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Lorentz Chern-Simons terms in Bianchi cosmologies and the cosmic no-hair conjecture

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The theory describing gravity and the Kalb-Ramond axion field with topological Lorentz Chern-Simons coupling is considered. The topological coupling induces an additional term in the energy-momentum tensor, which in general can lead to violations of the cosmic no-hair conjecture. These violations are shown explicitly by analyzing the solutions in a diagonal Bianchi type-II geometry.

I. INTRODUCTION

The long-standing cosmological problems of high-degree homogeneity, isotropy, and flatness of the observed universe have led to the proposal of the inflationary scenarios to explain them [1]. These scenarios all involve certain mechanisms to provide for a positive cosmological constant which is needed to produce fast growth of the cosmic scale factor. The cosmic no-hair conjecture, as it was dubbed by Hawking and Moss [2], then guarantees the success of the inflationary stage for a very large class of initially inhomogeneous and anisotropic universes. It asserts that under quite loose and physically realistic conditions any expanding universe satisfying them will evolve asymptotically towards a locally de Sitter geometry. In the course of evolution initial inhomogeneities and anisotropies get dissipated and flattened out. Thus, the no-hair conjecture in conjunction with the inflationary paradigm offers a viable explanation to homogeneity and isotropy in the universe as we know it.

In this paper we will not repeat the derivation of the cosmic no-hair conjecture. There already exists a number of excellent papers where the proof has been outlined in great detail [3–6]. Among these especially elegant is the paper by Wald [4], where he proved the no-hair conjecture for the Bianchi models, with the possible exception of Bianchi type IX. We will merely review his results here, as we will be investigating the Bianchi models too.

Most of the aforementioned papers dealt with standard general relativity with energy momentum carried solely by the matter sources. Superstring theory, on the other hand, indicates that the low-energy effective action for gravity may include curvature terms of higher order, like the Gauss-Bonnet and the Chern-Simons topological densities. The no-hair conjecture has been investigated for some models with higher-order curvature terms in the action [7], namely, the Starobinsky model. It was found to hold there. This should not come as a surprise, since it is well known that the Starobinsky model is conformally equivalent to the standard general relativity with a minimally coupled scalar field [8].

We will examine the cosmic no-hair conjecture in relation to the other superstring motivated abberation from Einstein's general relativity, the Lorentz Chern-Simons (LCS) coupling. This topological coupling is incorporated in the action through redefinition of the axion field strength $H_{\mu\nu\lambda}$ to ensure the gravitational anomally cancellation in the string field theory, as explained by Green and Schwarz [9]. One could ask what would happen if this term survives the era of quantum gravity and remains important at scales where inflation is likely to have occurred. Clearly it could influence evolution of the universe. Since it can account for interesting new effects in axion physics at smaller scales and put extra hair on black holes [10], it is a promising candidate to entertain similar effects at larger scales as well.

Below we will show that this is indeed the case. Although for certain Bianchi models (type I and type V and diagonal type III and type VI) the LCS terms allow evolution of the universe to proceed towards asymptotic locally de Sitter state, for others they can turn the situation around and force the universe to recollapse. We will demonstrate this by constructing numerical solutions of the equations of motion for the diagonal Bianchi type-II model with only one anisotropy function, $\beta_2=\beta_3=-\beta_1/2$. We will also obtain a class of particular solutions that experience the exponential growth of the scale factor, but retain hair, as the shear tensor does not vanish asymptotically.

The paper is organized as follows. In Sec. II we will present the equations of motion and the necessary background. In Sec. III we will discuss them in light of the cosmic no-hair conjecture and confirm its validity for general Bianchi type-I and type-V models as well as diagonal type II and type VI. We will inspect the Bianchi type-II model in Sec. IV and display that the no-hair conjecture is violated there. Conclusions will be addressed in Sec. V.

II. EQUATIONS OF MOTION

We start with the action that is a part of the effective action for light modes in superstring theory and includes the graviton and the Kalb-Ramond axion. Dilation self-interactions may append an effective cosmological constant Λ to it. So, in four dimensions (4D) our action is

$$S=\int d^4x\sqrt{g}\left[\frac{1}{2\kappa^2}R-H_{\mu\nu\lambda}H^{\mu\nu\lambda}-\Lambda\right] \tag{2.1}$$

and $H_{\mu\nu\lambda}$ is the 3-form Kalb-Ramond field strength, defined by

$$H = dB + \omega_L \ , \tag{2.2}$$

where B is the Kalb-Ramond two-form, with

$$\omega_L = \mathrm{Tr}(\omega \wedge d\omega + \tfrac{2}{3}\omega \wedge \omega \wedge \omega) \tag{2.3}$$

the LCS form and ω the spin connection.

In the action above we ignored explicit dilaton and Yang-Mills terms, as we are not interested in them here. Then, the standard variational procedure yields the equations of motion [11]:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 6 H_{\mu\lambda\rho} H_{\nu}{}^{\lambda\rho} - g_{\mu\nu} H_{\lambda\alpha\rho} H^{\lambda\alpha\rho} + 4\nabla_\sigma (H_{\mu\alpha\lambda} R^{\sigma}{}_{\nu}{}^{\alpha\lambda}) - \Lambda g_{\mu\nu} \ , \tag{2.4}$$

$$\nabla_\lambda H^{\mu\nu\lambda} = 0 \ , \tag{2.5}$$

and also from Eq. (2.2).

$$dH = d\omega_L \ . \tag{2.6}$$

The LCS term in (2.4) is symmetrized over μ and ν.

Our conventions are $g = (-,+,+,+)$, $R^{\mu}{}_{\nu\lambda\sigma} = \Gamma^{\mu}_{\nu\sigma,\lambda} - \cdots$, ∇_λ denotes covariant derivative with respect to basis vector fields X_λ. We work in the units $\kappa^2 = 1$. Summation over repeated indices, both greek and latin, is understood throughout the paper.

The equations of motion can be recast in a different form. Recall that in four dimensions an antisymmetric tensor is dual to a vector:

$$H = {}^*V \ . \tag{2.7}$$

From (2.5),

$$dV = 0 \ , \tag{2.8}$$

and as long as our manifold has vanishing first cohomology, which is true at least locally for Bianchi models, all closed one-forms are exact. Thus,

$$H = {}^*db \tag{2.9}$$

and we can rewrite the equations of motion as

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 12 X_\mu b X_\nu b - 6 g_{\mu\nu} X_\lambda b X^\lambda b + 4\nabla_\sigma [\sqrt{g}\, \epsilon_{\mu\alpha\lambda\rho} (X^\rho b) R^{\sigma}{}_{\nu}{}^{\alpha\lambda}] - \Lambda g_{\mu\nu} \ , \tag{2.10}$$

$$d\,{}^*db = d\omega_L \ . \tag{2.11}$$

Here we allow for the possibility of defining tensors in a general nonholonomic basis, where coordinate tangent vector fields ∂_λ are replaced by X_λ. In such bases, the exterior derivative operator is

$$d = \epsilon^\lambda X_\lambda \ , \tag{2.12}$$

where ϵ^λ are one-forms dual to vector fields X_λ [12]. The connexion forms can be computed in the usual way, as explained in [12].

Bianchi models encompass all possible realizations of spatially homogeneous metrics, i.e., those which admit isometry groups with three independent translational space-like vector fields. The vector fields are the Killing vectors of the metrics. The metrics can be written down as metrics on coset spaces of nine acceptable groups of motions, in 4D space-times,

$$ds^2 = -dt^2 + h_{ab}(t) \epsilon^a \otimes \epsilon^b \ , \tag{2.13}$$

with group properties included through

$$d\epsilon^a = -\tfrac{1}{2} C^{a}{}_{bc} \epsilon^b \wedge \epsilon^c \ , \tag{2.14}$$

where $C^{a}{}_{bc}$ are the structure constants of the group in question. These and further details can be found in [13].

I choose to work in the nonholonomic basis ($\epsilon^0 = dt$, ϵ^a, $a = 1,2,3$). The metric then depends only on time, and is conventionally given by

$$g \equiv h = \begin{bmatrix} -1 & 0 \\ 0 & R^2 (e^{2\beta})_{ab} \end{bmatrix} , \tag{2.15}$$

with $\mathrm{Tr}(\beta) = 0$. The matrix β measures anisotropy. This choice facilitates evaluation of the equations of motion, as it provides the same general expressions for all isometry groups. Formulas for a specific group are obtained upon substitution of the values of structure constants. Notice that homogeneity constrains the axion field b to depend only on time.

All this at hand, it is not difficult to compute the terms which appear in the equations of motion. For example, the Ricci tensor and the scalar curvature are [14]

$$\begin{aligned} R^{0}{}_{0} &= \dot{K} + \frac{K^2}{3} + \sigma_{ab}\sigma^{ab} \ , \\ R^{0}{}_{a} &= -C^{b}{}_{ac}\sigma^{c}{}_{b} - C^{d}{}_{db}\sigma^{b}{}_{a} \ , \\ R^{a}{}_{b} &= \frac{1}{\sqrt{h}} \left[\sqrt{h} \left[\sigma^{a}{}_{b} + \frac{K}{3}\delta^{a}{}_{b} \right] \right]^{\cdot} + {}^{(3)}R^{a}{}_{b} \ , \\ R &= 2\dot{K} + \tfrac{4}{3}K^2 + \sigma_{ab}\sigma^{ab} + {}^{(3)}R \end{aligned} \tag{2.16}$$

(overdot denotes time derivative and K and σ_{ab} are the volume expansion factor and the shear, respectively). However, I will not dwell on the general computation of the LCS contributions to the energy-momentum tensor here. The computation is tedious and not essential for the remainder of the paper. I will, however, discuss the axion equation of motion (2.11) in more detail, though.

It is easy to see from the structure of the metric (2.13) and the definition of the LCS form (2.3) that it can be written as

$$\omega_L = B_{abc}(t) \epsilon^a \wedge \epsilon^b \wedge \epsilon^c + A_{ab}(t) \epsilon^0 \wedge \epsilon^a \wedge \epsilon^b \ , \tag{2.17}$$

where B_{abc} and A_{ab} are antisymmetric time-dependent functions calculated from the metric and the structure constants. Then, using (2.14),

$$d\omega_L = [\dot{B}_{abc}(t) + A_{da}(t) C^{d}{}_{bc}] \epsilon^0 \wedge \epsilon^a \wedge \epsilon^b \wedge \epsilon^c \ . \tag{2.18}$$

Therefore (2.11) reads

$$(\sqrt{h}\, \dot{b})^{\cdot} = \tfrac{1}{6} [\dot{B}_{abc}(t) + A_{da}(t) C^{d}{}_{bc}] \epsilon^{abc} \ , \tag{2.19}$$

where ϵ^{abc} is just the standard antisymmetric symbol. At this point it is useful to recall the MacCallum-Ellis classification [14] of Bianchi models based on the form of the structure constants. The structure constants can be written out as

$$C^a{}_{bc}=M^{ad}\epsilon_{dbc}+A_c\delta^a{}_b-A_b\delta^a{}_c \quad , \tag{2.20}$$

where M is a symmetric matrix and A a 3-vector.

Bianchi models are classified according to whether A_a vanishes (class A) or does not vanish (class B). Inserting (2.20) in (2.19) and keeping in mind that A_{ab} is antisymmetric, we obtain

$$(\sqrt{h}\,\dot{b})^{\cdot}=\tfrac{1}{6}[\dot{B}_{abc}(t)+2A_{ba}(t)A_c]\epsilon^{abc} \ . \tag{2.21}$$

Equation (2.21) can be integrated immediately for class A models. This is not true for class B models, not even in the diagonal cases. It is easy to show that for diagonal Bianchi type IV and type $\mathrm{VII}_{h\neq 0}$ the second term in (2.21) is not a total derivative. I will hence concentrate on class A models. For them,

$$\dot{b}=\frac{C}{\sqrt{h}}+\frac{1}{6\sqrt{h}}B_{abc}(t)\epsilon^{abc} \ , \tag{2.22}$$

where C is an integration constant.

III. THE NO-HAIR CONJECTURE

Basic content of the no-hair conjecture has been mentioned in the Introduction. Roughly, it states the sufficient conditions under which an expanding universe continues to expand forever, with the Hubble parameter asymptotically approaching a constant. It has been proven for Bianchi models by Wald [4], and his result can be expressed as follows.

Theorem. All Bianchi models except Bianchi type IX dynamically described by Einstein's equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=T_{\mu\nu}-\Lambda g_{\mu\nu} \ , \tag{3.1}$$

which are initially expanding, and where the energy-momentum tensor satisfies the strong and dominant energy conditions, SEC and DEC, respectively.

$$(T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T)t^\mu t^\nu\geq 0 \ , \tag{3.2}$$

$$T_{\mu\nu}t^\mu t^\nu\geq 0 \ , \tag{3.3}$$

for all future directed timelike vectors t^μ, asymptotically approach isotropic expansion with $K=(3\Lambda)^{1/2}$ and appear to be matter free. Moreover, the conclusion holds true for the Bianchi type-IX model too if initially the cosmological constant is larger than the initial spatial curvature with $\beta=0$.

Wald obtained proof of the above theorem by inspecting the initial value constraint equation and the Raychaudhuri equation, which, with the metric (2.13) and with the help of (2.16), are

$$\tfrac{1}{3}K^2=\tfrac{1}{2}\sigma_{ab}\sigma^{ab}-\tfrac{1}{2}{}^{(3)}R+\Lambda+T_{00} \ , \tag{3.4}$$

$$\dot{K}+\tfrac{1}{3}K^2=\Lambda-\sigma_{ab}\sigma^{ab}-(T_{00}+\tfrac{1}{2}T) \ . \tag{3.5}$$

The SEC (3.2) and DEC (3.3) imply positivity of T_{00} and $T_{00}+\frac{1}{2}T$, and the 3-space curvature ${}^{(3)}R$ is negative for Bianchi types I–VIII. Hence the inequalities are derived:

$$\tfrac{1}{3}K^2\geq\Lambda \ , \tag{3.6}$$

$$\dot{K}+\tfrac{1}{3}K^2\leq\Lambda \ , \tag{3.7}$$

$$\sigma_{ab}\sigma^{ab}\leq\tfrac{1}{3}(K^2-3\Lambda) \ . \tag{3.8}$$

Simple examination of (3.6)–(3.8) leads to Wald's theorem. Note that the strong and dominant energy conditions were of crucial importance in the argument, as they led to the inequalities above. Formally, the inclusion of the LCS terms does not change the guise of the problem since the equations (2.10) closely resemble (3.1). Nevertheless, the LCS terms turn out to have quite disastrous effects on the no-hair conjecture. If we cast (2.10) in the form (3.1), we obtain the following expression for the energy-momentum tensor:

$$T_{\mu\nu}=12X_\mu bX_\nu b-6g_{\mu\nu}X_\lambda bX^\lambda b+A_{\mu\nu} \ , \tag{3.9}$$

$$A_{\mu\nu}=4\nabla_\sigma[\sqrt{g}\,\epsilon_{\mu\alpha\lambda\rho}(X^\rho b)R^\sigma{}_\nu{}^{\alpha\lambda}] \ . \tag{3.10}$$

It is implicitly assumed that the right-hand side of (3.10) is symmetrized in μ and ν. The tensor $A_{\mu\nu}$ is clearly traceless, which is a consequence of the first Bianchi identity for the curvature:

$$A^\mu{}_\mu=4\nabla_\sigma[\sqrt{g}\,\epsilon_{\mu\alpha\lambda\rho}(X^\rho b)R^{\sigma[\mu\alpha\lambda]}]\equiv 0 \ . \tag{3.11}$$

A quick look shows that this energy-momentum tensor does not always meet the SEC and DEC. Since b depends only on time,

$$T_{00}=6\dot{b}^2+A_{00} \ , \tag{3.12}$$

$$T_{00}+\tfrac{1}{2}T=12\dot{b}^2+A_{00} \ . \tag{3.13}$$

With the conventions layed out in the preceding section and after some straightforward but tedious algebra, A_{00} can be produced:

$$A_{00}=\frac{\dot{b}}{\sqrt{h}}(4M^{ab}\sigma_a^c\sigma_{bc}+\epsilon^{abc}\epsilon_{lmn}M^{lk}h_{ak}\sigma_b^m\sigma_c^n) \ . \tag{3.14}$$

In certain Bianchi models the SEC and DEC hold due to the properties of the structure constants. This is the case for general Bianchi type I and type V, as $M^{ab}=0$ there. Further scrutiny reveals that A_{00} vanishes for the diagonal Bianchi type-III and type-IV models, since for these M^{ab} is off-diagonal. Therefore, for these models Wald's proof is unaffected and the no-hair conjecture holds.

Still, (3.12) and (3.13) in general are not always positive. Rather, their sign is determined by the dynamics, and the axion equation of motion plays a significant role. This suggests that breakdown of the no-hair conjecture could occur [15]. A definite example featuring such behavior is constructed in the next section.

IV. THE BIANCHI TYPE-II MODEL

The simplest Bianchi model which entails breakdown of the cosmic no-hair conjecture incited by the LCS terms is the diagonal Bianchi type-II model with only one anisotropy function, $\beta_2=\beta_3=-\beta_1/2$. Its metric is [13]

$$ds^2=-dt^2+R^2\{\epsilon^{2\beta}(\epsilon^1)^2+\epsilon^{-\beta}[(\epsilon^2)^2+(\epsilon^3)^2]\} , \tag{4.1}$$

with

$$\begin{aligned}\epsilon^1&=dx^2-x^1dx^3 ,\\ \epsilon^2&=dx^3 ,\\ \epsilon^3&=dx^1 .\end{aligned} \tag{4.2}$$

The structure constants of the Bianchi type-II group are incorporated in the formulas above via the definitions of the nonholonomic basis one-forms.

Derivation of the equations of motion (2.10)–(2.11) is easy. Details can be found in the Appendix. The equations are

$$\dot K+K^2=3\Lambda+\frac{e^{4\beta}}{2R^2}+18\dot b\dot\beta^2\frac{e^{2\beta}}{R} , \tag{4.3}$$

$$\begin{aligned}\left[1+12\dot b\frac{e^{2\beta}}{R}-18\dot\beta^2\frac{e^{4\beta}}{R^2}\right]\ddot\beta+K\dot\beta&=-\frac{2}{3}\frac{e^{4\beta}}{R^2}-8\dot b\frac{e^{6\beta}}{R^3}-12\dot b\dot\beta^2\frac{e^{2\beta}}{R}+4K\dot b\dot\beta\frac{e^{2\beta}}{R}\\ &\quad-12\dot\beta^2\frac{e^{8\beta}}{R^4}+6K\dot\beta^3\frac{e^{4\beta}}{R^2}+18\dot\beta^4\frac{e^{4\beta}}{R^2} ,\end{aligned} \tag{4.4}$$

$$\frac{K^2}{3}=\Lambda+\frac{e^{4\beta}}{4R^2}+\frac{3}{4}\dot\beta^2+6\dot b^2+18\dot b\dot\beta^2\frac{e^{2\beta}}{R} , \tag{4.5}$$

$$\dot b=\frac{1}{R^3}\left[C+\frac{e^{6\beta}}{6}-\frac{3}{4}R^2\dot\beta^2\frac{e^{2\beta}}{R}\right] . \tag{4.6}$$

Equations (4.3) and (4.4) are the two independent spatial Einstein's equations, and Eq. (4.5) is the initial value constraint. Note that the axion equation (4.6) is first order. Indeed, (4.6) is the first integral of (2.11), as discussed at the end of Sec. II. The parameter C is the aforementioned constant of integration. Also note that the axion appears in the equations of motion only through $\dot b$, not b. This is exactly what one should expect since the theory is invariant under the global U(1) axial group, which shifts the value of the axion field by a constant. Furthermore, although the theory is apparently higher order in curvature by the presence of the LCS terms, the equations above still contain only first and second time derivatives. The higher-order curvature effects are manifested in appearance of the powers of derivatives greater than two. Before we embark on solving Eqs. (4.3)–(4.6), we digress a little and emphasize an extra symmetry that the metric (4.1)–(4.2) has. Our attention is placed upon the fact that a solution to the system of equations above, in principle, should involve five integration constants including C. However, the metric possesses an extra symmetry which eliminates one of them. Consider an anisotropic dilation of the spatial coordinates of the form

$$\begin{aligned}x^1&=\exp(\mu)x^{1'} ,\\ x^2&=\exp(2\mu)x^{2'} ,\\ x^3&=\exp(\mu)x^{3'} ,\end{aligned} \tag{4.7}$$

followed by a similar redefinition of the scale factor R, which could be interpreted as an isotropic dilation:

$$R=\exp(-\tfrac{4}{3}\mu)R' . \tag{4.8}$$

In terms of these new variables, the metric is formally the same as (4.1)–(4.2) with β replaced by

$$\beta'=\beta+\tfrac{2}{3}\mu . \tag{4.9}$$

The variables R' and β' represent a fully legitimate choice for the metric, as again $\mathrm{Tr}(\beta')=0$. Hence this indicates that one of the initial values of the two functions is physically irrelevant since it can be changed by the above transformation. Only the ratio $\eta=(e^{2\beta}/R)|_{t_0}$ is physically significant, since it is invariant under the transformation (4.7)–(4.9). If we define the scaled variables $r=R/R(t_0)$ and $\alpha=\beta-\beta(t_0)$, the parameter η plays the role of the effective coupling constant. Thus we can set $R(t_0)=1$ in conformity with the similar convention employed in the study of the Robertson-Walker models. Actually, this selection is the final step in gauge fixing the theory, since the freedom of choice of the initial value of R is a remnant of the general Gl(4) gauge invariance which has not been broken completely by picking the metric (4.1)–(4.2). Clearly, we could have also chosen to fix β_0 and leave R_0 arbitrary.

Having set forth all the necessary preliminaries, we can investigate Eqs. (4.3)–(4.6) at last. A class of exact solutions is easy to obtain. The form of the equations hints at an educated guess of trying

$$R=\frac{1}{\eta}e^{2\beta} , \tag{4.10}$$

where η is defined in the preceding paragraph with $C=0$. This ansatz makes the equations of motion independent of both R and β and further gives the identity

$$K=6\dot\beta . \tag{4.11}$$

It is not difficult to show that consistency of the system (4.3)–(4.6) requires $\dot\beta=\rho=\text{const}$, and then ρ and η are solutions of the algebraic equations below:

$$\begin{aligned}&6\rho^2\eta^4-27\rho^4\eta^2-72\rho^2+\eta^2+6\Lambda=0 ,\\ &36\rho^2\eta^4-243\rho^4\eta^2+4\eta^6-270\rho^2+6\eta^2+24\Lambda=0 .\end{aligned} \tag{4.12}$$

The solutions can be represented in the parametric form with the introduction of a positive parameter $p>\frac{2}{9}$:

$$\begin{aligned}\eta^2&=\left[\frac{18p+2}{15(3p+\frac{2}{15})^2-\frac{64}{15}}\right]^{1/2} ,\\ \rho^2&=p\left[\frac{18p+2}{15(3p+\frac{2}{15})^2-\frac{64}{15}}\right]^{1/2} ,\\ \Lambda&=\frac{3(126p^2-15p-\frac{2}{9})}{2(15p+2)}\eta^2 ,\end{aligned} \tag{4.13}$$

and they exist for all $\Lambda \geq 1.64144$. The axion equation reads

$$\dot{b} = \frac{\eta^3}{6} - \frac{3}{4}\rho^2\eta \ . \qquad (4.14)$$

Our interest in these solutions is self-explanatory, as they actually violate the no-hair conjecture. Namely, although the cosmological constant is positive, and the universe can be initially expanding, and at large times the volume grows exponentially, according to

$$\sqrt{g} = \exp(6\rho t) \ , \qquad (4.15)$$

it is not matter free. The key to such behavior are the identity (4.11) and the LCS terms, which provide for constant nonvanishing shear, whence the resulting axion hair.

The violation of the no-hair conjecture we found above indicates that even more severe departures from it can emerge. We can concoct the mechanism to account for possible departures. Look at the initial value constraint (4.5), and remember that in Wald's proof it led to the inequality (3.6), which prohibited the change of sign of K. Then note that in our model the LCS term could reverse the argument. It is linear in $\dot{b}$, and thus could become negative and large if the evolution drives $\dot{b}$ towards negative values, and simultaneously keeps $\dot{\beta}^2 e^{2\beta}/R$ approximately constant for a finite time. So it is contrievable that the LCS term might prevail over the positive terms in (4.5) and by continuity force K to change sign. Furthermore, the negative $\dot{K}$ at this moment can decrease K down to $|K| > \Lambda$ and diminish the LCS term. The converse of the no-hair theorem would then lead to recollapse, as the model would satisfy the same inequalities of Wald's proof, only with negative K.

Complexity of the equations of motion forces us to resort to numerical methods to investigate the possibility elaborated above. Indeed, we confirm the conclusion of our naive argument. To obtain a complete description of the solutions, we recall they are labeled by four independent parameters, Λ, C, β_0, and $\dot{\beta}_0$. The initial value of the Hubble parameter is determined from (4.5). We elect to fix β_0 and $\dot{\beta}_0$ and observe how solutions behave if we vary Λ and C. The results are presented in Figs. 1–4.

Figures 1 and 2 correspond to $\beta_0=0$, $\dot{\beta}_0=0$. We find the half-plane of positive Λ to be divided in two regions, one where the no-hair conjecture holds and the other where the universe recollapses (Fig. 1). A typical recollapsing solution is given in Fig. 2. The time when initial singularity is reached is $\Delta t \simeq t_P$, where t_P is the Planck time. Qualitatively the same picture is obtained for nonvanishing but small $\beta_0, \dot{\beta}_0$.

When initial values are large, the situation becomes more complicated. It turns out that all solutions violate the no-hair conjecture. Besides the solutions which feature recollapse, we find those which asymptotically tend to the exact solutions (4.13) instead of obeying the no-hair scenario (Fig. 3). This is no surprise since the solution (4.13) actually replaces the corresponding asymptotia $K=\text{const}$, $\beta=0$ of the Einstein's theory. The numerical investigation of these solutions displays that they are stable in a large region of the parameter plane.

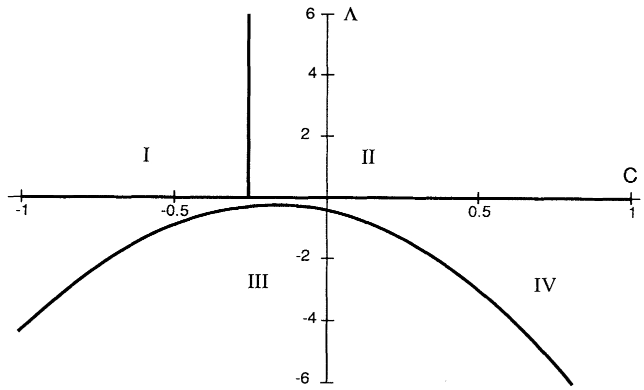

FIG. 1. $\Lambda - C$ parameter plane for the $\beta_0=\dot{\beta}_0=0$ case: for positive Λ, solutions either experience recollapse (region I) or feature the no-hair conjecture (region II). Region III is kinematically forbidden and region IV corresponds to negative Λ.

There is yet another type of solutions that seem to have both R and β singular at some finite time (Fig. 4). In the $\Lambda - C$ parameter plane these appear where we would expect behavior of the type of Fig. 3. They originate from the fact that $\ddot{\beta}$ not only controls the evolution of the anisotropy, but also of the axion, as can be seen form taking a derivative of (4.6). So, a positive large $\ddot{\beta}$ implies negative large $\dot{b}$, which in turn increases the value of $\dot{b}^2$ and further speeds up the increase of magnitude of $\ddot{\beta}$, leading to the behavior such as we observe. An effect of such type is a manifestation of the higher-order curvature terms in the equations. More detailed study of it seems to us to be physically unjustified, if we accept that the terms that produce it are really perturbative corrections in the superstring action.

V. CONCLUSION

I have discussed the validity of the cosmic no-hair conjecture in the superstring motivated theory of gravity and Kalb-Ramond axion field. In the definition of the axion field strength the Lorentz Chern-Simons form, whose ap-

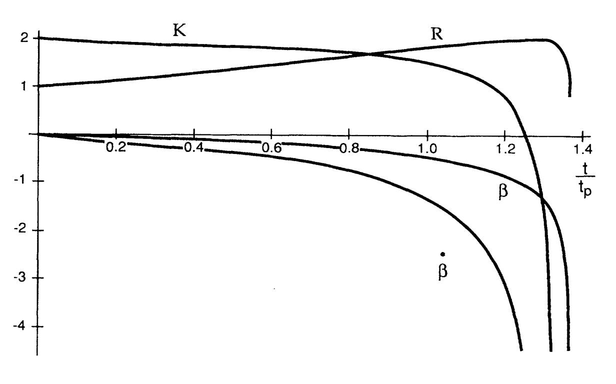

FIG. 2. A typical recollapsing solution: $\Lambda=1$, $C=-0.26$.

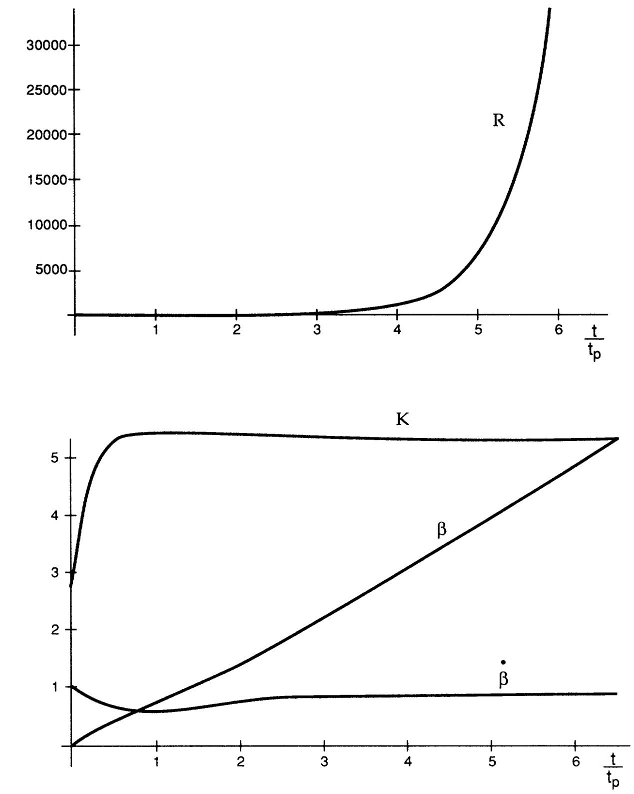

FIG. 3. A solution with axion hair: $\beta_0=0$, $\dot{\beta}_0=1$, $\Lambda=10$, $C=0$.

pearance is necessitated by the requirement of the anomaly cancellation in the string field theory, has been included. It produces extra contributions to the equations of motion which in general can dispute the cosmic no-hair conjecture. I have demonstrated their effects in a diagonal Bianchi type-II cosmology and have found they lead to the breakdown of the cosmic no-hair theorem for Bianchi models. This phenomenon is tractable back to the aforementioned extra contributions in the equations of

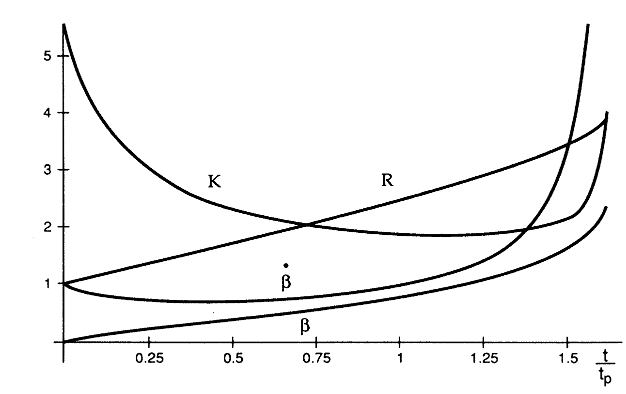

FIG. 4. Superexponential growth: $\beta_0=0$, $\dot{\beta}_0=1$, $\Lambda=1$, $C=1$.

motion, which formally can be looked upon as quantum corrections to the energy-momentum tensor arising from the gauge-dependent topological axion-gravity coupling, and do not satisfy the strong and dominant energy conditions. It has already been noted that such violations lead to instabilities of the asymptotic de Sitter phase and disprove the no-hair conjecture [15].

The failure of the cosmic no-hair theorem occurs in two different ways. One of them is the recollapse of the universe, which would be a more expected type in light of the previous results. We have also found another class of solutions which feature asymptotically anisotropic exponential growth of the proper volume and retain axion hair. The hair is carried by the nonvanishing anisotropy. Apparently, such solutions are specific to the theory with the LCS form coupled to the axion—the hair is produced by the geometric source $\mathrm{Tr}(R\wedge R)$ in the axion equation of motion [10]. On the basis of our result, we would expect a similar situation in other Bianchi models.

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APPENDIX

Here we present a short derivation of the equations of motion (4.3)–(4.6). We employ the tetrad method, which is the simplest for diagonal Bianchi models. For the metric (4.1)–(4.2) the tetrads are $e^0=dt$, $e^1=R\exp(\beta)\epsilon^1$, $e^2=R\exp(-\beta/2)\epsilon^2$, $e^3=R\exp(-\beta/2)\epsilon^3$. The spin connexion can be computed with the help of the metricity and vanishing torsion requirements. It is given by

$$\begin{aligned}
\omega_{10}&=\left[\dot{\beta}+\frac{K}{3}\right]e^1\,,\\
\omega_{20}&=\left[\frac{K}{3}-\frac{\dot{\beta}}{2}\right]e^2\,,\\
\omega_{30}&=\left[\frac{K}{3}-\frac{\dot{\beta}}{2}\right]e^3\,,\\
\omega_{12}&=-\frac{\exp(2\beta)}{2R}e^3\,,\\
\omega_{13}&-\frac{\exp(2\beta)}{2R}e^2\,,\\
\omega_{23}&=\frac{\exp(2\beta)}{2R}e^1\,,
\end{aligned}\tag{A1}$$

and we recall that the connexion coefficients are defined by $\omega^{\mu}{}_{\nu}=\Gamma^{\mu}_{\nu\lambda}e^{\lambda}$. From the connexion (A1) it is easy to compute the curvature forms. They are

$$R_{12}=-\frac{3}{4R}\exp(2\beta)\dot{\beta}e^0\wedge e^3+\left[\left[\dot{\beta}+\frac{K}{3}\right]\left[\frac{K}{3}-\frac{\dot{\beta}}{2}\right]+\frac{1}{4R}\exp(4\beta)\right]e^1\wedge e^2\,,$$

$$R_{13}=\frac{3}{4R}\exp(2\beta)\dot{\beta}e^0\wedge e^2 + \left[\left[\dot{\beta}+\frac{K}{3}\right]\left[\frac{K}{3}-\frac{\dot{\beta}}{2}\right]+\frac{1}{4R}\exp(4\beta)\right] e^1\wedge e^3 ,$$

$$R_{23}=\frac{3}{2R}\exp(2\beta)\dot{\beta}e^0\wedge e^1 + \left[\left[\frac{K}{3}-\frac{\dot{\beta}}{2}\right]^2-\frac{3}{4R}\exp(4\beta)\right] e^2\wedge e^3 , \quad \text{(A2)}$$

$$R_{10}=\left[\frac{\ddot{\beta}+\dot{K}}{3}+\left[\dot{\beta}+\frac{K}{32}\right]^2\right]e^0\wedge e^1 - \frac{3}{2R}\exp(2\beta)\dot{\beta}e^2\wedge e^3 ,$$

$$R_{20}=\left[\frac{\dot{K}}{3}-\frac{\ddot{\beta}}{2}+\left[\frac{K}{3}-\dot{\beta}\right]^2\right]e^0\wedge e^2 ,$$

$$R_{30}=\left[\frac{\dot{K}}{3}-\frac{\ddot{\beta}}{2}+\left[\frac{K}{3}-\dot{\beta}\right]^2\right]e^0\wedge e^3 .$$

Computation of the Ricci tensor from the formulas above is trivial; it gives the well-known results [4–6,12,13]. We will calculate the LCS energy momentum. If we define the tensor $\beta_{\mu\sigma\nu}$ by

$$B_{\mu\sigma\nu}=H_{\alpha\lambda(\mu}R_{\nu)\sigma}{}^{\alpha\lambda} , \quad \text{(A3)}$$

where brackets denote symmetrizing over the enclosed indices, the LCS energy momentum can be computed by formula (3.11) as its covariant derivative. Using the identities (A1) and (A2) we find the nonvanishing components of $B_{\mu\sigma\nu}$:

$$B_{231}=B_{132}=-B_{123}=-B_{321} =\dot{b}\left[\frac{3}{2}\left[\frac{K}{3}-\frac{\dot{\beta}}{2}\right]\dot{\beta}+\frac{e^{4\beta}}{R}\right] ,$$

$$B_{101}=-2B_{202}=-2B_{303}=-2B_{011} =4B_{022}=4B_{033}=3\dot{b}\dot{\beta}\frac{e^{2\beta}}{R} . \quad \text{(A4)}$$

Then we use (A1), (A4), and (3.11) to evaluate the LCS energy momentum:

$$A_{00}=18\dot{b}\dot{\beta}^2\frac{e^{2\beta}}{R} ,$$

$$A_{11}=-12(\dot{b}\dot{\beta})^{\cdot}\frac{e^{2\beta}}{R}-8K\dot{b}\dot{\beta}\frac{e^{2\beta}}{R}-6\dot{b}\dot{\beta}^2\frac{e^{2\beta}}{R}-8\dot{b}\frac{e^{6\beta}}{R^3} , \quad \text{(A5)}$$

$$A_{22}=A_{33}=6(\dot{b}\dot{\beta})^{\cdot}\frac{e^{2\beta}}{R}+4K\dot{b}\dot{\beta}\frac{e^{2\beta}}{R}+12\dot{b}\dot{\beta}^2\frac{e^{2\beta}}{R}+4\dot{b}\frac{e^{6\beta}}{R^3} .$$

To compute the Einstein's equations we rewrite them as

$$R_{\mu\nu}=T_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}T+\Lambda g_{\mu\nu} , \qquad R=4\Lambda-T . \quad \text{(A6)}$$

After a little algebra, these, (2.16) and (A5) yield

$$\frac{K^2}{3}=\Lambda+\frac{e^{4\beta}}{4R^2}+\frac{3}{4}\dot{\beta}^2+18\dot{b}\dot{\beta}^2\frac{e^{2\beta}}{R} ,$$

$$\dot{K}+K^2=3\Lambda+\frac{e^{4\beta}}{2R^2}+18\dot{b}\dot{\beta}^2\frac{e^{2\beta}}{R} , \quad \text{(A7)}$$

$$\ddot{\beta}+K\dot{\beta}=-\frac{2}{3}\frac{e^{4\beta}}{R^2}-12(\dot{b}\dot{\beta})^{\cdot}\frac{e^{2\beta}}{R} -12\dot{b}\dot{\beta}^2\frac{e^{2\beta}}{R}-8K\dot{b}\dot{\beta}\frac{e^{2\beta}}{R}-8\dot{b}\frac{e^{6\beta}}{R^3} .$$

At the end we obtain the axion equation (4.6) from (2.3), (2.22), and (A1). Since the LCS form is

$$\omega_L=6\left[\frac{e^{6\beta}}{6}-\frac{3}{4}\dot{\beta}^2R^2e^{2\beta}\right]dx^1\wedge dx^2\wedge dx^3 , \quad \text{(A8)}$$

Eq. (2.22) leads directly to (4.6). Then, with a little more manipulation, (A8) and (4.6) lend to deriving (4.3)–(4.5). Q.E.D.

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