PHYSICAL REVIEW D VOLUME 44, NUMBER 8 15 OCTOBER 1991

Chaos in mixmaster models

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(Received 31 October 1990; revised manuscript received 22 April 1991)

We apply the methods developed in our previous papers to investigate chaos in mixmaster cosmological models. We show that these models have an infinite relaxation time and zero Kolmogorov entropy near the singularity. Therefore they exhibit a power-law instability instead of an exponential one. From the point of view of ergodic theory these systems should be classified as the so-called K systems.

I. INTRODUCTION

There is a class of chaotic dynamical systems in general relativity which describes the dynamics of Bianchi type-IX or type-VIII homogeneous cosmological models. Their chaotic behavior near the singularity consists in the fact that the evolution of three scale factors which oscillate anisotropically is very sensitive to initial conditions.

One can distinguish at least four approaches to the problem of chaos in the mixmaster model. The first is associated with papers by Belinskii *et al.* [1]. In this approach the evolution of mixmaster models is approximated by a series of Kasner epochs.

The evolution consists in consecutive transitions from one Kasner epoch to another. During such a transition one of the oscillating scale factors starts decreasing and the decreasing one begins to oscillate.

In Belinskii's representation one is drawing conclusions about chaotic behavior not from the precise evolution of the model but from its approximation. Both oscillating scale factors will decrease or the number of oscillations during the given epoch will be random.

The second approach traces back to Misner's paper [1] where the Hamiltonian [Arnowitt-Deser-Misner (ADM)] description of mixmaster models was developed. In the ADM formalism the universe particle moves inside the potential well with expanding walls. This can also be regarded as a billiard ball on expanded walls. Hence, in Misner's representation chaos is a consequence of elastic collisions of the universe particle with walls of the potential well. A collision corresponds to the transition to a new Kasner epoch. The third line of argument follows Barrow's idea of describing the evolution of mixmaster models by certain difference equations [1]. These equations establish the dependence of parameters characterizing the system in two consecutive Kasner epochs.

In the fourth approach one approximates the dynamics of mixmaster models by separatrices of the corresponding dynamical systems. This method was developed by Bogoyavlenskii [2] and is often referred to as the maximally nondegenerate compactification method. Its key idea is to construct a compact manifold S with boundary Γ so that the corresponding dynamical system on S can be smoothly continued to Γ in such a way that it will have maximally nondegenerate critical points on Γ. Then the problem is to investigate the nondegenerate critical points (or sets of them) on S along with their separatrices. In mixmaster models the separatrices join the unstable critical points. These series of separatrices approximate the complex dynamical regimes of real physical trajectories. In some cases the critical points form continuous sets. Then the separatrices define a transformation from one set to another. In this way we obtain a discrete combinatorial model of chaos. The second mechanism of stochastization is associated with indeterminacy of transitions along separatrices in the case when there is a pencil of separatrices $X_1,\ldots,X_m$ at a certain critical point. This indeterminacy implies the sensitivity of trajectories with respect to initial conditions.

In our approach we represent mixmaster models as geodesic flows on a certain Riemannian manifold and then investigate their local instability [3–5]. First remarks about the stability of geodesic flows and its connection with curvature have appeared in Barrow's paper [1] in 1982. Further studies by Barrow and Chernoff [1] suggested that chaos may appear in mixmaster models due to existence of chaotic one-dimensional maps associated with them. Although these results are correct and mathematically rigorous there is a significant difference between the discrete dynamics studied in Ref. [1] and the full mixmaster dynamics.

We use the precise dynamical equations instead of their approximations, which is the main advantage of our approach. A criterion for local instability of geodesic flows is formulated by means of the averaged geodesic deviation equation. It implies that the negativity of the Ricci scalar is a sufficient condition for local instability [1,2]. As a consequence this method allows us to describe such properties of the model either near or far from the singularity that cannot be seen within the frame of other approaches. In particular we show that the Lyapunov exponents are zero near the singularity and so is the Kolmogorov entropy. Therefore the mixmaster models exhibit a power-law instability near the singularity instead of an exponential one. (It is obvious that the divergence rate of adjacent trajectories depends on the time gauge adopted. There is a freedom of time gauge in general relativity which makes the notion of Lyapunov exponents useless since they are not invariant under time reparametrization. The detailed discussion of this problem is presented in Ref. [6].) A class of such systems is known in mathematics [7]. The outcome of our investigations is confirmed by numerical studies of mixmaster models.

II. REDUCTION OF HAMILTONIAN SYSTEMS TO A GEODESIC FLOW

We shall consider the dynamical systems with the Lagrangian

$$L(q,\dot q)=\tfrac{1}{2}a_{ij}(q)\dot q^{i}\dot q^{j}-V(q) , \tag{1}$$

where $B(\xi,\xi)=\frac{1}{2}a_{ij}\xi^i\xi^j$ is a positive-definite quadratic form, q^i are the generalized coordinates, and $V(q)$ is the potential.

From (1) we obtain

$$p_i=\frac{\partial L}{\partial \dot q^{i}}=a_{ij}\dot q^{j} . \tag{2}$$

Hence, if we define a matrix $a^{ij}(q)$ so that $a^{ij}a_{jk}=\delta^i_k$, then

$$\dot q^{i}=a^{ik}p_k \tag{3}$$

and we can immediately write the Hamiltonian

$$H(p,q)=p_i\dot q^{i}-L(q,p)=\tfrac{1}{2}a^{ij}(q)p_ip_j+V(q) . \tag{4}$$

Hamilton's equations read

$$\dot p_i=-\frac{\partial H}{\partial q_i}=-\frac{1}{2}\left[\frac{\partial}{\partial q^i}a^{rs}(q)\right]p_rp_s-\frac{\partial}{\partial q^i}V(q) ,$$
$$\dot q^{i}=a^{ij}(q)p_j . \tag{5}$$

On the other hand we have $\partial H/\partial t=0$ along the trajectories. This implies that $H(p,q)=h=\text{const}$. Therefore the corresponding constraint is

$$H(p,q)=\tfrac{1}{2}a^{ij}(q)p_ip_j+V(q)=h$$

and (6)

$$L(q,p)=p_i\dot q^{i}-H(p,q)=p_i\dot q^{i}-h .$$

Thus the equations of motion can be obtained by extremizing the action

$$I=\int_{t_b}^{t_a}p_i\dot q^{i}dt \tag{7}$$

with the Hamiltonian constraint $\frac{1}{2}a_{ij}(q)\dot q^{i}\dot q^{j}+V(q)=h$. The constraint condition which reflects the conservation of energy can be satisfied in the following way. Namely, we have

$$a_{ij}\dot q^{i}\dot q^{j}=2[h-V(q)]=2W . \tag{8}$$

Let us introduce a new time parameter λ. Then one can rewrite the formula (8) in the form

$$a_{ij}(q(t(\lambda)))\frac{dq^idq^j}{d\lambda\, d\lambda}\left[\frac{d\lambda}{dt}\right]^2=2W(q(t(\lambda))) . \tag{9}$$

We define the λ parameter so that

$$a_{ij}(q(t(\lambda)))\frac{dq^idq^j}{d\lambda\, d\lambda}=1 . \tag{10}$$

Hence,

$$\left[\frac{d\lambda}{dt}\right]^2=2W(q(t(\lambda)))$$

or

$$dt=\frac{d\lambda}{\sqrt{2W(q(t(\lambda)))}} . \tag{11}$$

This choice of parameter λ assures the conservation of energy $H(p,q)=h$. The action (6) assumed now the form

$$I=\int_C\sqrt{2[h-V(q)]a_{ij}dq^idq^j} , \tag{12}$$

where C is a trajectory of the system. Let us now notice that the action (12) is independent of the parametrization of the curve C. Therefore the equations of motion are equivalent to extremizing the length of a curve in space with the metric

$$q_{ij}=2[h-V(q)]a_{ij}(q) . \tag{13}$$

If we introduce a new parameter s, so that

$$\frac{ds}{dt}=2W , \tag{14}$$

then the equations of motion assume the form

$$\frac{d^2q^i}{ds^2}+\Gamma^i_{jk}\frac{dq^j}{ds}\frac{dq^k}{ds}=0 . \tag{15}$$

The system moves along the geodesic and

$$\Gamma^i_{lm}(q)=-\frac{1}{2[h-V(q)]}(\partial_l V\delta^i_m+\partial_m V\delta^i_l-\partial_j Va^{ji}a_{lm})+\tfrac{1}{2}a^{ij}(\partial_l a_{jm}+\partial_m a_{jl}-\partial_j a_{lm}) . \tag{16}$$

If $a_{jm}=\delta_{jm}$, then $a^{ir}=\delta^{ir}$ and

$$\Gamma^i_{lm}(q)=-\frac{1}{2[h-V(q)]}(\partial_l V\delta^i_m+\partial_m V\delta^i_l-\partial_i V\delta_{lm}) . \tag{17}$$

Returning back to the Newtonian time t, Eq. (15) reads

$$\ddot q^{i}+\Gamma^i_{km}\dot q^{k}\dot q^{m}=\frac{1}{W}\dot q^{k}\partial_k W\dot q^{i} , \tag{18}$$

where a dot denotes differentiation with respect to t. It can be easily checked that Eq. (18) coincides with the Hamiltonian equations (5), i.e.,

$$\ddot q^{i}+\Gamma^i_{jk}\dot q^{j}\dot q^{k}=-a^{ir}\partial_r V , \tag{19}$$

where

$$\Gamma^m_{sk}(q)=\frac{1}{2}a^{mi}\left[\frac{\partial a_{ik}}{\partial q^s}+\frac{\partial a_{is}}{\partial q^k}-\frac{\partial a_{sk}}{\partial q^i}\right].$$

In this way the dynamics of a system described by the Lagrangian (1) is reduced to a problem of geodesics described by Eq. (15).

The motion of the system is restricted to the subspace Γ_h of phase space Γ:

$$\Gamma_h=((q,p):\tfrac{1}{2}a^{ij}(q)p_ip_j+V(q)=h) \tag{20}$$

or to a subspace Q of the tangent fiber bundle TM, where M is the configurational space of the system:

$$Q=((q,\dot q):a_{ij}(q)\dot q^i\dot q^j=2W) . \tag{21}$$

Therefore vector fields normalized in the sense of the metric

$$g_{ij}=2Wa_{ij} \tag{22}$$

belong to Q:

$$Q=((q,v_q):g(v_q,v_q)=1) .$$

Hence the problem is reduced to solving the geodesic equation in Riemannian space (M,g). The equations of motion read

$$\nabla_u u=0 , \tag{23}$$

where u is the vector tangent to a geodesic and ∇ denotes the covariant derivative. We should bear in mind that the new parameter s, such that $ds/dt=2W$, is now measured along the geodesic. The Ricci scalar for the space with metric (22) is

$$R=4W^{-3/2}(W^{1/2})_{,ij}g^{ij}-2W^{-2}(W^{1/2})_{,i}(W^{1/2})_{,j}g^{ij} . \tag{24}$$

The formula (24) will be useful later on.

III. MIXMASTER MODELS AS HAMILTONIAN SYSTEMS

Mixmaster models belong to the so-called class-A models of the Hamiltonian [4,8]:

$$H=T+V_G , \quad T=g^{ab}p_ap_b ,$$

$$g^{ab}=\begin{bmatrix}-q_1^2 & q_1q_2 & q_1q_3\\ q_2q_1 & -q_2^2 & q_2q_3\\ q_3q_1 & q_3q_2 & -q_3^2\end{bmatrix}, \tag{25}$$

$$V_G=-W=\frac{1}{4}\left[2\sum_{i<j}^{3}n_in_jq_iq_j-\sum_{i=1}^{3}n_i^2q_i^2\right],$$

where T and V_G are the kinetic and potential energies, respectively, $n_i=1$ for the Bianchi type-IX model and $n_1=n_2=-n_3=1$ for the Bianchi type-VIII model. q_i are the squared scale factors. In the case of vacuum models the Hamiltonian constraint is

$$H=0 . \tag{26}$$

The metric of a Riemannian space on which the mixmaster model generates a geodesic flow assumes the form

$$ds^2=-V_Gg_{ab}dq^adq^b=Wg_{ab}dq^a\,dq^b , \tag{27}$$

where

$$g_{ab}=\frac{1}{2q_1q_2q_3}\begin{bmatrix}0 & q_3 & q_2\\ q_3 & 0 & q_1\\ q_2 & q_1 & 0\end{bmatrix}.$$

The Hamiltonian (24) can be rewritten as

$$H=2\sum_{i<j}^{3}p_ip_j-\sum_{i=1}^{3}p_j^2+2\sum_{i<j}^{3}\exp(q_i+q_j)-\sum_{i=1}^{3}\exp(2q_i) . \tag{28}$$

One can notice that it is a particular case of the so-called disturbed Toda lattice [4]:

$$H=\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}p_ip_j+\sum_{k,l=1}^{n+1}b_{kl}\exp(\{\alpha_k,q\}+\{\alpha_lq\}) , \tag{29}$$

where $\alpha_1,\ldots,\alpha_{n+1}$ are vectors in n-dimensional space $\mathbb{R}^n$, $\alpha_k=(\alpha_{k_1},\ldots,\alpha_{k_n})$, q is the vector $(q_1,\ldots,q)_n$ in $\mathbb{R}^n$, and there are two scalar products $(x,y)=a_{ij}x_iy_j$ and $\{x,y\}=x_iy_i$.

One can show that systems described by the Hamiltonian (29) are the hydrodynamical-type systems of the form

$$\dot u^i=\Gamma^i_{jk}u^ju^k . \tag{30}$$

Such systems appear in approximating the hydrodynamical equations by Galerkin's method. According to Ref. [9] the hydrodynamical systems (30) with constant Γ^i_{jk} have an energy integral quadratic in u and the flux-conserving phase volume $\mathrm{div}\dot u^i=0$.

The above considerations give rise to three other approaches to the problem, which have been announced by Bogoyavlenskii [4]: namely, (i) mixmaster models as disturbed Toda lattices [10], (ii) mixmaster models as hydrodynamical-type systems, and (iii) mixmaster models as hydrodynamical-type systems.

Let us notice that insofar as the kinetic energy is positively definite in classical Hamiltonian systems it is indefinite for the system (27).

It is a specific effect arising from the relativistic nature of Bianchi cosmological models. The corresponding dynamical system with the Hamiltonian (27) after the transformation of time $t\rightarrow\tau$:

$$d\tau=(q_1q_2q_3)^{-1/2}dt$$

assumes the form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -[2p_i(p_jq_j + p_kq_k - p_iq_i) + \tfrac{1}{2}n_i(n_jq_j + n_kq_k - n_iq_i)] ,$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i} = 2q_i(p_jq_j + p_kq_k - p_iq_i) , \tag{31}$$

where $i,j,k = 1,2,3$ and the overdot denotes the differentiation with respect to τ.

IV. CHAOS IN MIXMASTER MODELS

Local instability of a geodesic flow on Riemannian space with metric (27) depends on the Ricci scalar [Misner and Chitre introduced a coordinate system in which the mixmaster system becomes the motion of a massless particle in a hyperbolic triangle in the Lobatchevskii unit disc with negative curvature. This unit disc approach has been discussed by Pullin [11]. However this approach is less convenient for analytical studies being equivalent to certain billiard problems. We emphasize that the criterion of negativity of the Ricci tensor is a nontrivial conclusion from our previous works cited above. It is not a mere statement that geodesics on a space with negative curvature diverges (this is true only for spaces with constant curvature)]

$$R = -\frac{3}{4}W^{-3}\left[\frac{1}{4}W\sum_{i=1}^{3} n_i^2q_i^2 + \sum_{i<j}^{3} n_i^2n_j^2q_i^2q_j^2 - \frac{1}{2}\sum_{i=1}^{3} n_i^4q_i^4\right] . \tag{32}$$

In the case of the Bianchi type-IX model ($n_i = n_2 = n_3 = 1$) formula (32) reads

$$R = -\frac{3}{2}W^{-3}\left[\frac{1}{4}\sum_{i=1}^{3} q_i^4 + \sum_{i<j}^{3} q_i^2q_j^2 - \sum_{(i,j,k)} q_i^2(q_j+q_k)^2\right] , \tag{33}$$

$(i,j,k) = \{(1,2,3),(2,3,1).(3,1,2)\}$. From formula (33) we immediately see that the following three conclusions are valid.

(1) When $W = \text{const}$, we have $R = 0$. This corresponds to the Bianchi type-I model. The fact that the Ricci scalar is equal to zero reflects the integrability of this model. More generally, if the model admits the constant potential near the singularity, then its dynamics is always regular. In terms of Lyapunov exponents $R = 0$ implies that the Lyapunov exponent is also zero.

(2) In Belinskii's first approximation when $q_i \gg q_2, q_3$, R is negative, i.e., the system has the property of local instability. In terms of Lyapunov exponents it implies the existence of a positive exponent.

(3) In Belinskii's second approximation when $q_i \simeq q_2 \gg q_3$, R is also negative generating in this way the local instability. There is a positive Lyapunov exponent in this case, too.

In general it can be shown that the negativity of the potential V_G determines the local instability of a geodesic flow. Then the condition that $V_G < 0$ is equivalent to $P_i^i < 0$; i.e., it means that the Ricci scalar on hypersurfaces of constant time is negative. In other words the local instability is associated with the averaged curvature of hypersurfaces of constant time rather than with that of a spacetime as a whole. On the other hand we have shown in Ref. [2] that the negativity of the Ricci scalar on compact three-dimensional spaces is sufficient for creating chaos in geodesics. It illustrates a strong connection between the chaotic behavior of scale factors and the behavior of geodesics in mixmaster models. It is an interesting feature because chaos in geodesics representing histories of real observers has a clear physical meaning in contrast to the chaotic behavior of scale factors.

As discussed in Ref. [11], the Ricci scalar R allows us to determine the mean time scale for mixing:

$$\tau = \left|-\frac{R}{3}\right|^{-1/2} . \tag{34}$$

According to the idea developed in Ref. [12] τ is a natural time scale determined by the chaotic dynamics. The minimal value of the Ricci scalar is reached at $R \to 0$ which implies the infinite relaxation time $\tau = \infty$. The Lyapunov exponent is equal to

$$\lambda = \sqrt{2k} , \tag{35}$$

where $k = \min|-R/3|$. Therefore $\lambda \to 0$ as $R \to 0$. This Lyapunov exponent is determined from the averaged geodesic deviation equation in a Fermi basis, which in our case describes a harmonic oscillator with frequency $\omega = (-R/3)^{1/2}$. The mean time scale for mixing is inversely proportional to this frequency.

It can be seen from Eqs. (32) and (33) that the Ricci scalar is zero for the asymptotically axially symmetric solutions $q_1 = q_2$, $q_3 = 0$. It is associated with the integrability of the system in this case. On the other hand for $q_1 - q_2 = \epsilon \ll 1$, $q_3 = 0$ we are in the neighborhood of axially symmetric solutions known as the so-called Taub solutions, and then $R < 0$. It means that the irregular chaotic behavior is concentrated around unstable critical points (saddles) $q_i = q_j$, $q_k = 0$, $p_i = \text{const}$. It is also concentrated around separatrices which join these points. However, we are not able to show it explicitly because we do not know the exact solutions corresponding to separatrices.

The mechanism creating chaos around separatrices has been demonstrated for some simple dynamical systems [7]. It turns out that the phase space is organized by separatrices around which irregular motion concentrates. Thus however small and weak the perturbation of the system is it always significantly modifies the system's behavior at the vicinity of separatrices.

The divergence rate for neighbor trajectories is given by the relation

$$\frac{D(\tau)}{D(0)} \propto \exp\left[\sqrt{-R/3}\int_0^\tau W\,d\tau\right] , \tag{36}$$

where $d\tau = dt/V$, $V = \sqrt{q_1q_2q_3}$ is the volume of hypersurface of constant time. When we are in Kasner epoch, when $W \to$const, $d\tau = dt/t$, and thus

$$\frac{D(t)}{D(0)} \propto (t-t_0)^{\text{const} \times \sqrt{-R/3}} , \tag{37}$$

where const is positive. The detailed discussion of Eq. (37) is possible if we know how the Ricci scalar behaves, i.e., if we know the behavior of scale factors q_i. Therefore we shall examine what it tells us about mixmaster models in regions where the dynamics is qualitatively known. It can be explicitly shown from formula (37) that if we approximate the evolution of the mixmaster model by a series of Kasner epochs, then the system is characterized by a power law t^α, $\alpha > 0$ divergence rate of adjacent trajectories instead of an exponential one. It implies that going backward in time toward the initial singularity $D(t)$ tends to zero. In terms of Lyapunov exponents it means that they also tend to zero near the singularity. Of course the result about the rate of divergence of nearby trajectories depends on a time gauge adopted. In particular a power-law instability in time t corresponds to a hyperexponential one in time τ. A more detailed discussion of this problem is presented in a separate paper [6]. Several valuable comments may be found in Ref. [11]. However it turns out that hyperexponents vanish near the singularity and so do the Lyapunov exponents.

We shall now give some other theoretical arguments in favor of this. These arguments have been confirmed numerically in Ref. [13]. The first argument arises from the observation that for a harmonic oscillator with the total energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2$ we can define the probability density $p(x)$ for finding the particle within the interval $x_{\min} \le x \le x_{\max}$ ($x_{\min} \neq x_{\max}$ are turning points) in the following manner:

$$p(x) = \frac{C}{v(x)} ,$$

where C is a normalizing constant and v is the velocity. After simple calculations we obtain

$$p(x) = \frac{\pi/\omega}{\left[\dfrac{2E}{m} - \omega^2 x^3\right]^{1/2}} . \tag{38}$$

As could be expected this density is maximal near the turning points where the velocity is zero. It means that the particle can be localized with the greatest probability in the neighborhood of turning points. An analogous consideration of mixmaster models yields the corresponding density function is concentrated in regions where the potential is zero. In this case the Ricci scalar is zero and so is the Lyapunov exponent. Therefore the high concentration of chaotic motions around separatrices is associated with vanishing Lyapunov exponents near the singularity.

The second argument is based on some general reasons. It is valid for Bianchi type-IX models only. A substantial difference between the evolution of Bianchi type-IX and Bianchi type-VIII models lies in thier long-term behavior. Bianchi type-IX models represent a closed universe, while Bianchi type-VIII models are open. Because both models exhibit chaos near the initial singularity, the problem lies in the nature of their final states. In the case of a Bianchi type-IX model we can use a time-reversal argument [14]. Namely, the vacuum Einstein equations are invariant under time reversal. Since the vacuum Bianchi type-IX model begins and ends in a singularity, it would seem that the approaches to these singularities are similar. This is not really clear in the Bianchi type-VIII case. The initial singularity of the Bianchi type-IX model is reached at $\tau = -\infty$ whereas the final one at $\tau = +\infty$. The Lyapunov exponents in these two states are

$$\lambda(-\infty) = \lim_{\tau \to -\infty} \frac{\sqrt{-R/3}}{\tau} \int_0^\tau W\, d\tau ,$$

$$\lambda(+\infty) = \lim_{\tau \to +\infty} \frac{\sqrt{-R/3}}{\tau} \int_0^\tau W\, d\tau .$$

This implies that $\lambda(-\infty) = -\lambda(+\infty)$, but from the time-reversal argument we have $\lambda(-\infty) = \lambda(+\infty)$ and thus it must be $\lambda(-\infty) = \lambda(+\infty) = 0$. It can also be seen from formula (37) that the initial ($t \to 0$) and final ($t \to t_0$) singularities may be indistinguishable only if $R \to 0$.

The Kolmogorov entropy is the measure of loss of information about initial conditions. For typical chaotic systems the following quantities are of the same order: $\tau_c^{-1} \sim \lambda \sim h$ where τ_c denotes the correlation splitting time, λ is the Lyapunov exponent, and h is the Kolmogorov entropy. This relation is true for systems which exhibit an exponential instability. However, the correlation splitting can occur not exponentially but according to a power-law formula. In such systems the relaxation time is finite but despite this they still have the mixing property.

On the other hand we can measure the stochasticity of the system by ratio of the configurational space volume for which $R < 0$ to the volume for which $V(q) < 0$. This ratio tends to zero while moving toward the singularity which can be interpreted as the vanishing entropy near the singularity.

Concluding our considerations we can state that the mixmaster models are not the typical examples of chaotic Hamiltonian systems. Their Kolmogorov entropy and Lyapunov exponents vanish near the singularity and at the same time the mean time scale for mixing is infinite. In case of typical chaotic systems all these quantities are finite. Nevertheless such exotic dynamical systems are known in mathematics [7] and they should be regarded as the so called K systems.

ACKNOWLEDGMENTS

This work was partly supported by the Polish Interdisciplinary Research Project No. CPBP 01.03.

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