{e}{4\pi} \frac{L(D)}{R_0(S)} . \tag{3.31}$$

To obtain an estimate of the capacity of the surface S, we use the result [40] that

$$R_0(S) \le \frac{1}{4\pi} \int_S \bar{\kappa}\, d^2S , \tag{3.32}$$

where $\bar{\kappa}$ is the mean curvature of the surface. If we take D to be specified by the equation $\rho = R(z)$ for $z_0 \le z \le z_1$, with $R(z_0) = R(z_1) = 0$, then we obtain

$$\bar{\kappa} = \frac{1}{2} \left[\frac{1/R}{(1 + R'^2)^{1/2}} - \frac{R''}{(1 + R'^2)^{3/2}}\right] . \tag{3.33}$$

In this expression the second term in large parentheses is the curvature κ of the curve D in the ρz plane, and primes denote differentiation with respect to z. Inserting this into the bound (3.32) gives

$$R_0(S) \le \tfrac{1}{4}\Delta z + \tfrac{1}{4} \int_D \rho\kappa\, dl , \tag{3.34}$$

where $\Delta z \equiv z_1 - z_0$, so that from Eq. (3.31)

$$\frac{L_p}{4\pi m} \ge \frac{e}{\pi} \left[\frac{\Delta z}{L(D)} + \langle \rho\kappa \rangle\right]^{-1} \ge \frac{e}{\pi} \frac{1}{1 + \langle \rho\kappa \rangle} , \tag{3.35}$$

where

$$\langle \rho\kappa \rangle \equiv \frac{1}{L(D)} \int_D \rho\kappa\, dl . \tag{3.36}$$

Now for a general curve D, the quantity $\langle \rho\kappa \rangle$ may be arbitrarily large, but when D is convex it is possible to show that it is bounded above. As $R'' \le 0$ for D convex, we have

$$\langle \rho\kappa \rangle = \frac{1}{L(D)} \int dz \frac{-R''\rho}{1+R'^2} \le \frac{\rho_m}{L(D)} \int dz \frac{-R''}{1+R'^2}$$
$$= -\frac{\rho_m}{L(D)} \arctan(R') \Big|_{z_0}^{z_1}$$
$$= \frac{\rho_m}{L(D)} \pi \; . \tag{3.37}$$

But it is easy to see that $L(D) \ge 2\rho_m$ so that from Eqs. (3.35) and (3.37) we finally obtain

$$\frac{L_p}{4\pi m} \ge \frac{e}{\pi} \frac{1}{1+\pi/2} \approx \frac{1}{3} \; . \tag{3.38}$$

It is possible to improve this bound to give the one quoted in the theorem by combining Eq. (3.37) with the first inequality in (3.35) and considering the geometric relationships between Δz, ρ_m, and $L(D)$; see Fig. 7.

In the course of the above proof we showed that condition (ii) in the theorem is equivalent to $\lambda^{-1} \ge 2\mu_m$, where $\mu = -\gamma_0$ and $\mu_m = \sup_D \mu$. By using the inequality $\int_0^1 e^f \ge \exp(\int_0^1 f)$ it is possible to weaken this condition to $\lambda^{-1} \ge 2\langle \mu \rangle$, where $\langle \mu \rangle \equiv \int_0^{L(D)} d\xi\, \mu(\xi)/L(D)$.

D. Future directions for Weyl spacetimes

In this section we have considered the ratio $\Theta \equiv \mathcal{C}(s)/4\pi m$ for convex equipotential surfaces S in Weyl geometries as a function of (i) the curve D that generates S by a rotation, and (ii) the value $-\lambda$ of the potential ψ on this surface. We found that $\Theta \ge 1$ for all positive values of λ whenever the eccentricity of the curve is greater than $2/e$. For smaller values of the eccentricity we found that the ratio Θ is bounded below by a positive constant of the order of a half when $\lambda \ge \lambda_1(\epsilon(D))$ and when $\lambda \le (2\langle \mu \rangle)^{-1}$, where $\lambda_1(x)$ is the larger of the two roots of $2\lambda e^{-\lambda} = x$ (see Fig. 8). Now in general $(2\langle \mu \rangle)^{-1}$ is not greater than $\lambda_1(\epsilon(D))$, and thus there is a range of values of λ for which none of the bounds obtained for Θ apply. For example, for ellipses whose ratio of semiminor axis to semimajor axis is $t \ll 1$, asymptotically $\lambda_1(\epsilon(D)) \approx |\ln t|$ and $(2\langle \mu \rangle)^{-1} \approx \frac{1}{2}|\ln t|$. Further progress towards showing $\Theta \gtrsim 1$ generically would necessitate finding better estimates for the lengths of polar curves with λ in this intermediate range.

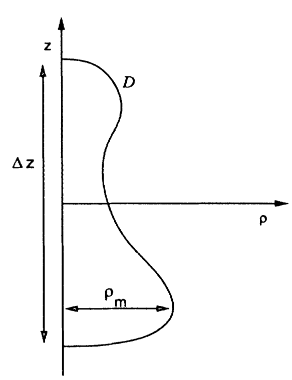

FIG. 7. The length of the curve $L(D)$ from the diagram will be greater than $\sqrt{(\Delta z - \alpha)^2 + \rho_m^2} + \sqrt{\alpha^2 + \rho_m^2}$ for some α.

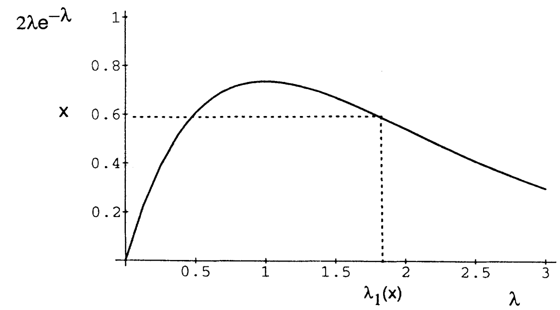

FIG. 8. An illustration of the definition of the function $\lambda_1(x)$ which is defined only for $x < 2/e$.

Also, we have restricted attention to Weyl spacetimes with isolated singular sources. If one considers nonisolated sources, one can obtain spacetimes with non-Schwarzschild event horizons. A complete classification of such "distorted black holes" in static, axisymmetric spacetimes has been given by Geroch and Hartle [41]. It might be interesting to investigate the circumferences of the distorted horizons.

IV. CONCLUSION

In this paper we have reviewed some of the evidence in favor of the HC, and presented some tentative but suggestive results which tend to support it. We have also advocated the following two points of view: (i) that the conjecture can be interpreted as a quasilocal statement in the spirit of the theorem of Schoen and Yau, and (ii) that the geometric constraints embodied in the conjecture may very well be consequences of the exterior vacuum field equations and independent of the complicated interior physics.

The dynamical collapse simulations of Shapiro and Teukolsky [8] apparently exhibit violations of cosmic censorship, violations which are in accordance with the HC. A proof of some criterion along the lines of the HC would both bolster our faith in those calculations, and also possibly increase our understanding of when such violations can occur.

ACKNOWLEDGMENTS

The author's thanks go to Kip Thorne for many helpful and detailed discussions about aspects of this work, and about it's presentation.

APPENDIX A: DEFINITIONS OF CIRCUMFERENCE

In this appendix we show that for axisymmetric surfaces S, the definition (2.2) of circumference satisfies

$$\mathcal{C}_2(S) = L_p \; , \tag{A1}$$

where L_p is twice the distance from the north pole to the south pole. In particular we have that $\mathcal{C}_2(S) \neq \mathcal{C}_1(S)$

$\equiv \max(L_e, L_p)$, which is the quantity that seems to be appropriate for the HC in axisymmetric spacetimes. Here L_e denotes the maximum of the lengths of closed azimuthal curves as in Eq. (3.7).

An axisymmetric surface S can be described by a line element of the form

$$ds^2 = \alpha(\theta)^2 d\theta^2 + \beta(\theta)^2 \sin^2\theta \, d\phi^2 \ .$$

Given points $\mathcal{P}$, $\mathcal{Q}$, in S, let Γ_1 be the curve joining $\mathcal{P}$ to the north pole $\mathcal{N}$ along a line of longitude ($\phi=$ const). Let Γ_2 be a similar curve joining $\mathcal{N}$ to $\mathcal{Q}$, and let $\Gamma_{\mathcal{N}}$ be Γ_2 joined onto Γ_1. Similarly construct $\Gamma_{\mathcal{S}}$ joining $\mathcal{P}$ to $\mathcal{Q}$ via the south pole $\mathcal{S}$. Then the distance from $\mathcal{P}$ to $\mathcal{Q}$ satisfies $d(\mathcal{P},\mathcal{Q}) \le L(\Gamma_{\mathcal{N}})$, and similarly for the curve $\Gamma_{\mathcal{S}}$. However it is easy to see that $L(\Gamma_{\mathcal{N}}) + L(\Gamma_{\mathcal{S}}) = L_p$, so that $2d(\mathcal{P},\mathcal{Q}) \le L_p$. Now taking the maximum over all points $\mathcal{P}$ and $\mathcal{Q}$ and using the definition (2.2) yields that $\mathcal{C}_2(S) \le L_p$, and Eq. (A1) then follows.

Now consider a surface of constant lapse in the spacetime given by Eq. (3.12). From Eq. (3.8) we obtain that

$$\begin{aligned} L_p(S) &= 4\sinh^{\Gamma^2}(u)\tanh^{-\Gamma}(u/2) \\ &\quad \times \int_0^{\pi} dv\,(\sinh^2 u + \sin^2 v)^{-\Gamma^2} \\ &\le 4\pi \tanh^{-\Gamma}(u/2)\sinh^{\Gamma^2}(u) \ , \end{aligned} \tag{A2}$$

which tends to zero as u tends to zero if $\Gamma > 1$. Hence $\mathcal{C}_2(S)/m$ can be arbitrarily small.

APPENDIX B: WORLD-TUBE CIRCUMFERENCE PROOF

In this appendix we show that the definition (2.5) of the circumference of a world tube reduces to the usual definition in the case of a static spacetime. Such a spacetime can be described by a line element of the form

$$ds^2 = g_{00}(x^k)dt^2 + g_{ij}(x^k)dx^i dx^j \ , \tag{B1}$$

where i, j, and k range from 1 to 3, and $g_{00}(x^k) < 0$. Any spacelike hypersurface is given (not just locally) by an equation of the form

$$t = F(x^1, x^2, x^3) \ , \tag{B2}$$

and the induced metric on Σ is

$${}^{(\Sigma)}ds^2 = (g_{ij} + g_{00}F_{,i}F_{,j})dx^i dx^j \ . \tag{B3}$$

If we define Σ_o to be the hypersurface $t=0$, then the diffeomorphism from Σ_o to Σ obtained by identifying the coordinates x^1, x^2, and x^3 will be a contraction, i.e., lengths of curves on Σ_o will always be longer than the lengths of the corresponding curves on Σ. This is because g_{ij} is positive definite and $g_{00} < 0$. A similar statement applies to distances between points. This shows that the maximum over all Σ of $\mathcal{C}_e(\Sigma \cap T)$ will be achieved by Σ_o, which establishes the desired result.

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