Chaotic inflation driven by a minimally coupled scalar field in Brans-Dicke models

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We consider inflation driven by the energy density of a conventional scalar field minimally coupled to gravity, which has an initial vacuum expectation value which is large compared with the value at the minimum of its potential ("chaotic inflation"), in the context of Brans-Dicke-type models with a time-dependent Newton's constant. The equations of motion for the scalar field driving inflation and the Brans-Dicke scalar are solved in the slow-rolling approximation for the case of a $\lambda\sigma^4$ potential driving inflation, and the magnitude and spectrum of density perturbations produced during inflation are calculated. Sufficient inflation to account for the observed homogeneity and isotropy of the Universe is found to occur only if the coupling ϵ between the Brans-Dicke scalar and the Ricci scalar is smaller than 1. It is shown that only in the case where the initial value of the scalar field driving inflation is large compared with the initial value of the Brans-Dicke scalar can ϵ be larger than 10^{-3}, and that in this case the self-coupling λ is constrained to be much smaller than in the case of conventional chaotic inflation models. It is shown also that the spectrum of density perturbations is sufficiently flat to explain galaxy formation only if $\epsilon < 0.04$.

I. INTRODUCTION

There has recently been some interest in models of inflation [1] in which the value of Newton's constant is determined by the value of a scalar field which is evolving in the early Universe according to its equation of motion in a homogeneous and isotropic background [2]. (We refer to these generically as Brans-Dicke models [3].) In particular, it has been suggested that the "graceful exit" problem [1] of inflationary models in which inflation is driven by a scalar field which undergoes a first-order phase transition can be solved by having a time-dependent Newton's constant. An alternative approach to inflation was suggested by Linde ("chaotic inflation") [4], in which a scalar field rolls down a potential from some initial large value and drives inflation by its potential energy. It is the purpose of the present paper to consider this method of driving inflation in the context of models with a time-dependent Newton's constant. In particular, we are interested in the case where the scalar field driving the inflation corresponds to a matter scalar which is minimally coupled to gravity, rather than the Brans-Dicke scalar itself. Our motivation for considering this class of inflationary models comes from a number of observations.

(i) The case of inflation driven by a potential for the Brans-Dicke scalar itself has been extensively studied previously [5–7]. It is of interest to consider the case of inflation driven by the potential of a conventional matter scalar in order to compare with the case of inflation driven purely by the Brans-Dicke scalar.

(ii) It has been pointed out that the case of inflation driven by a potential for the Brans-Dicke sclar may have a problem with reheating [5]. This is true in the simplest models in which there is a conformally invariant action up to terms in the potential of the Brans-Dicke scalar. In this case, the Brans-Dicke scalar decouples from the light matter fields when a conformal transformation to the frame with fixed Newton's constant is made [5], preventing the Brans-Dicke scalar from decaying to quarks and leptons and so reheating the Universe. Thus, it is of interest to consider inflation driven by a matter scalar which will not decouple from light matter fields and so, in principle, may decay to these and reheat the Universe.

This paper is organized as follows. In Sec. II, we discuss the equations of motion and their approximate solution. We consider in detail the example of a quartic self-coupling potential for the scalar driving inflation. In Sec. III, we discuss the duration of inflation and the nature of the density perturbations produced. The magnitude of density perturbations is shown to depend strongly on the value of the Brans-Dicke parameter and on whether the initial value of the field driving inflation is larger or smaller than the initial value of the field determining Newton's constant. We give our conclusions in Sec. IV.

II. APPROXIMATE SOLUTION OF THE EQUATIONS OF MOTION

In the following we consider models described by the action

$$S=\int d^4x\sqrt{-g}\,[-\tfrac{1}{2}\epsilon\phi^2R-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-V(\phi)-\tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma-V(\sigma)]\ . \quad (1)$$

(Our conventions follow those of Ref. [8].) The equations of motion are then

$$H^2\left[1+2H^{-1}\frac{\dot\phi}{\phi}\right]=\frac{1}{3\epsilon\phi^2}[V(\sigma)+V(\phi)+\dot\sigma^2/2+\dot\phi^2/2]-\frac{k}{a^2}\ , \tag{2a}$$

$$\ddot\phi+3H\dot\phi+\dot\phi^2/\phi+\frac{1}{1+6\epsilon}\{\dot\sigma^2/\phi+V'(\phi)-4[V(\phi)+V(\sigma)]/\phi\}=0\ , \tag{2b}$$

$$\ddot\sigma+3H\dot\sigma+V'(\sigma)=0\ , \tag{2c}$$

where a is the scale factor. In the following we take $k=0$. We consider the solution of these equations under the following three assumptions (slow-rolling approximations) which will be seen to hold consistently throughout chaotic inflation: (I) $\ddot\phi \ll 3H\dot\phi$; (II) $V(\sigma)\gg\dot\sigma^2/2$, $V(\phi)$, $\phi V'(\phi)/4$; (III) $\ddot\sigma \ll 3H\dot\sigma$. Equations (2) become

$$H^2[1+2H^{-1}(\dot\phi/\phi)]=\frac{V(\sigma)}{3\epsilon\phi^2}+\frac{1}{6\epsilon}\left[\frac{\dot\phi}{\phi}\right]^2\ , \tag{3}$$

$$3H\dot\phi+\dot\phi^2/\phi=\frac{4V(\sigma)}{(1+6\epsilon)\phi}\ , \tag{4}$$

$$\dot\sigma=-\frac{V'(\sigma)}{3H}\ . \tag{5}$$

Note that no assumption of $\dot\phi/\phi$ being small compared with H or of $\dot\phi^2/2$ being small compared with $V(\sigma)$ has been made. Equations (3) and (4) can be consistently solved by assuming

$$H=\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\frac{\dot\phi}{\phi}\ . \tag{6}$$

This gives

$$\dot\phi=B\phi_0\ , \tag{7}$$

where

$$B=\left[\frac{32\epsilon V(\sigma)}{(1+6\epsilon)(20\epsilon+6)\phi_0^2}\right]^{1/2}\ . \tag{8}$$

(A subscript zero denotes the value at $t=0$.) Thus,

$$\phi=\phi_0\left[1+\int_0^t Bdt\right]\ , \tag{9}$$

$$H=\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\frac{B}{1+\int_0^t Bdt}\ . \tag{10}$$

These are similar to the expressions considered in discussions of the extended inflation models in which σ and B are constants [2]. Although it is not necessary to consider $\dot\phi/\phi$ small compared with H or $\dot\phi^2/2$ small compared with $V(\sigma)$, it will be convenient to have the simple expression for H which follows if we do make these assumptions (which we refer to as assumption IV):

$$H=\left[\frac{V(\sigma)}{3\epsilon\phi^2}\right]^{1/2}\ . \tag{11}$$

The correct value of H from (3) will differ from this by only a small factor if $\epsilon\gg1$ [H from (3) is about $\frac{1}{2}$ the value in (11)] and the two values become equal for ϵ smaller than 1. Thus, the results derived using (11) will generally differ only slightly from the exact results. We emphasize that the results obtained using the above equations may be applied to the case with $\epsilon>1$. [Note that this model has also been studied in Ref. [9], although in Ref. [9] the equation corresponding to (4) omits the $(1+6\epsilon)$ factor and so is strictly valid only for $\epsilon<1$. We shall compare the results of the present paper and those of Ref. [9] in our conclusions.]

In order to discuss the physical implications of inflation driven by $V(\sigma)$ we need to solve (5) for the σ field evolution. Using (10) we write (5) as

$$\dot\sigma=\frac{-V'(\sigma)\left[1+\int_0^t Bdt\right]}{3(\frac{1}{2}+1/4\epsilon)B}\ . \tag{12}$$

We consider potentials of the form

$$V(\sigma)=\lambda\sigma^n\ ,\quad n=2,4,\ldots\ . \tag{13}$$

On differentiating both sides of (12) with respect to t and using (13), one obtains

$$\ddot\sigma-(n/2-1)\dot\sigma^2/\sigma=\frac{-\lambda n\sigma^{(n-1)}}{3(\frac{1}{2}+1/4\epsilon)}\ . \tag{14}$$

The behavior of the solution for σ can best be seen by changing variables as follows:

$$\sigma=\rho^{-\gamma}\ ,\quad \gamma=\frac{1}{n-2}\ . \tag{15}$$

Then (14) becomes

$$\ddot\rho-\frac{n\dot\rho^2}{2(n-2)\rho}=(n-2)K\ , \tag{16}$$

where

$$K=\frac{n\lambda}{3(\frac{1}{2}+1/4\epsilon)}\ . \tag{17}$$

Note that $\rho>0$. Since the right-hand side of (16) is independent of ρ, it is easy to see how ρ evolves. There are two possible types of behavior.

(i) At early times the $\dot\rho^2/\rho$ term in (16) is small compared with $(n-2)K$. The ρ and $\dot\rho$ will then increase until the $\dot\rho^2/\rho$ term becomes larger than the $(n-2)K$ term. Thereafter, the evolution is described by (16) with K effectively zero.

(ii) It is possible that the $\dot\rho^2/\rho$ term dominates the K term from the beginning of inflation ($t=0$). In this case the K term in (16) may be neglected.

In the following we will consider the solution of (16) for the case $n=4$ (we refer to this as $\lambda\sigma^4$ chaotic inflation). In this case (16) becomes

$$\ddot{\rho}-\frac{\dot{\rho}^2}{\rho}=2K \ . \tag{18}$$

We consider solving (18) for values of t where the $2K$ term is dominant and for values of t where the $\dot{\rho}^2/\rho$ term is dominant. Thus, we solve

$$\ddot{\rho}=\frac{\dot{\rho}^2}{\rho} \tag{19b}$$

and

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We will match these solutions at the time when the $\dot{\rho}^2/\rho$ and $2K$ terms in (18) are of a similar magnitude. Although this solution gives a very crude approximation to the numerical value of $\rho(t)$, it will be seen to allow us to easily calculate analytically the main quantities of physicial interest. The solution of (19a) gives

$$\rho=\left[\frac{1}{\sigma_0^2}+\frac{8\lambda t}{3H_0}+Kt^2\right] , \tag{20a}$$

$$\sigma=\frac{\sigma_0}{[1+8\lambda\sigma_0^2 t/3H_0+K\sigma_0^2 t^2]^{1/2}} \ . \tag{20b}$$

The coefficients are determined by the initial conditions at $t=0$:

$$\sigma(t=0)=\sigma_0 \ , \tag{21a}$$

$$\dot{\sigma}(t=0)=-\frac{4\lambda\sigma_0^3}{3H_0} \ , \tag{21b}$$

where the latter follows from (5). The solution of (19b) gives

$$\rho=c_1 e^{c_2(t-t_a)} \ , \tag{22}$$

where c_1 and c_2 are constants. We match (22) to (20a) at $t=t_a$ where

$$t_a=\left[\frac{2}{K\sigma_0^2}\right]^{1/2} . \tag{23}$$

At t_a (20) gives $\dot{\rho}^2/\rho=8K/3$. Matching at t_a then gives the solution

$$\rho=\frac{3}{\sigma_0^2}\exp[\tfrac{4}{3}\alpha t_a^{-1}(t-t_a)] \ , \tag{24a}$$

$$\sigma=\frac{\sigma_0}{\sqrt{3}}\exp[-\tfrac{2}{3}\alpha t_a^{-1}(t-t_a)] \ . \tag{24b}$$

Because the matching at t_a determines the exponent in (24), any inaccuracy in the approximation will result in a large inaccuracy in the value of σ when $t>t_a$. We have included a factor α in (24), which is generally of order 1, to account for this. (We find from numerical solution that, typically, $\alpha\approx 1-3$.) It will be seen that quantities of physical interest are not significantly affected by α. This solution is of interest when initially the $2K$ term in (18) is large compared to the $\dot{\rho}^2/\rho$ term. If this is not the case, then one must consider (22) as applying from the outset. Using the initial conditions (21), this gives

$$\sigma=\sigma_0 e^{-4(\lambda\epsilon/3)^{1/2}\phi_0 t} \ . \tag{25}$$

The condition for (24) to apply is $\dot{\rho}^2/\rho<2K$ at $t=0$. Using (21) this gives the condition

$$\sigma_o^2>8\epsilon\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\phi_0^2 \ . \tag{26}$$

Thus, solution (24) applies if $\sigma_0\gtrsim\phi_0$ when $\epsilon\lesssim 1$ ($\sigma_0\gtrsim\epsilon^{1/2}\phi_0$ when $\epsilon\gtrsim 1$). Otherwise (25) applies. Since it will be seen that $\epsilon<1$ is necessary in order to have sufficient inflation, it is interesting that the nature of the solution for $\sigma(t)$ is determined just by whether or not σ is greater than ϕ initially.

III. INFLATION AND DENSITY PERTURBATIONS

We next consider the conditions for sufficient inflation to explain the homogeneity and isotropy of the observed Universe, and the nature of the density perturbations and gravitational-wave perturbations of the cosmic microwave background.

A. Inflation

The number N_e of e-foldings of inflation is given by

$$N_e\equiv\int_0^t H\,dt=\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\ln\left[1+\int_0^t B dt\right] . \tag{27}$$

We consider the two cases where (i) $\sigma_0\gtrsim\epsilon^{1/2}\phi_0$ and (ii) $\sigma_0\lesssim\epsilon^{1/2}\phi_0$ (with ϵ in these replaced by 1 when $\epsilon<1$).

(i) $\sigma_0>\epsilon^{1/2}\phi_0$. ($\sigma_0>\phi_0$ if $\epsilon<1$.) In this case we split the integral in (27) into that from 0 to t_a and that from t_a to t. Between 0 and t_a $\sigma\approx\sigma_0$, so

$$\int_0^{t_a} B\,dt\approx B_0 t_a \ . \tag{28}$$

From t_a to t one has

$$\int_{t_a}^{t} B\,dt=\frac{B_0 t_a}{4\alpha}(1-e^{-(4/3)\alpha t_a^{-1}(t-t_a)}) \ . \tag{29}$$

Thus, once $(t-t_a)$ is larger than t_a/α, the contribution to the total number of e-foldings is small and the total number of e-foldings is N_{eT}, where

$$N_{eT}\approx\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\ln(1+B_0 t_a) \tag{30}$$

and

$$B_0 t_a=\frac{2\sigma_0}{(4\epsilon+2)^{1/2}\phi_0} \ . \tag{31}$$

For σ_0 large compared with $\epsilon^{1/2}\phi_0$, this will be large compared with 1. From (30) we see that ϵ less than 1 will be necessary in order to have the minimum $\sim(60)$ e-foldings needed to account for the homogeneity of the observed Universe without requiring an extremely large value for the ratio σ_0 to ϕ_0 (the need for ~ 60 e-foldings is discussed further in Appendix A).

We also note that although most of the e-foldings occur for $t<t_a$, if the total number of e-foldings was sufficiently large, it would be possible for the final 60 or so e-foldings to occur at $t \gtrsim t_a$, when $\sigma(t)$ is evolving exponentially according to (24). This is important in the discussion of density perturbations given later. For $t>t_a$, the time at which the final ΔN_e e-foldings begin is obtained from

$$\Delta N_e \approx \frac{1}{\alpha}\left[\frac{1}{2}+\frac{1}{4\epsilon}\right] e^{-(4/3)\alpha t_a^{-1}(t-t_a)} , \tag{32}$$

where the exponential is taken to be small compared with 1. Thus, with $\Delta N_e \approx 60$, we find that the time at which the final 60 e-foldings begin (t_{60}) is

$$t_{60}-t_a \approx \frac{3t_a}{4\alpha}\ln\left[\frac{1}{960\epsilon\alpha}\right] . \tag{33}$$

Thus, if $\epsilon \lesssim 10^{-3}$, then $t_{60}>t_a$, when σ is decreasing exponentially with t. If $\epsilon \gtrsim 10^{-3}$ then the final 60 e-foldings begin at $t_{60} \lesssim t_a$, when σ is approximately constant. (As discussed in Appendix A, t_{60} is approximately the time at which perturbations responsible for galactic and large-scale structure formation cross the horizon. In the following we will use t_{60} as an estimate of the horizon crossing time for all scales between the scale of galaxies and the scale of the observed Universe).

The inflation will end when the slow-rolling approximations I–III break down. It is straightforward to show that they all fail at approximately the same time, given by

$$t-t_a \approx \frac{3t_a}{4\alpha}\ln\left[\frac{1}{\epsilon}\right] . \tag{34}$$

(Approximation IV also holds up to this time.) At this time, only about 1 e-folding of inflation remains, thus the results for ΔN_e and t_{60} given above are consistent with the slow-rolling approximations.

(ii) $\sigma_0 < \epsilon^{1/2}\phi_0$. ($\sigma_0<\phi_0$ if $\epsilon<1$.) In this case, the solution (25) gives, for the integral in (27),

$$\int_0^t B\,dt = \frac{B_0}{8\phi_o}\left[\frac{3}{\lambda\epsilon}\right]^{1/2}\left(1-e^{-8(\lambda\epsilon/3)^{1/2}\phi_0 t}\right) . \tag{35}$$

This is small compared with 1. Thus, the total number of e-foldings in this case is

$$N_{eT}=\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\left[\frac{3}{2(1+6\epsilon)(20\epsilon+6)}\right]^{1/2}\left[\frac{\sigma_0}{\phi_0}\right]^2 . \tag{36}$$

In order to have $N_{eT} \gtrsim 60$, one requires

$$\epsilon \lesssim \frac{1}{480}\left[\frac{\sigma_0}{\phi_0}\right]^2 \lesssim 10^{-3} . \tag{37}$$

The final ΔN_e e-foldings begin at a time obtained from

$$\Delta N_e = \left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\frac{B_0}{8\phi_0}\left[\frac{3}{\lambda\epsilon}\right]^{1/2} e^{-8(\lambda\epsilon/3)^{1/2}\phi_0 t} \tag{38}$$

which holds if the exponential is small compared with 1, i.e., $T>\theta$, where $\theta=(1/8\phi_0)(3/\lambda\epsilon)^{1/2}$. If $t<\theta$, the remaining ΔN_e e-foldings at time t are given by

$$\Delta N_e=(1-t/\theta)N_{eT} . \tag{39}$$

Note that Δ_e will differ from N_{eT} by only a small factor if $t<\theta$, and so $\Delta N_e \approx 60$ would require N_{eT} to be only slightly greater than 60 in this case. Using (36), this constrains ϵ to be

$$\epsilon \approx \frac{1}{8\Delta N_e}\left[\frac{\sigma_0}{\phi_0}\right]^2 , \quad \Delta N_e \approx 60 . \tag{40}$$

From these results we see that only if $\sigma_0>\phi_0$ is $\epsilon \gtrsim 10^{-3}$ possible. The approximations I–III (and IV) all hold until the final e-folding of inflation; thus, the expressions derived are consistent with the slow-rolling approximation.

We see that, in the case of chaotic inflation driven by a matter scalar, it is necessary for ϵ to be less than 1 in order to have a significant number of e-foldings of inflation. This may be contrasted with the case of chaotic inflation driven by a potential energy for the Brans-Dicke scalar itself[5–7]. In this case, if one considers $\epsilon \gg 1$, then inflation can be successfully implemented. Moreover, if ϵ is sufficiently large, the problem of fine-tuning of couplings required to evade large energy density perturbations is overcome in these models.[5,6] Thus, provided there is no problem with reheating the pure Brans-Dicke chaotic inflation models, they may provide an attractive framework for inflation. The case of chaotic inflation driven by a matter scalar will be of particular interest if reheating is a problem for inflation driven by the potential energy of the Brans-Dicke scalar itself.

So far we have considered the possibility that ϵ can take any value. However, if one does not introduce a potential for ϕ, then ϵ is constrained by time-delay experiments to be less than 5×10^{-4} [10]. If one introduces a potential for ϕ so as to fix its value at late times and so avoid the time-delay bound, then it is likely that, even if the initial period of inflation is dominated by $V(\sigma)$ and the Universe subsequently reheats, there will be a period during which $V(\phi)$ comes to dominate the energy density and drives inflation. (This is discussed in Appendix B.) This would be a problem if reheating was a problem for $V(\phi)$-driven inflation [5], since the Universe would become cool due to the inflation following the decay of the $V(\sigma)$ energy density. However, as discussed in Appendix B, it is conceivable that, in some models, a subsequent period of $V(\phi)$-driven inflation may necessarily have to be avoided, for example, if reheating of the ϕ energy density cannot regenerate a baryon asymmetry. As a result, one should not rule out the possibility that $\epsilon>10^{-3}$ with $\lambda\sigma^4$ chaotic inflation scenario as described above remaining unaltered by subsequent ϕ-driven inflation. Therefore, in the following we will consider all values of ϵ up to order 1.

B. Density perturbations

In order to calculate the magnitude of density perturbations when they reenter the horizon during the Friedmann-Robertson-Walker era, it is necessary to have a generalization of the analysis of Ref. [11], which deals with the gauge dependence of $\delta\rho$ for perturbations larger than the horizon, to the case of inflation driven by a matter scalar in the context of a Brans-Dicke gravity theory. However, at present such a generalization is not available. Therefore, in this section we follow previous treatments of inflation models based on Brans-Dicke gravity [2,5–7] and make the assumption that the results of Ref. [11] for the evolution of density perturbations on scales larger than the horizon may be applied to the Brans-Dicke case with the substitution of Newton's constant G_N by the effective time-dependent Newton's constant $G_{N\mathrm{eff}}$. Physically we expect this to be a reasonable assumption for superhorizon-sized perturbations (in the limit where Newton's constant changes slowly compared with the expansion rate of the Universe) since such perturbations are unaffected by microphysics and evolve purely kinematically. Therefore, for small changes in time, Newton's constant may be regarded as fixed when considering the evolution of the perturbations, with the effect of the time dependence taken into account by replacing G_N by $G_{N\mathrm{eff}}$. In particular, from the discussion in Ref. [11], one finds that the constancy of the quantity $\delta\rho/(\rho+p)$ is not affected by having a time-dependent Newton's constant. This is supported by the analysis of Ref. [12], which considers in some detail the evolution of the density perturbations for the case of a pure induced gravity model. Thus, with

$$\rho+p=\dot{\sigma}^2+\dot{\phi}^2 , \tag{41}$$

during inflation one obtains, for the magnitude of density perturbations on the scale of the observed Universe,

$$\frac{\delta\rho}{\rho}=\left[\frac{\delta\rho}{\dot{\sigma}^2+\dot{\phi}^2}\right]_{t_{60}} , \tag{42}$$

where the expression is evaluated at the time when the radius of the region which evolved into the observed Universe crossed the horizon during inflation, which we estimate by t_{60} in the following. [There may be a small factor multiplying the right-hand side of (42) [12]. We take this to be 1 in the following.] In order to calculate $\delta\rho$ we must consider the contributions due to fluctuations in the fields $\delta\sigma$ and $\delta\phi$. The contribution due to $\delta\sigma$ is given by

$$\delta\rho_\sigma=\frac{\partial V(\sigma)}{\partial\sigma}\delta\sigma . \tag{43}$$

Using (5) one finds

$$\delta\rho_\sigma=-3H\dot{\sigma}\delta\sigma . \tag{44}$$

For the case of $\delta\phi$ one expects that the initial perturbation in Newton's constant will give rise to a similar perturbation in the energy density $|\delta\rho/\rho|\approx|\delta\phi/\phi|$. This may also be seen by considering the curvature induced by a perturbation $\delta\phi$ around a flat spacetime. From the Friedmann equation (2a) we find that the curvature k produced by $\delta\phi$ is the same as that produced by an energy-density fluctuation of magnitude

$$\delta\rho_\phi=-\frac{2\rho}{\phi}\delta\phi . \tag{45}$$

With (45) giving the density perturbation due to $\delta\phi$ and using (4) in the limit where $|\dot{\phi}/\phi| \ll H$, one finds

$$\delta\rho_\phi=\frac{3H\dot{\phi}}{2}\delta\phi . \tag{46}$$

With the perturbation in the fields during inflation given by $\delta\sigma=\delta\phi\approx H/2\pi$ the magnitude of the density perturbation is then given by

$$\frac{\delta\rho}{\rho}\approx H^2\frac{\dot{\sigma}+\dot{\phi}}{\dot{\sigma}^2+\dot{\phi}^2}\Bigg|_{t_{60}} . \tag{47}$$

In the following we apply (47) to the four distinct cases corresponding to the $\sigma(t)$ solutions given above. (In accordance with our discussion of inflation we can restrict attention to the case $\epsilon<1$.)

(i) $\sigma_0>\phi_0$, $t_{60}<t_a$ ($\epsilon\gtrsim10^{-3}$). From Eqs. (5) and (6) we find that

$$\dot{\sigma}/\sigma=4\epsilon\left[\frac{\phi^2}{\sigma^2}\right]H , \tag{48}$$

$$\dot{\phi}/\phi=4\epsilon H . \tag{49}$$

From these we see that if initially $\sigma>\phi$ then $\dot{\phi}>\dot{\sigma}\,\forall t$ and vice versa. Thus, in case (i), one has (up to unimportant factors)

$$\frac{\delta\rho}{\rho}=\left[\frac{H^2}{\dot{\phi}}\right]_{t_{60}} . \tag{50}$$

Up to a small factor, $\sigma\approx\sigma_0$ at the time of horizon crossing. Thus, from (5),

$$|\dot{\sigma}|\approx\frac{4\lambda\sigma_0^3}{3H} \tag{51}$$

and so, using (11), (49), and (50), we find

$$\frac{\delta\rho}{\rho}=\frac{3\lambda^{1/2}\sigma_0^2}{4(3\epsilon)^{3/2}\phi_0^2(1+B_0t)^2} . \tag{52}$$

Using the expression for N_e (27) and N_{eT} (30), one obtains

$$\frac{\delta\rho}{\rho}=\frac{3\lambda^{1/2}(4\epsilon+2)\exp[2\Delta N_e/(\frac{1}{2}+1/4\epsilon)]}{16(3\epsilon)^{3/2}} . \tag{53}$$

Requiring that $\delta\rho/\rho\lesssim10^{-4}$ for consistency with bounds on the isotropy of the microwave background gives the constraint

$$\lambda\lesssim8\times10^{-6}\frac{\epsilon^3}{(4\epsilon+2)^2}\exp\left[\frac{-4\Delta N_e}{\frac{1}{2}+1/4\epsilon}\right] . \tag{54}$$

Thus, we find that the upper bound on λ has an exponential dependence on ϵ. With $\Delta N_e=60$, we find, for example, $\lambda<10^{-46}$ when $\epsilon=0.1$ and $\lambda<10^{-15}$ when $\epsilon=2\times10^{-3}$. Thus, as ϵ becomes large, the density perturbations require a much more extreme suppression of the σ self-coupling than in the case of conventional chaotic inflation models [4].

(ii) $\sigma_0>\phi_0$, $t_{60}>t_a$ ($\epsilon\lesssim10^{-3}$). In this case one finds that $\delta\rho/\rho$ is given by (52) with the substitutions $\sigma_0\rightarrow\sigma$

and $(1+B_0 t)\rightarrow(1+B_0 t_a)$. Using (24b), (31), and $B_0 t_a \gg 1$, we find

$$\frac{\delta\rho}{\rho}=\frac{(4\epsilon+2)\lambda^{1/2}}{48\sqrt{3}\epsilon^{3/2}}e^{-(4/3)\alpha t_a^{-1}(t-t_a)} . \tag{55}$$

Using the expression for ΔN_e (32) gives

$$\frac{\delta\rho}{\rho}=\frac{\lambda^{1/2}\alpha\Delta N_e}{6\sqrt{3}\epsilon^{1/2}} . \tag{56}$$

Thus, requiring that $\delta\rho/\rho \lesssim 10^{-4}$ gives the constraint

$$\lambda \lesssim \frac{1.1\times10^{-6}\epsilon}{\alpha^2\Delta N_e^2} . \tag{57}$$

With $\Delta N_e=60$ this gives the upper bound on λ:

$$\lambda \lesssim 3.1\times10^{-10}\frac{\epsilon}{\alpha^2} . \tag{58}$$

We see from the results that as the value of ϵ is reduced, the constraint on λ becomes weaker until $\epsilon \lesssim 10^{-3}$, and then gets stronger as ϵ is further reduced. Note also, that, in this case, there is an explicit ϵ dependence, so the result differs nontrivally from the case of a model with a fixed Newton's constant.

(iii) $\phi_0>\sigma_0$, $t_{60}>\theta$. In this case $\dot\sigma \geq \dot\phi$, thus from (47) one finds

$$\frac{\delta\rho}{\rho}=\left[\frac{H^2}{\dot\sigma}\right]_{t_{60}} . \tag{59}$$

Using (25) one obtains

$$\frac{\delta\rho}{\rho}=(\tfrac{1}{3})^{1/2}\lambda^{1/2}\epsilon^{-3/2}(\sigma_0/\phi_0)^3 e^{-12(\lambda\epsilon/2)^{1/2}\phi_0 t} , \tag{60}$$

where $\phi\approx\phi_0\forall t$ has been used, which follows from (9) and (35). Using (38) one obtains

$$\frac{\delta\rho}{\rho}=16(\tfrac{2}{3})^{1/2}\lambda^{1/2}\Delta N_e^{3/2} . \tag{61}$$

This gives the upper bound on λ:

$$\lambda \lesssim \frac{6\times10^{-11}}{\Delta N_e^3} . \tag{62}$$

Thus, with $\Delta N_e\approx60$ one obtains $\delta\rho/\rho \lesssim 3\times10^{-16}$. The limit $\epsilon\rightarrow0$ corresponds to the case of a fixed Newton's constant. Since (62) is ϵ independent, it should be the same result as that obtained in the case of a fixed Newton's constant.

(iv) $\phi_0>\sigma_0$, $t_{60}<\theta$. In this case $\phi\approx\phi_0$ and $\sigma\approx\sigma_0$ at t_{60}. Using (5) and (59) one obtains

$$\frac{\delta\rho}{\rho}\approx(\lambda/3)^{1/2}\epsilon^{-3/2}\left[\frac{\sigma_0}{\phi_0}\right]^3 . \tag{63}$$

$\delta\rho/\rho \lesssim 10^{-4}$ then gives the upper bound on λ:

$$\lambda \lesssim 3\times10^{-8}\epsilon^3\left[\frac{\phi_0}{\sigma_0}\right]^6 . \tag{64}$$

The value of ϵ is constrained in this case according to (40). Using (40) gives

$$\lambda \lesssim \frac{6\times10^{-11}}{\Delta N_e^3} . \tag{65}$$

$\Delta N_e=60$ then gives the upper bound $\lambda \lesssim 3\times10^{-16}$, the same as in the case of a fixed Newton's constant. We see that, in cases (i) and (ii), corresponding to $\sigma_0>\phi_0$, there is an explicit ϵ dependence, whereas in cases (iii) and (iv), corresponding to $\phi_0>\sigma_0$, the results for density perturbations on the scale of the observed Universe are ϵ independent and so are essentially the same as those obtained in the case of a fixed Newton's constant.

C. Gravitational waves and the microwave background isotropy

The amplitude of gravitational-wave perturbations on wavelengths of order the present horizon scale is [13,14]

$$A_{\rm GW}=\left.\frac{H}{M_{\rm Pl_{eff}}}\right|_{t_{60}} \tag{66}$$

where $M_{\rm Pl_{eff}}=(8\pi\epsilon)^{1/2}\phi$. $A_{\rm GW}$ is constrained to be less than 2×10^{-5} in order to leave the isotropy of the cosmic microwave background undisturbed. We merely state the resulting bounds on λ for cases (i)–(iv) considered above in the discussion of density perturbations.

$$\text{Case (i):}\quad \lambda \lesssim 5\times10^{-7}\frac{\epsilon^2}{(4\epsilon+2)^2}\exp\left[\frac{-4\Delta N_e}{\frac{1}{2}+1/4\epsilon}\right] . \tag{67}$$

$$\text{Case (ii):}\quad \lambda \lesssim \frac{5\times10^{-9}}{\Delta N_e^2} . \tag{68}$$

$$\text{Case (iii):}\quad \lambda \lesssim \frac{5\times10^{-10}}{\Delta N_e^2} . \tag{69}$$

$$\text{Case (iv):}\quad \lambda \lesssim \frac{5\times10^{-10}}{\Delta N_e^2} . \tag{70}$$

Comparing these with the bounds from density perturbations, one finds that, in all cases, the bounds from density perturbations are more severe. Thus, in general, $\delta\rho/\rho\approx10^{-4}$ as required by galaxy formation can be generated by scalar field fluctuations during $\lambda\sigma^4$ chaotic inflation without conflicting with the microwave isotropy due to gravitational waves.

D. Spectrum of density perturbations

As discussed in Appendix A, the dependence of ΔN_e on the scale R of the perturbation at present in cases (i)–(iii) above is

$$\Delta N_e=\ln R+k , \tag{71}$$

where k is a constant. Thus, substituting into the expressions for $\delta\rho/\rho$ gives

$$\text{case (i):}\quad \frac{\delta\rho}{\rho}\propto R^{8\epsilon} , \tag{72}$$

$$\text{case (ii):}\quad \frac{\delta\rho}{\rho}\propto(\ln R+k) , \tag{73}$$

$$\text{case (iii):}\quad \frac{\delta\rho}{\rho}\propto(\ln R+k)^{3/2} . \tag{74}$$

In case (iv), since $\phi \approx \phi_0$ and $\sigma \approx \sigma_0$, we find that H is constant at t_{60}, and so for scales corresponding to $t \lesssim \theta$ the density perturbations will be scale independent. From (72)–(74) we see that the spectrum of $\delta\rho/\rho$ is such that it increases with R. Thus, there is no problem with primordial black holes in this model [13,15]. However, if one is interested in galaxy formation, one must check that the increase in $\delta\rho/\rho$ from the scale of galaxy formation (when $\delta\rho/\rho \approx 10^{-4}$) to the scale of the observed Universe does not result in an unacceptable anisotropy of the microwave background ($\delta\rho/\rho \lesssim 10^{-3}$ on the scale of the presently observed Universe). This requires that $\alpha \lesssim 0.1$, where α is defined by $\delta\rho/\rho \propto R^{3\alpha}$. From (72) we then obtain in case (i) the constraint $\epsilon < 0.04$. In cases (ii) and (iii) we may express the log variation with R as an effective power law around $t = t_{60}$ by taking the power to be

$$3\alpha = R\frac{d}{dR}\ln\left[\frac{\delta\rho}{\rho}\right] . \tag{75}$$

Applying this to (73) and (74) gives, for the effective power in case (ii),

$$\alpha = \frac{1}{3\Delta N_e} , \tag{76}$$

and, in case (iii),

$$\alpha = \frac{1}{2\Delta N_e} . \tag{77}$$

With $\Delta N_e \approx 60$, these are an order of magnitude below the upper bound. Hence, in cases (ii) and (iii) we find that the spectrum of density perturbations is, in general, consistent with the isotropy of the microwave background.

IV. CONCLUSIONS

From the equations of motion we expect that, as $\epsilon \to 0$, the results will tend to those obtained with a fixed Newton's constant. We see from the above that, in cases (iii) and (iv) (which have $\epsilon \lesssim 10^{-3}$), the results for $\delta\rho/\rho$ and A_{GW} are independent of ϵ and so will be the same as the results in standard $\lambda\sigma^4$ chaotic inflation which occur in the limit $\epsilon \to 0$. Cases (i) and (ii), on the other hand, give results that are explicitly dependent on ϵ, and so are altered from the standard case by the time dependence of Newton's constant during inflation. We also find that, in order to have sufficient inflation, ϵ must be less than 1. This is different from the case of chaotic inflation driven by a potential energy for the Brans-Dicke scalar itself, in which case ϵ can take any value (with $\epsilon \gg 1$ being preferred in order to suppress density perturbations). We find that, if $\epsilon \gtrsim 10^{-3}$, then only case (i) can give sufficient inflation (requiring that $\sigma_0 > \phi_0$ in this case), whilst if $\epsilon \lesssim 10^{-3}$, then both $\sigma_0 > \phi_0$ and $\phi_0 > \sigma_0$ can give sufficient inflation. In general, the spectrum of density perturbations has $\delta\rho/\rho$ increasing with the scale of the perturbations. We find that, so long as $\epsilon \lesssim 0.04$, it is possible to have density perturbations $\delta\rho/\rho \approx 10^{-4}$ on scales associated with galaxy formation without producing large-scale anisotropies in the microwave background.

Although time-delay experiments put bounds [10] on ϵ in the case of a pure Brans-Dicke theory ($\epsilon < 5 \times 10^{-4}$), it is possible to add a potential for ϕ which would fix Newton's constant at present. As discussed in Appendix B, one would, in general, expect to have a period of inflation driven by $V(\phi)$ at some time after σ driven chaotic inflation ends. However, there may be models in which a period of ϕ driven inflation must necessarily be avoided. For instance, it has been suggested [5] that Brans-Dicke scalar-driven inflation may have a problem with reheating. In this case, it would be necessary for ϕ at the end of σ-driven chaotic inflation to be close to the minimum of its potential so as to avoid $V(\phi)$-driven inflation and the associated cooling of the Universe with no subsequent reheating (or with insufficient reheating to regenerate a baryon asymmetry). This would correspond to an anthropic principle determination of the value of ϕ at the end of inflation. Thus, the possibility that $\epsilon > 10^{-3}$ with no significant $V(\phi)$ driven inflation should not be discounted, although it is unattractive. However, we find on studying in detail the σ driven chaotic inflation scenario that even if ϵ could be greater than 10^{-3} for some reason, the value of the σ self-coupling is constrained to be extremely small for large ϵ (less than 10^{-46} for $\epsilon = 0.1$). This would probably make reheating in this model difficult (since the couplings of σ to quarks and leptons would also have to be small in order not to generate a large λ at one loop), so giving little improvement over the case of $V(\phi)$-driven inflation. Thus, chaotic inflation driven purely by a minimally coupled scalar in a model with ϵ much larger than $\sim 10^{-3}$ is disfavored even if it can be made consistent with the time-delay experiment bound.

Chaotic inflation in the context of Brans-Dicke theory has also been discussed by Linde [9]. He considers essentially the same model as discussed in Sec. II and gives solutions for σ and ϕ in a general parametric form for an arbitrary $V(\sigma)$. The main emphasis of the present discussion has been on the constraints originating from the physical consequences of density perturbations generated during inflation, which is not analyzed in Ref. [9]. The main results of Ref. [9] deal with the self-regeneration of inflationary domains due to quantum fluctuations of the scalar fields in the model, and with the possibility of the present large value of the Planck mass being determined by probability arguments (what initial region inflates the most) or by the anthropic principle. The topics discussed in the present paper are mostly distinct from those discussed in Ref. [9] and so complement the discussion of Ref. [9]. We believe that the explicit (though approximate) solutions for σ given in this paper are particularly convenient for the analysis of density perturbations and inflation in this model.

We conclude that chaotic inflation in Brans-Dicke theory with a pure quartic potential for the matter scalar driving inflation can differ significantly from the case of gravity with fixed Newton's constant when initially $\sigma > \phi$ (and, in particular, when $\epsilon \gtrsim 10^{-3}$), but does not conflict with the observed isotropy of the Universe provided that $\epsilon < 0.04$. Sufficient inflation is found to require that $\epsilon < 1$. However, in the range $10^{-3} \ll \epsilon < 0.04$, the density perturbations produced during inflation will require a much smaller value for the σ self-coupling than in the case of conventional chaotic inflation models with fixed Newton's constant.

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APPENDIX A: VALUES OF ΔN_e AT HORIZON CROSSING IN $\lambda\sigma^4$ CHAOTIC INFLATION

In this Appendix we give expressions for the value of ΔN_e at which perturbations corresponding to scales of size R in the present Universe leave the horizon during inflation. The radius of a volume of the present Universe arising from a region crossing the horizon at time t is

$$R(t)=\left[\frac{g(T_{\rm rh})}{g(T_\gamma)}\right]^{1/3}\left[\frac{T_{\rm rh}}{T_\gamma}\right]H^{-1}(t)e^{\Delta N_e(t)}\ , \qquad \text{(A1)}$$

where the horizon at time t in $\lambda\sigma^4$ driven inflation is

$$H^{-1}=\left[\frac{3\epsilon\phi_0^2}{\lambda\sigma_0^4}\right]^{1/2}\left[1+\int_0^t B\,dt\right]\left[\frac{\sigma_0}{\sigma}\right]^2 . \qquad \text{(A2)}$$

$\Delta N_e(t)$ is the number of e-foldings remaining until the end of inflation, $T_{\rm rh}$ is the reheat temperature at the end of inflation, T_γ is the temperature of the photons at present (2.7 K), and $g(T)$ is the effective number of massless degrees of freedom at temperature T. In the following we give the expressions for ΔN_e corresponding to $R(t)=R_p$, where R_p is the size of the observable Universe at present. We consider the four cases for the time of horizon crossing corresponding to the cases discussed in Sec. III.

(i) $\sigma_0>\phi_0$, $t<t_a$. In this case the horizon during inflation is given by

$$H^{-1}=\left[\frac{3\epsilon\phi_0^2}{\lambda\sigma_0^4}\right]^{1/2}(1+B_0t)\ , \qquad \text{(A3)}$$

where we have used $\sigma\approx\sigma_0$ for $t<t_a$. From (27) we find

$$(1+B_0t)\approx(1+B_0t_a)\exp\left[\frac{-\Delta N_e}{\frac{1}{2}+1/4\epsilon}\right] . \qquad \text{(A4)}$$

Thus, from (A1) we find

$$R(t)=\left[\frac{g(T_{\rm rh})}{g(T_\gamma)}\right]^{1/3}\left[\frac{T_{\rm rh}}{T_\gamma}\right]\left[\frac{3\epsilon\phi_0^2}{\lambda\sigma_0^4}\right]^{1/2}\times(1+B_0t_a)\times\exp\left\{\Delta N_e\left[1-\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]^{-1}\right]\right\} . \qquad \text{(A5)}$$

Using (31) we find

$$\Delta N_e=\frac{1}{1-(\frac{1}{2}+1/4\epsilon)^{-1}}\ln\left[R_pT_\gamma\left[\frac{g(T_{\rm rh})}{g(T_\gamma)}\right]^{1/3}\right]-\ln(T_{\rm rh})+\tfrac{1}{2}\ln\left[\frac{4\epsilon+2}{12}\right]+\ln\sigma_0+\tfrac{1}{2}\ln\lambda-\tfrac{1}{2}\ln\epsilon+\ln[V(\sigma_0)^{1/4}/T_{\rm rh}] . \qquad \text{(A6)}$$

Therefore, with $R_p=3000$ Mpc $(=5\times10^{41}$ GeV) and $T_\gamma=2.7$ K $(=2\times10^{-13}$ GeV), and assuming that $g(T_{\rm rh})/g(T_\gamma)\approx100$, we find that ΔN_e corresponding to the present Universe is given by

$$\Delta N_e=\left[\frac{2-4\epsilon}{2+4\epsilon}\right]\{61.7+\tfrac{1}{4}\ln\lambda-\tfrac{1}{2}\ln\epsilon+\ln[V(\sigma_0)^{1/4}/T_{\rm rh}]\} . \qquad \text{(A7)}$$

Thus, one finds that, up to corrections typically $\lesssim10$, the scales corresponding to the observable Universe cross the horizon at ≈60 e-foldings before the end of inflation, assuming that λ is not very small compared with the value which gives density perturbations of magnitude $\delta\rho/\rho\approx10^{-4}$. It was shown in Sec. III that acceptable density perturbations can constrain λ to be very small if ϵ is larger than 10^{-3}. One might ask if the value of ΔN_e from (A7) could be much smaller than 60 in this case. In fact, this turns out not to be the case. If we substitute the value of λ from (54) which gives $\delta\rho/\rho\approx10^{-4}$ into (A7), we obtain the expression for ΔN_e:

$$\Delta N_e(1+6\epsilon)=58.6+\tfrac{1}{4}\ln\epsilon+\ln[V(\sigma_0)^{1/4}/T_{\rm rh}]\ , \qquad \text{(A8)}$$

which shows that, in general, $\Delta N_e\approx60$ for scales corresponding to the observable Universe.

(ii) $\sigma_0>\phi_0$, $t_{60}>t_a$. In this case, using (24) in (A2), one finds that

$$H^{-1}=H_0^{-1}3B_0t_ae^{(4/3)t_a^{-1}\alpha(t-t_a)} . \qquad \text{(A9)}$$

Then using (32) and (A9) in (A1), one obtains

$$R(t)=\left[\frac{g(T_{\rm rh})}{g(T_\gamma)}\right]^{1/3}\left[\frac{T_{\rm rh}}{T_\gamma}\right]\frac{3B_0t_a}{H_0}\times\left[\frac{1}{2}+\frac{1}{4\epsilon}\right]\left[\frac{e^{\Delta N_e(t)}}{\Delta N_e}\right] . \qquad \text{(A10)}$$

Using (31) and the above values for R_p and T_γ, we find that ΔN_e corresponding to the observable Universe is

$$\Delta N_e=66.0+\ln\Delta N_e+\tfrac{1}{4}\ln\lambda+\tfrac{1}{2}\ln\epsilon+\ln[V(\sigma_0)^{1/4}/T_{\rm rh}] . \qquad \text{(A11)}$$

Again this is typically ≈60, assuming that ϵ is not extremely small and that λ is not too much smaller than the value which gives $\delta\rho/\rho\approx10^{-4}$.

(iii) $\phi_0>\sigma_0$, $t>\theta$. In this case the horizon is given by

$$H^{-1}=\left[\frac{3\epsilon\phi_0^2}{\lambda\sigma_0^4}\right]^{1/2}\exp\left[8\left[\frac{\lambda\epsilon}{3}\right]^{1/2}\phi_0t\right]\ , \qquad \text{(A12)}$$

where we have used (25) and $1+\int B\,dt\approx1$. Thus, from

(A1) we find

$$R(t)=\left[\frac{g(T_{\rm rh})}{g(T_\gamma)}\right]^{1/3}\left[\frac{T_{\rm rh}}{T_\gamma}\right]\frac{1}{8\Delta N_e}\left[\frac{3}{\lambda\epsilon}\right]^{1/2}\frac{e^{\Delta N_e(t)}}{\phi_0}, \quad (A13)$$

where (38) has been used. Thus, we find that ΔN_e corresponding to the observable Universe is given by

$$\Delta N_e = 68.3+\ln\Delta N_e+\tfrac{1}{4}\ln\lambda+\tfrac{1}{2}\ln\epsilon+\ln(\phi_0/\sigma_0) +\ln[V(\sigma_0)^{1/4}/T_{\rm rh}], \quad (A14)$$

which again is typically ≈ 60, with the assumptions of not too small ϵ and λ as in case (ii).

(iv) $\phi_0<\sigma_0$, $t<\theta$. In this case, $\phi\approx\phi_0$ and $\sigma\approx\sigma_0$ during inflation. Therefore, $H^{-1}\approx H_0^{-1}$ and so, from (A1),

$$\Delta N_e=\ln\left[R_pT_\gamma\left[\frac{g(T_{\rm rh})}{g(T_\gamma)}\right]^{1/3}\frac{H_0}{T_{\rm rh}}\right]. \quad (A15)$$

Thus, we find for ΔN_e corresponding to the observable Universe,

$$\Delta N_e=65.2+\tfrac{1}{2}\ln\Delta N_e+\tfrac{1}{4}\ln\lambda+\ln[V(\sigma_0)^{1/4}/T_{\rm rh}], \quad (A16)$$

which, as in the previous cases, is typically ≈ 60.

Thus, in general, when the scales corresponding to the observable Universe cross the horizon during inflation the number of e-foldings of inflation remaining is $\Delta N_e\approx 60$ (assuming λ and ϵ are not extremely small). The scales corresponding to galaxies have $R\approx 10^{-3}R_p$, and so have ΔN_e smaller by about 7.

APPENDIX B: POST-INFLATION EVOLUTION OF THE UNIVERSE IN THE $\lambda\sigma^4$ CHAOTIC INFLATION MODEL

In this appendix we consider the effect on the post-inflation evolution of the $\lambda\sigma^4$ chaotic inflation model of including a nonzero $V(\phi)$ in the model. Including a nonzero $V(\phi)$ is necessary in the case where $\epsilon>5\times10^{-4}$ in order to avoid the bound on ϵ coming from time-delay experiments [10]. In particular, we consider the question of whether or not a period of $V(\phi)$-driven inflation can be avoided following the reheating of the $V(\sigma)$ energy density.

In order to discuss the evolution of the Universe in the presence of a nonzero $V(\phi)$, we will consider in the following a simple symmetry-breaking potential:

$$V(\phi)=\frac{\lambda_\phi}{4}(\phi^2-v^2)^2 . \quad (B1)$$

The full equations of motion including an energy density ρ_m a pressure p_m for the matter and radiation energy are then

$$H^2\left[1+2H^{-1}\frac{\dot\phi}{\phi}\right]=\frac{1}{3\epsilon\phi^2}[V(\phi)+\dot\phi^2/2+\rho_m], \quad (B2)$$

$$\ddot\phi+3H\dot\phi+\dot\phi^2/\phi+\frac{1}{(1+6\epsilon)}[V'(\phi)-4V(\phi)/\dot\phi-(\rho_m-3p_m)/\phi]=0 . \quad (B3)$$

The initial $V(\sigma)$-driven inflation period will come to an end once the σ field starts to oscillate about its minimum. For the case of a pure $\lambda\sigma^4$ potential, the resulting energy density will evolve exactly like a radiation energy density [16]. As a result, in this case we need not consider the question of reheating, since the evolution will be essentially the same before and after reheating. For a radiation energy density one has $\rho_m=3p_m$, thus, with the potential (B1) the equations of motion in the slow-rolling approximation are

$$H^2=\frac{\dot\phi^2/2+V(\phi)+\rho_m}{3\epsilon\phi^2}, \quad (B4)$$

$$3H\dot\phi=\frac{\lambda_\phi v^2(v^2-\phi^2)}{(1+6\epsilon)\phi} . \quad (B5)$$

We can solve these equations exactly for the case where the Universe is dominated by a radiation energy density (or by an effective radiation density for the oscillating σ case). The radiation energy density can be written in terms of the scale factor as $\rho_m=k_r/a^4$. Using this and (B4) and (B5), we obtain an equation for ϕ in terms of a:

$$\frac{d\phi}{da}=\frac{3\epsilon\lambda_\phi v^2\phi(v^2-\phi^2)a^3}{(1+6\epsilon)k_r} . \quad (B6)$$

This equation has the solution

$$\phi^2=\frac{v^2e^x}{e^x+v^2/\phi_e^2-1}, \quad (B7)$$

where X is defined by

$$X=\frac{3\epsilon\lambda_\phi v^4(a^4-a_e^4)}{2(1+6\epsilon)k_r}, \quad (B8)$$

and ϕ_e and a_e are the values of ϕ and a at the end of $V(\sigma)$-driven inflation.

We first consider whether or not it is possible for the ϕ field to begin oscillating before the radiation energy density becomes dominated by $V(\phi)$, thus avoiding a period of $V(\phi)$-driven inflation. We first consider the case where, initially, $\phi>v$. From (B7) we see that, for X small compared to 1, ϕ can be written as

$$\phi^2\approx v^2/(X+v^2/\phi_e^2) . \quad (B9)$$

Thus, once X is larger than v^2/ϕ_e^2, ϕ^2 will become proportional $1/X$. Let the time at which this occurs be denoted by t_x. Assuming that $a^4 \gg a_e^4$ at t_x, we see that, after t_x, the energy density in the ϕ field ($\propto \phi^4$) will drop more rapidly than the radiation energy density. Thus, if the radiation energy density is dominant at t_x then it will remain dominant until the ϕ field begins oscillating. Using (B8) and $\rho_m = k_r/a^4$, we find that the condition for the radiation energy density to be dominant when the ϕ field begins to rapidly decrease is

$$\epsilon/(1+6\epsilon) > \frac{\phi_e^2}{6v^2} . \tag{B10}$$

This is not satisfied if $\epsilon < 1$. Thus, if initially $\phi > v$, the Universe will become dominated by $V(\phi)$ before ϕ changes significantly from its value at the end of inflation ϕ_e. (The slow-rolling approximation for ϕ will hold up to $X \approx v^2/\phi_e^2$ in this case.)

In the case ϕ less than v, we see from (B7) that once X is greater than order 1, ϕ will start to grow rapidly. Once X is larger than v^2/ϕ^2, we find that $\phi \approx v$. We see from (B1) that, until ϕ is close to v, the potential does not change much from $\lambda_\phi v^4/4$. The condition for the radiation energy density to dominate $V(\phi)$ at $X \approx 1$ (when ϕ begins to change significantly from its initial value) is $\rho_m > \lambda_\phi v^4/4$, which gives

$$\epsilon > \frac{1+6\epsilon}{6} . \tag{B11}$$

We see again that this is not satisfied if $\epsilon < 1$. (In this case the slow-rolling approximation for ϕ is valid up to $X \approx 1$).

Thus, we conclude that, in general, $V(\phi)$ will come to dominate the energy density of the Universe before it or ϕ change significantly from their values at the end of inflation. Therefore, if ϕ is slow-rolling once $V(\phi)$ becomes dominant, a period of ϕ-driven inflation will occur. This would be a problem if either the ϕ field lacks the couplings to light fields allowing the energy density in $V(\phi)$ to decay and reheat the Universe, or if the temperature at the end of $V(\phi)$ reheating was too low to allow nucleosynthesis or baryogensis to occur (the original baryon density being diluted away by the ϕ driven inflation). These possibilities are clearly very model dependent. However, if in some model they do occur then it would be necessary for the region of the Universe in which we live to be such that little or no ϕ-driven inflation occurs. This amounts to a fine-tuning of the initial value of ϕ following inflation, justified by the anthropic principle. (The initial period of σ-driven inflation is crucial in creating an ensemble of regions the size of the observable Universe, in each of which the value of ϕ is expected to be different, depending on its initial value in the region of the preinflation Universe which inflates to the observable Universe-sized region.) Thus, we consider the conditions under which little or no $V(\phi)$-driven inflation occurs after the $V(\sigma)$-driven inflation ends.

No $V(\phi)$-driven inflation will occur once $V(\phi)$ becomes dominant if ϕ at this time satisfies the condition (Accetta *et al.*, Ref. [7])

$$|\phi - v| < \epsilon^{1/2} v . \tag{B12}$$

In this case the slow-rolling approximation for ϕ breaks down and the ϕ field will begin oscillating around the minimum of $V(\phi)$. Alternatively, one could simply require that there was not too much ϕ-driven inflation, so preventing a preexisting baryon asymmetry from being diluted by too large a factor. From Accetta *et al.* the total number of e-foldings of inflation which occur as ϕ from its initial value to $\phi \approx v$ is

$$N_e \approx \frac{1}{8\epsilon}\left[\left[\frac{\phi}{v}\right]^2 - 1\right], \quad \phi > v , \tag{B13}$$

$$N_e \approx \frac{1}{4\epsilon}\left[\ln\left[\frac{v}{\phi}\right] - \frac{1}{2}\right], \quad \phi < v . \tag{B14}$$

Thus, we see that if ϵ is large (greater than about 0.05), then there will be no more than about 10 e-foldings of inflation due to $V(\phi)$ if ϕ is of order v when $V(\phi)$ becomes the dominant energy density. So, in this case there can be relatively little cooling and expansion of the Universe during $V(\phi)$-driven inflation, leaving the $V(\sigma)$-driven inflation scenario largely unaffected by the subsequent period of ϕ-driven inflation.

We conclude that, although in general there will be a period of significant $V(\phi)$-driven inflation following the end of the chaotic $V(\sigma)$-driven inflation, it is conceivable that there are models in which the cooling and expansion of the Universe due to $V(\phi)$ is physically inconsistent with the Universe as it exists at present. In this case, anthropic principle considerations require that the $V(\phi)$ inflation should either not occur or should have little effect on the overall evolution of the Universe following the end of σ-driven chaotic inflation. For this reason one should consider the possibility that $\epsilon > 10^{-3}$ in the $V(\sigma)$-driven chaotic inflation scenario, with the existence of $V(\phi)$ giving a mass to the Brans-Dicke scalar, allowing the time-delay bound $\epsilon < 5\times 10^{-4}$ to be evaded but otherwise not altering the inflation scenario significantly.

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