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Effects of magnetohydrodynamic accretion flows on global structure of a Kerr black-hole magnetosphere

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In relation to black-hole models for central engines in active galactic nuclei, we discuss some effects of accreting matter on the global structure of the Kerr black-hole magnetosphere. Based on the steady axisymmetric magnetohydrodynamic (MHD) equations in Kerr geometry, we analytically study the general-relativistic Grad-Shafranov equation and the poloidal wind equation. The field line geometry is clearly given in the region where the plasma can be magnetically supported. Near the black hole, the magnetic support breaks down and the plasma begins to flow toward the event horizon. We consider the magnetohydrodynamic conditions for the plasma inflows which should pass through the Alfvén critical point. Then it is found that the accumulation of magnetic field lines threading the horizon is suppressed for the rapidly rotating holes. This result means that the global shape of the field lines and the MHD energy extraction from the rotating black hole can drastically change as the spin down of the black hole proceeds. We present an evolutionary model for the activity and the structure of the black-hole magnetosphere.

I. INTRODUCTION

It is widely believed that supermassive black holes work as an engine of very energetic and compact phenomena observed in active galactic nuclei (AGN's). The mechanism of energy generation is roughly divided into two categories. One is essentially due to a release of the gravitational energy of accreting matter, called "fuel" models, and the other is essentially due to an extraction of the rotational energy from accretion disks or black holes, called "flywheel" models. Both of these mechanisms may be important in actual AGN's for explaining the enormous luminosity and other properties of observed phenomena (e.g., short time variation of activity). In this paper, we are concerned with the flywheel model in which the rotational energy and the angular momentum of the black hole are extracted by magnetic dragging and transferred in the form of electromagnetic flux. A key problem for this model is to reveal the structure of the black-hole magnetosphere. This structure is determined through some interaction between the black hole and the ambient plasmas which may be described within the framework of magnetohydrodynamics (MHD). The basic equations for the steady and axisymmetric magnetohydrodynamic system can be reduced to the poloidal wind equation and the Grad-Shafranov (GS) equation. The former denotes the development of physical quantities along a stream line while the latter denotes the transfield equilibrium giving a geometry of field lines. But these equations are very complicated and the analysis has been developed using some approximations.

An important step for studying the magnetosphere in Kerr background space-time was originated by Blandford and Znajek [1] (BZ) in the force-free limit. Their work has clearly shown a mechanism for electromagnetic energy extraction (which we call the BZ mechanism) which produces power proportional to $\Omega_F(\omega_H-\Omega_F)$ and the square of poloidal magnetic flux at the horizon, where Ω_F and ω_H are angular velocities of the field lines and the black hole, respectively. The radial or paraboloidal magnetic-field solution was found by a perturbation method which is applicable to the case in which both Ω_F and ω_H are very small.

The next step should be to contain fluid contributions to the magnetospheric structure. In the black-hole magnetosphere, the outer region will be filled with "transmagnetosonic" out-going plasma flows observed as jets or winds, but the inner region must be filled with "transmagnetosonic" in-going plasma flows transporting energy and angular momentum to the black hole. Phinney [2] emphasized the importance of a role of such plasma flows by showing how Ω_F can be determined through a condition at the fast magnetosonic critical point for the poloidal wind equation. Further, Takahashi *et al.* [3] generally proved that the position of the Alfvén critical point for the inflows is crucial to transport the negative-energy influx into the black hole.

In a Newtonian analysis recent works revealed some interesting effects of the matter part (plasma's inertia) on field line geometry. For example Sakurai [4] numerically showed that poloidal magnetic-field lines connected with a rotating compact object are forced to bend azimuthally by the plasma's inertia as the distance from the central object increases and strong toroidal magnetic field is piled up near the equatorial plane. Hence the magnetic pressure induces a pinch effect and collimates the flow along the spin axis. Heyvaerts and Norman [5] obtained analogous results, by using a purely analytic method under general boundary conditions. Here we are interested in the magnetosphere under the effects of plasma accretion flows onto the Kerr black hole where a general-relativistic treatment should be essential. The purpose of this paper is to understand the global magnetospheric structure without assuming the force-free limit and the Newtonian limit.

We consider a Kerr black hole immersed in an axisymmetric and steady magnetic field that becomes asymptotically uniform. In Sec. II, we introduce the GS equation and the poloidal wind equation for the MHD system in Kerr geometry. In Sec. III, we solve the basic equations in the low-poloidal flow limit. We find the existence of a stagnation region where inertial and electromagnetic forces acting on plasma particles are balanced in the Kerr background geometry. Our approach is to assume that the plasma can be magnetically supported in a wide region of the magnetosphere. This requires that the dimensionless parameter $\epsilon \equiv m\Omega_F$ is very small, where m is the black hole's mass in geometrical units. In this case, the field lines turn out to be considerably bent from a cylindrical configuration. In Sec. IV, we consider the region where such a poloidal quasiequilibrium breaks down. The poloidal accretion flows must be accelerated to a super-Alfvénic speed in this region near the black hole. We estimate physical quantities in order of magnitude as a power of the small parameter ϵ by analyzing the Alfvén critical condition and the boundary condition at the event horizon. Then we arrive at our main conclusion that the magnetic field threading the horizon is less amplified as the hole's angular velocity ω_H becomes larger than Ω_F. In Sec. V, as an application of the results obtained here, we discuss the MHD version of energy extraction from a rotating black hole and the evolution of a black-hole magnetosphere.

II. BASIC EQUATIONS AND CONSTANTS OF MOTION

We study the general-relativistic equations for steady and axisymmetric ideal magnetohydrodynamics (no resistivity and no viscosity), which are summarized in the Appendix. In this paper the MHD equations are treated in the cold limit (i.e., the pressureless limit $P=0$) as the first step for understanding some effects due to the accretion flow on the shape of the field lines (the pressureless and viscousless assumption can remove any direct interaction between neighboring fluid elements).

The background geometry is given by the Kerr metric in the Boyer-Lindquest coordinates as follows:

$$ds^2=\left[1-\frac{2mr}{\Sigma}\right]dt^2+\frac{2mar\sin^2\theta}{\Sigma}2\,dt\,d\phi - \frac{A\sin^2\theta}{\Sigma}d\phi^2-\frac{\Sigma}{\Delta}dr^2-\Sigma\,d\theta^2\ , \tag{2.1}$$

where $\Sigma=r^2+a^2\cos^2\theta, \Delta=r^2-2mr+a^2, A=(r^2+a^2)^2-\Delta a^2\sin^2\theta$, and a and m are the spin and mass parameters, respectively. We use geometrical units $c=G=1$.

The set of MHD equations in Kerr geometry has been completely solved except the transfield components of the momentum equations. In the cold limit we have the four constants of motion along magnetic-field lines (stream lines), and the poloidal velocity of plasma flows is related to these constants through the poloidal wind equation (see, for example, Takahashi *et al.* [3] and Camenzind [6,7]). Here we refer to the relevant points.

The four constants of motion can be defined by the electromagnetic field $F_{\alpha\beta}$ and the four-velocity u^α as follows. The angular velocity of magnetic-field line Ω_F and the injection rate η are

$$\Omega_F \equiv -\frac{F_{tr}}{F_{\phi r}} = -\frac{F_{t\theta}}{F_{\phi\theta}}\ , \tag{2.2}$$

$$\eta \equiv -\frac{\sqrt{-g}\,nu^r}{F_{\theta\phi}} \tag{2.3}$$

$$= -\frac{\sqrt{-g}\,nu^\theta}{F_{\phi r}} \tag{2.4}$$

$$= -\frac{\sqrt{-g}\,nu^t(\Omega-\Omega_F)}{F_{r\theta}}\ , \tag{2.5}$$

where

$$\Omega=\frac{u^\phi}{u^t} \tag{2.6}$$

is the angular velocity of plasma flows, and n is the proper number density. The total energy E and the total angular momentum L of a plasma particle are

$$E\equiv \mu u_t - \frac{\Omega_F B_\phi}{4\pi\eta}\ , \tag{2.7}$$

$$L\equiv -\mu u_\phi - \frac{B_\phi}{4\pi\eta}\ , \tag{2.8}$$

where μ is the specific enthalpy, which in the cold limit reduces to the rest-mass energy of a plasma particle, and B_ϕ is the ϕ component of the magnetic field defined by $B_\alpha \equiv \frac{1}{2}\sqrt{-g}\,\epsilon_{\alpha\beta\rho\sigma}k^\beta F^{\rho\sigma}$ (k^β is the timelike Killing vector).

By using the metric tensor $g_{\mu\nu}$ and the constants of motion Ω_F, η, E, and L, we have

$$u_t=\frac{(g_{tt}+g_{t\phi}\Omega_F)e-EM^2}{\mu(k_0-M^2)}\ , \tag{2.9}$$

$$u_\phi=\frac{(g_{t\phi}+g_{\phi\phi}\Omega_F)e+LM^2}{\mu(k_0-M^2)}\ , \tag{2.10}$$

and

$$B_\phi=\frac{-4\pi\eta}{(k_0-M^2)}[(g_{tt}+g_{t\phi}\Omega_F)L+(g_{t\phi}+g_{\phi\phi}\Omega_F)E]\ , \tag{2.11}$$

where

$$k_0\equiv g_{tt}+2g_{t\phi}\Omega_F+g_{\phi\phi}\Omega_F^2 \tag{2.12}$$

and

$$e\equiv E-\Omega_F L\ . \tag{2.13}$$

These constants of motion should be understood as functions of the ϕ component Ψ of the electromagnetic vector potential. The function $\Psi=\Psi(r,\theta)$ determines the poloidal field line geometry and is called the flux function.

The poloidal velocity u_P defined by $u_P^2 = -u_A u^A$ $(A = r, \theta)$ can be derived by solving the poloidal wind equation

$$(1+u_P^2)(k_0 - M^2)^2 = \left[\frac{E}{\mu}\right]^2 (k_0 k_2 - 2k_2 M^2 - k_4 M^4) , \tag{2.14}$$

where k_2 and k_4 are given by the metric tensor and the constants of motion as

$$k_2 \equiv \left[\frac{e}{E}\right]^2 , \tag{2.15}$$

$$k_4 \equiv \frac{g_{\phi\phi} + 2g_{t\phi}(L/E) + g_{tt}(L/E)^2}{\rho_W^2} , \tag{2.16}$$

and $\rho_W^2 \equiv g_{t\phi}^2 - g_{tt} g_{\phi\phi}$ $(= \Delta \sin^2\theta)$ is the invariant cylindrical radius. The Alfvén Mach number defined by

$$M^2 \equiv \frac{4\pi\mu\eta^2}{n} \tag{2.17}$$

can be rewritten in the form

$$M^4 = 16\pi^2\mu^2\eta^2(g_{tt} + g_{t\phi}\Omega_F)^2 \frac{u_P^2}{B_P^2} , \tag{2.18}$$

where n is the proper number density of plasma particles and B_P is the poloidal magnetic field

$$B_P^2 \equiv -B_A B^A = \frac{-(g_{tt} + g_{t\phi}\Omega_F)^2}{\rho_W^2} \times \left[\frac{1}{g_{rr}}(\partial_r \Psi)^2 + \frac{1}{g_{\theta\theta}}(\partial_\theta \Psi)^2\right] . \tag{2.19}$$

The poloidal wind equation (2.14) may become singular at the Alfvén point where

$$(k_0 - M^2)_A = 0 . \tag{2.20}$$

The subscript A means quantities evaluated at the Alfvén point. The flows can smoothly pass through this critical point only if the condition

$$(k_2 + k_0 k_4)_A = 0 \tag{2.21}$$

is simultaneously satisfied. Hence we call Eqs. (2.20) and (2.21) the Alfvén critical conditions. From Eqs. (2.9), (2.10), and (2.20) we obtain

$$E = \frac{e(g_{tt} + g_{t\phi}\Omega_F)_A}{k_{0A}} \tag{2.22}$$

and

$$L = -\frac{e(g_{t\phi} + g_{\phi\phi}\Omega_F)_A}{k_{0A}} . \tag{2.23}$$

The flux function must satisfy the poloidal transfield component of the momentum equation which is called the Grad-Shafranov equation. A detailed derivation of the GS equation is given in the Appendix. In the cold limit it can be written in the form

$$-\frac{\rho_W^2(k_0 - M^2)}{4\pi\sqrt{-g}}\left[\partial_r\left[\frac{\sqrt{-g}\,(k_0 - M^2)}{\rho_W^2}\frac{\partial_r\Psi}{g_{rr}}\right] + \partial_\theta\left[\frac{\sqrt{-g}\,(k_0 - M^2)}{\rho_W^2}\frac{\partial_\theta\Psi}{g_{\theta\theta}}\right]\right]$$
$$= -2\pi\left[\frac{\rho_W^2}{M^2}(e^2\eta^2)' + g_{tt}(L^2\eta^2)' + 2g_{t\phi}(EL\eta^2)' + g_{\phi\phi}(E^2\eta^2)'\right]$$
$$+ \frac{4\pi\mu^2\rho_W^2(k_0 - M^2)}{M^2}\eta\eta' + \frac{4\pi\mu^2\rho_W^2}{M^4}(g_{t\phi} + g_{\phi\phi}\Omega_F)\eta^2\left[\left[\frac{e}{\mu}\right]^2 - (k_0 - M^2)\right]\Omega_F' , \tag{2.24}$$

where the prime denotes the derivative with respect to Ψ because E, L, η, and Ω_F are functions of Ψ only. The GS equation is a highly nonlinear partial differential equation for Ψ and is coupled with the poloidal wind equation. Equation (2.24) seems to give an additional condition to eliminate the singularity at the Alfvén point and the horizon. However, it is easy to prove that if the regularity requirement for the poloidal wind equation, conditions (2.20) and (2.21), is satisfied, it is automatically satisfied for the GS equation. Let us rewrite Eq. (2.24) in the form

$$-\frac{\rho_W^2}{4\pi\sqrt{-g}}\left\{(k_0 - M^2)\left[\partial_r\left[\frac{\sqrt{-g}}{\rho_W^2}\frac{\partial_r\Psi}{g_{rr}}\right] + \partial_\theta\left[\frac{\sqrt{-g}}{\rho_W^2}\frac{\partial_\theta\Psi}{g_{\theta\theta}}\right]\right] + \frac{\sqrt{-g}}{\rho_W^2}\left[\frac{\partial_r\Psi}{g_{rr}}\partial_r(k_0 - M^2) + \frac{\partial_\theta\Psi}{g_{\theta\theta}}\partial_\theta(k_0 - M^2)\right]\right\}$$
$$= \frac{4\pi\mu^2\rho_W^2}{M^2}\eta\eta' - \frac{4\pi\mu^2\rho_W^2}{M^4}(g_{t\phi} + g_{\phi\phi}\Omega_F)\eta^2\Omega_F' + \frac{2\pi}{M^2}[(E^2\eta^2)'g_{\phi\phi} + (L^2\eta^2)'g_{tt} + (EL\eta^2)'2g_{t\phi}]$$
$$- \frac{4\pi\eta}{M^2}[(g_{t\phi} + g_{\phi\phi}\Omega_F)(E\eta)' + (g_{tt} + g_{t\phi}\Omega_F)(L\eta)']\frac{E(g_{t\phi} + g_{\phi\phi}\Omega_F) + L(g_{tt} + g_{t\phi}\Omega_F)}{k_0 - M^2}$$
$$- \frac{4\pi\rho_W^2}{M^4}\eta^2 L(E - L\Omega_F)\Omega_F'\frac{[(g_{t\phi} + g_{\phi\phi}\Omega_F)/(g_{t\phi} + g_{\phi\phi}\Omega_F)_A]k_{0A} - M^2}{k_0 - M^2} . \tag{2.25}$$

It is now clear that the right-hand side of Eq. (2.25) does not diverge even at the Alfvén point because the equality

$$[E(g_{t\phi}+g_{\phi\phi}\Omega_F)+L(g_{tt}+g_{t\phi}\Omega_F)]^2=E^2\rho_W^2(k_2+k_0k_4) \tag{2.26}$$

holds. We need only the Alfvén critical conditions (2.20) and (2.21) which result from the poloidal wind equation.

At the horizon ($\rho_W^2=0$) Eq. (2.14) reduces to the relation

$$-\left[\frac{k_0-M^2}{\sin\theta}\partial_\theta\Psi\right]_H=4\pi(r_H^2+a^2)[(E\eta)-\omega_H(L\eta)] , \tag{2.27}$$

where the subscript H means quantities evaluated at the horizon and r_H is the horizon radius. However, Eq. (2.24) also requires the derivative with respect to θ of Eq. (2.27).

Hence we find that the GS equation gives no additional condition for regularity both at the Alfvén point and at the horizon. In Sec. IV, we will discuss some effects due to the existence of the Alfvén point and the horizon by using these conditions.

III. STAGNATION REGION

Many out-going jets observed in active astrophysical phenomena are believed to be generated from magnetospheres surrounding compact objects. Here we consider a black-hole magnetosphere. The whole region of the black-hole magnetosphere cannot be occupied only by the outflow region because of the existence of the horizon. There exists some boundary region where the inflows are separated from the outflows and the poloidal velocity of plasma particles becomes very low, and we call this the stagnation region.

The accretion flows toward a black hole start from the stagnation region with very low-poloidal velocity. Here plasma particles can move only in the toroidal direction. Let us study the magnetospheric structure in the low-poloidal flow limit ($u_P^2\ll 1$, $M^2\ll 1$). In this limit, the field line geometry can be determined by the poloidal wind equation which reduces to

$$\left[\frac{e}{\mu}\right]^2=k_0(r,\theta) . \tag{3.1}$$

In the asymptotic region where $z\to\infty$ and R ($\equiv r\sin\theta$) remains finite, we assume a uniform magnetic field

$$\lim_{z\to\infty} B_z=\frac{\partial_R\Psi}{R}=B_0=\text{const} , \tag{3.2}$$

which means

$$\lim_{z\to\infty}\Psi=\frac{R^2}{2}B_0 . \tag{3.3}$$

By noting that the function k_0 has the form $k_0=1-R^2\Omega_F^2$ in the asymptotic region (except the polar region where the inequality $R^2\Omega_F^2\gg m/r$ does not hold), we can give the function $e=e(\Psi)$ (which is valid in the whole region of the magnetosphere) as

$$\left[\frac{e}{\mu}\right]^2=1-\frac{\Psi}{\Psi_0} , \tag{3.4}$$

where

$$\Psi_0\equiv\frac{B_0}{2\Omega_F^2} . \tag{3.5}$$

This leads to the flux function

$$\Psi(r,\theta)=\Psi_0[1-k_0(r,\theta)] , \tag{3.6}$$

which gives the shape of field lines ($\Psi=$const) in the stagnation region.

Now the role of the GS equation (2.24) is to give the proper number density of plasma particles. The left-hand side of Eq. (2.24) describes electromagnetic forces which work in the transfield direction and balance with inertial forces in the right-hand side. In the low-poloidal flow limit ($M^2\ll 1$), the terms with the factor M^{-2} become dominant in the right-hand side. Then, by virtue of Eq. (2.17), we find

$$n(r,\theta)=\frac{-B_0k_0}{4\pi\mu\rho_W^2\Omega_F^2}\times\left[g^{rr}\partial_r(k_0\partial_r\Psi)+\sin\theta g^{\theta\theta}\partial_\theta\left[\frac{k_0}{\sin\theta}\partial_\theta\Psi\right]\right]\times\left[1+\frac{B_0\Omega_F'}{\Omega_F^3}(1-g_{tt}-g_{t\phi}\Omega_F)\right]^{-1} . \tag{3.7}$$

If we set $\Omega_F=$const, then Eq. (3.7) reduces to

$$n(r,\theta)=\frac{B_0^2k_0}{8\pi\mu\rho_W^2\Omega_F^4}\times\left[g^{rr}\partial_r(k_0\partial_rk_0)+\sin\theta g^{\theta\theta}\partial_\theta\left[\frac{k_0}{\sin\theta}\partial_\theta k_0\right]\right] , \tag{3.8}$$

for which the shape of field lines and the line $n=0$ are shown in Fig. 1. The solution obtained here is physically meaningful only in the region where $n>0$. We find such a region (the dotted region in Fig. 1) between two light surfaces. In this stagnation region, asymptotically uniform magnetic field lines (the solid lines in Fig. 1) are considerably bent because of the balance between inertial forces and electromagnetic forces in Kerr background geometry. The very similar shape of field lines was numerically obtained by Wilson [8]. Notice that such a field line geometry is due to the condition for the quasiequilibrium of matter. In the force-free assumption, Blandford and Znajek [1] solved the GS equation and obtained the solutions with a monopole or paraboloidal geometry. We find that the effect of matter which is neglected in the force-free limit is very important for determining the global shape of field lines.

It is clear that the stagnation region cannot occupy any regions near the horizon and the polar axis where the

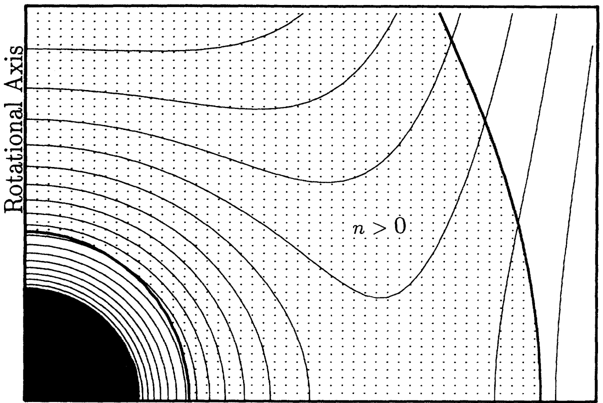

FIG. 1. Poloidal magnetic-field lines and density contour in the low-poloidal flow limit ($m\omega_H = m\Omega_F = 0.06$). The vertical axis is parallel to the spin axis of the black hole and the magnetosphere. The horizontal axis denotes the equatorial plane. The thin solid lines denote the field lines, and the dotted region corresponds to the region $n > 0$, with the boundary $n = 0$ shown by the thick solid lines. The shape of the field lines becomes unphysical near the polar region and near the horizon. These properties do not change even if $\omega_H \gg \Omega_F$ or $\omega_H \ll \Omega_F$.

plasma particles cannot be sustained by the electromagnetic forces (see Fig. 1). We observe the unphysical results that the magnetic-field lines do not thread the horizon and do cross with the polar axis. Then the proper number density n becomes negative near the horizon and the toroidal current density diverges at the polar axis. [Note that we cannot use Eq. (3.6) in the polar region where $R^2\Omega_F^2 \leq m/r$.] In these regions, poloidal plasma flows must occur to deform the field line geometry shown in Fig. 1.

The inner part of the stagnation region is bounded by the line $n = 0$ (located at a distance of several times the horizon radius r_H) and the polar region. These boundaries can be understood as the "plasma horizon" pointed out by Damour *et al.* [9]. Their discussion was based on the condition for the electromagnetic force balance $\mathbf{E}\cdot(\mathbf{E}+\mathbf{V}\times\mathbf{B})=0$ for a test particle where **E**, **B**, and **V** are the electric field, the magnetic field, and the velocity of test particle, respectively. Our result is an extension of their work to the MHD system. Notice that the outer boundary of the stagnation region also exists. This is due to the rotation of field lines and is characteristic of our MHD treatment.

The magnetic-field lines in the low-poloidal flow limit are approximately given by $k_0 = \text{const}$, because these lines are just "equipotential" lines along which the flow acceleration is not effective. As was discussed by Blandford and Payne [10], the plasma inflows with very low velocity will be generated in a region near (and inside) the surface $\partial_r k_0 = 0$, which corresponds to a local maximum of the potential k_0 along a radial direction. Except near the polar region, the maximum surface exists at the distance

$$r = r_S = \left[(1 - a\Omega_F)^2 \frac{m}{\Omega_F^2} \right]^{1/3} . \tag{3.9}$$

Then, from Eqs. (3.7), (2.17), and (2.18), we can estimate in order of magnitude of the Alfvén Mach number M^2 and the poloidal velocity u_P^2 in the stagnation region near the surface $r = r_S$,

$$M^2 \sim \frac{\mu^2\eta^2}{\Psi_0^2}\frac{r_S^5}{m} \sim \frac{\mu^2\eta^2}{\Psi_0^2\Omega_F^4}(1-a\Omega_F)^{10/3}(m\Omega_F)^{2/3} , \tag{3.10}$$

$$u_P^2 \sim \frac{B_P^2 M^4}{(g_{tt} + g_{t\phi}\Omega_F)^2(4\pi\mu\eta)^2} \sim \frac{\mu^2\eta^2}{\Psi_0^2\Omega_F^4}(1-a\Omega_F)^{8/3}(m\Omega_F)^{4/3} . \tag{3.11}$$

Some mechanism which leads to the usual result that $\Omega_F \sim \omega_H$ has been discussed (e.g., see Phinney [2]). However, we have still a controversy [11] over the validity of such a mechanism in terms of the consistency with MHD causality. Physical conditions in the stagnation region may be important for determining the value of Ω_F. In this paper we do not pursue this problem, and we treat Ω_F as a free parameter independent of ω_H. Nevertheless, we must recall that any accretion flows in the black-hole magnetosphere start from the region $k_0 > 0$. For example, on the equatorial plane, k_0 can become positive only if $\epsilon < \frac{1}{2}$ for $a = m$ and $\epsilon < 1/\sqrt{27}$ for $a = 0$, where $\epsilon \equiv m\Omega_F$. Hence we assume that

$$\epsilon \; (\equiv m\Omega_F) \ll 1 . \tag{3.12}$$

This means that the stagnation region can occupy a large part of the "slowly rotating" magnetosphere (i.e., the surface $r = r_S$ is very remote from the horizon).

Our additional assumption is

$$\left[\frac{\mu\eta}{\Psi_0\Omega_F^2} \right]^2 \sim 1 , \tag{3.13}$$

which gives the consistent results that $u_P^2 \sim M^4$ ($\sim \epsilon^{4/3}$) and $1 - k_0 \sim M^2$ ($\sim \epsilon^{2/3}$) in order of magnitude. Equation (3.3) obtained in the low-poloidal velocity limit from the poloidal wind equation (2.14) turns out to be valid up to the order of M^2.

It is interesting to note that the flows in the magnetosphere considered here are neither magnetically dominated nor particle dominated. In fact, from Eq. (3.13), we find

$$\eta \sim \frac{B_0}{\mu} , \tag{3.14}$$

which, by virtue of Eq. (2.3), leads to the result

$$\left| \frac{\gamma n \mu \, dr/dt}{cB_0 \partial_\theta \Psi / \sqrt{-g}} \right| \sim 1 , \tag{3.15}$$

where γ is the Lorentz factor. The left-hand side of Eq. (3.15) can be regarded as the degree of magnetization of flows (i.e., the ratio of particle energy flux to electromagnetic energy flux). It is clear that Eq. (3.13) is a plausible assumption because we are mainly interested in the effects due to matter accretion on the magnetospheric structure.

IV. ACCRETION FLOWS FROM STAGNATION REGION

At the stagnation region ($u_P^2 \ll M^2 \ll 1$) the poloidal wind equation was approximated by Eq. (3.1), while in a region near the inner boundary accretion flows may be accelerated to the velocity $u_P^2 \sim M^2$. Then it should be modified to the form (keeping the terms up to the order of M^2)

$$(1+u_P^2)k_0 = \left[\frac{e}{\mu}\right]^2 = 1 - \frac{\Psi}{\Psi_0} . \qquad (4.1)$$

In Eq. (4.1) k_0 is larger than M^2; hence, the flows are still sub-Alfvénic. Equation (4.1) represents the effect of plasma flows on the shape of field lines in the sub-Alfvénic region. Along a field line Ψ=const the function k_0 decreases as u_P^2 increases. The plasma acceleration makes the field line turn in the direction of the inner light surface $r_L^{\rm in}$, where $k_0=0$, which is close to the ergosurface $g_{tt}=0$ in the case $m\Omega_F \ll 1$ (see Fig. 2). This effect will be amplified as the plasma acceleration goes on.

Then, the magnetic field lines that originate from the stagnation region may be able to cross the inner light surface and finally reach the horizon. However, this is possible only when the flows satisfy the critical conditions at the Alfvén and fast magnetosonic points and the boundary condition at the horizon. These conditions will put some constraints on the values of Ψ, u_P^2, M^2, etc., along the field lines threading the horizon. In this section we will estimate in order of magnitude these quantities to reveal the magnetospheric structure in the super-Alfvénic region. Our approach is based on the assumptions (3.12) and (3.13). The conditions (2.20) and (2.21) that result from the poloidal wind equation or the GS equation are used to estimate the order of a physical quantity A in a power of ϵ like $A \sim \pm\epsilon^\alpha$ (including the sign of A).

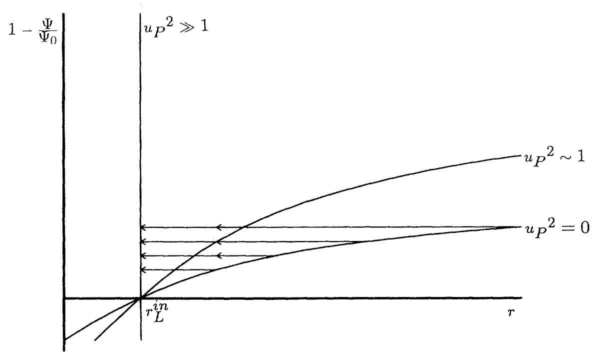

FIG. 2. Schematic picture of effects of inflows in the sub-Alfvénic region. The vertical axis denotes $1-\Psi/\Psi_0$ and the horizontal axis denotes the distance r from the black hole. In the low-poloidal flow limit, roughly speaking, the function $1-\Psi/\Psi_0$ has a shape determined by $u_P^2=0$. When the flows are accelerated to the velocity $u_P^2 \sim 1$ or $u_P^2 \gg 1$, the shape of field lines is changed. Because flows are fixed in the line of Ψ=const, the flow lines or the field lines must bend inward.

We are interested in how the field line configuration depends on the angular velocity ω_H of the black hole. Hence we denote the order of the Kerr spin parameter a as $a\Omega_F \sim \epsilon^\gamma$. The accretion flows must pass through the critical points (the Alfvén point and the fast magnetosonic point) to reach the horizon. We denote the order of the Alfvén Mach number at the Alfvén point as $M_A^2 \sim \epsilon^\alpha$ (i.e., the Alfvén point is located at the region where $k_0 \sim \epsilon^\alpha$). The stream lines start from the stagnation region where Eqs. (3.4) and (3.6) hold. Then we have the flux function Ψ and the conserved quantity e satisfying the relations that $\Psi < \Psi_0$ and $e \sim +\mu$. Some part of the filed lines may not thread the horizon. We specify the field lines threading the horizon by denoting the flux function as $\Psi \sim \Psi_0 \epsilon^\mu$. If the field lines happen to enter into the polar region of the black hole, we must evaluate the order of the polar angles θ_A and θ_H at the Alfvén point and the horizon. The Alfvén point is very close to the horizon in the polar region; hence, it is plausible to assume $\theta_A \sim \theta_H$ and we denote their order as $\sin^2\theta_A \sim \sin^2\theta_H \sim \epsilon^\lambda$. Now we show how μ,α,λ depend on γ.

First let us consider the case for a rapidly rotating black hole $\omega_H \gg \Omega_F$, which corresponds to the range $1 \le \gamma < 2$. We start with the analysis of the Alfvén critical conditions (2.20) and (2.21), in which k_0 consists of three terms. The orders of $2g_{t\phi}\Omega_F$ and $g_{\phi\phi}\Omega_F^2$ are $\epsilon^{\gamma+\lambda}$ and $\epsilon^{2+\lambda}$, respectively, but the order of g_{tt} cannot be fixed as yet. Hence we must consider the following cases: (i) $\alpha < \gamma+\lambda$, (ii) $\alpha = \gamma+\lambda$, (iii) $\gamma+\lambda < \alpha$.

For case (i), we obtain in order of magnitude $M^2 \sim g_{tt}$, which leads to $g_{tt}+g_{t\phi}\Omega_F \sim +\epsilon^\alpha$, $\rho_W^2\Omega_F^2 \sim +\epsilon^{2(\gamma+\lambda)}$ or $+\epsilon^{\alpha+\lambda+2}$. Then the orders of E/μ and $L\Omega_F/\mu$ become $+1$ and $-\epsilon^{\gamma+\lambda-\alpha}$, respectively, from Eqs. (2.22) and (2.23), and finally we obtain $k_2/k_0 \sim \epsilon^{-\alpha}$. On the other hand, from the definition of k_4, we obtain $k_4 \sim -\epsilon^{-\alpha}$. Note that these results are consistent with condition (2.21). In this case the accretion flows can pass through the Alfvén critical point. However, the total energy E must be positive, and so no energy extraction occurs.

Through a similar discussion for case (ii), we can obtain the results that $g_{tt} \sim \pm\epsilon^{\gamma+\lambda}$, $g_{tt}+g_{t\phi}\Omega_F \sim \pm\epsilon^{\gamma+\lambda}$, $\rho_W^2\Omega_F^2 \sim +\epsilon^{2(\gamma+\lambda)}$, $E/\mu \sim \pm 1$, $L\Omega_F/\mu \sim -1$, and finally, $k_2/k_0 \sim \epsilon^{-(\gamma+\lambda)}$. From the definition of k_4, we obtain $k_4 \sim \pm\epsilon^{-(\gamma+\lambda)}$. In this case also, condition (2.21) can be satisfied if the Alfvén point exists in the region $k_4 < 0$. This case is astrophysically interesting, because the energy and angular momentum of the black hole can be extracted. (The accretion flows with $E<0$ are allowed.) For the case (iii) condition (2.21) is never satisfied. Therefore we do not consider this unphysical case.

From the above discussion we find the result $\alpha \le \gamma+\lambda$ for the Alfvén critical condition. Further it should be noted that the angular momentum L given on the field lines passing through the polar region will be proportional to $\sin^2\theta_A$. Then the ratio of $L\Omega_F/\mu$ ($\sim -\epsilon^{\gamma+\lambda-\alpha}$) to $\sin^2\theta_A$ ($\sim\epsilon^\lambda$) should not depend on the parameter λ. This means that $\gamma-\alpha$ does not depend on λ. Here we are interested in an active black hole from which the rotational energy can be extracted. Therefore we assume that $\alpha=\gamma$. This relation gives $E\sim L\Omega_F$ in order of magnitude, except on the field lines passing through the polar region. The negative-energy influx is impossible in the polar region [$\alpha<\gamma+\lambda$, i.e., case (i)], where the electromagnetic contribution to E becomes very small. Now we can estimate the poloidal velocity u_P at the Alfvén point. From Eq. (2.18) we have the result $u_P^2\sim\epsilon^{2\mu-4-2\lambda+\gamma}$ for $\lambda>2-\gamma$ ($\sin^2\theta\ll\Omega_F/\omega_H$) and $u_P^2\sim\epsilon^{2\mu-2-3\gamma}$ for $0\le\lambda<2-\gamma$ ($\sin^2\theta\ge\Omega_F/\omega_H$), while by analyzing the poloidal wind equation (2.14) in the limit $k_0\to M^2$ and $k_2+k_0k_4\to 0$ we have the result $u_P^2\sim\epsilon^{-\gamma}$ and $u_P^2\sim\epsilon^{2-2\gamma-\lambda}$, respectively. These two estimates become consistent only if

$$\mu=2-\gamma+\lambda\ . \tag{4.2}$$

This is an important result which determines the field lines threading the horizon.

Next let us discuss the case for a slowly rotating black hole $\omega_H\ll\Omega_F$ ($\gamma>2$), and analyze conditions (2.20) and (2.21) by noting that $g_{t\phi}\Omega_F\ll|g_{\phi\phi}\Omega_F^2|$. Now we can set up the following cases: (i) $\alpha<2+\lambda$, (ii) $\alpha=2+\lambda$, (iii) $2+\lambda<\alpha$.

We follow the same procedure as with the rapidly rotating case. For case (i), we can obtain the results that $g_{tt}\sim+\epsilon^{\alpha}$, $g_{tt}+g_{t\phi}\Omega_F\sim+\epsilon^{\alpha}$, $\rho_W^2\Omega_F^2\sim+\epsilon^{\alpha+\lambda+2}$, which leads to $E/\mu\sim+1$ and $L\Omega_F/\mu\sim+\epsilon^{2+\lambda-\alpha}$. Hence we obtain $k_2/k_0\sim\epsilon^{-\alpha}$, which is consistent with condition (2.21) because $k_4\sim-\epsilon^{-\alpha}$. In this case the energy influx must be positive.

Even for case (ii), which can be consistent with condition (2.21) because $k_2/k_0\sim+\epsilon^{2+\lambda}$ and $k_4\sim\pm\epsilon^{-(2+\lambda)}$, both E and L of the inflows become positive, i.e., $E/\mu\sim L\Omega_F/\mu\sim+1$. This is because the BZ mechanism does not work for the slowly rotating ($\omega_H<\Omega_F$) black holes.

Case (iii) is unphysical because the condition (2.21) is not satisfied. Then we obtain the result that $\alpha\le 2+\lambda$ for the Alfvén critical condition. In the same way as with the rapidly rotating case, let us require that $E\sim L\Omega_F/\sin^2\theta_A$, which leads to the relation $\alpha=2$. By calculating u_P^2 at the Alfvén point from Eqs. (2.14) and (2.18), we can derive the result

$$\mu=\lambda\ . \tag{4.3}$$

It should be noted that Eq. (4.2) changes smoothly to Eq. (4.3) when $\gamma=2$ (i.e., $\omega_H\sim\Omega_F$). In fact, this result is explicitly verified by the analysis of the case $\gamma=2$.

Before reaching the horizon, the super Alfvénic inflows must pass through the fast magnetosonic point too. However, we find no additional constraints from the calculation in order of magnitude at the critical point.

At the horizon, the GS equation and the poloidal wind equation reduce to the same equation (2.27), from which we can obtain the following results: When $\omega_H\gg\Omega_F$, we have $M_H^2\sim+\epsilon^{\gamma}$ for the polar region where $\sin^2\theta\ll\Omega_F/\omega_H$, and $M_H^2\le+\epsilon^{\lambda-2}$ for the other region where $\sin^2\theta\ge\Omega_F/\omega_H$. It will be plausible to choose $M_H^2\sim\epsilon^{\gamma}$ ($\ll\epsilon^{\lambda-2}$) throughout the horizon because $M_A^2\sim\epsilon^{\gamma}$ and the Alfvén point is close to the horizon (if $\gamma>1$). When $\omega_H\ll\Omega_F$, we have $M_H^2\sim+\epsilon^2$. Notice that these results are consistent with the results obtained at the Alfvén point if the Alfvén Mach number does not drastically change between the Alfvén point and the horizon.

By comparing Eq. (4.2) for the rapidly rotating case ($\omega_H\gg\Omega_F, 1\le\gamma<2$) and Eq. (4.3) for the slowly rotating case ($\omega_H\ll\Omega_F, 2<\gamma$), we find the interesting result that the flux function for the field lines threading the horizon depends crucially on the ratio ω_H/Ω_F:

$$\frac{\Psi}{\Psi_0}\sim\frac{\Omega_F}{\omega_H}\sin^2\theta_H \tag{4.4}$$

for $\omega_H\gg\Omega_F$ and

$$\frac{\Psi}{\Psi_0}\sim\sin^2\theta_H \tag{4.5}$$

for $\omega_H\ll\Omega_F$. From Eq. (4.4), the averaged poloidal magnetic-field strength at the horizon is estimated as $B_{P|H}\sim\Psi_0\Omega_F/(m^2\omega_H)\sim B_0\epsilon^{-\gamma}$. The black hole can amplify the magnetic field (always $B_{P|H}\gg B_0$), but its rotation suppresses this amplification (because $B_{P|H}$ decreases as γ decreases). Such a remarkable change of the magnetic field does not occur when $\omega_H\sim\Omega_F$ and $\omega_H\ll\Omega_F$.

Magnetic-field lines coil around the black hole due to a strong dragging of the inertial frame, and the toroidal component of the magnetic field is strengthened near the horizon. The toroidal magnetic field B_T near the horizon can be calculated from Eq. (2.11). At the horizon it become negative for the slowly rotating hole ($\omega_H<\Omega_F$), which generates a "leading" type of field line, while it becomes positive for the rapidly rotating hole ($\omega_H>\Omega_F$), which generates a "trailing" type of field line. The former corresponds to the injection of angular momentum into the horizon, and the extraction of angular momentum from the black hole occurs only for $\omega_H>\Omega_F$. Plasma particles have a tendency to comove with the inertial frame as a result of their inertia, so in the vicinity of the horizon, their angular velocity is forced to approach ω_H. However, they must be frozen on the field line rotating with angular velocity Ω_F. Then, if $\omega_H>\Omega_F$, the dragging of the inertial frame works to make the field lines of a trailing type through the plasma's inertia, and as a result the angular momentum of the black hole is transferred into the magnetic field. If $\omega_H<\Omega_F$, the dragging of the inertial frame extracts angular momentum from the magnetic field of a leading type

The analogous effect of dragging was discussed in the hydrodynamic limit and the force-free limit. Ruffini and Wilson [12] assumed the hydrodynamical pressureless limit, in which the fluid accretes freely along geodesics, and the process determining the field line geometry is dominated by the fluid motion. Their result corresponds to the case $\omega_H\gg\Omega_F$. Blandford and Znajek [1], in the force-free limit, found that the angular momentum extraction from the black hole and the toroidal magnetic field on the horizon are proportional to $\omega_H-\Omega_F$. Thus our result obtained here is consistent with these results in the different limits.

In summary, the Alfvén critical conditions (2.20) and (2.21) severely constrain the accretion flows onto the black hole, and the magnetic field on the horizon depends crucially on the ratio ω_H/Ω_F as a result of the effect of the plasma's inertia. Though the black hole amplifies the magnetic field by the effect of gravity, its rotation suppresses this amplification. When $\omega_H > \Omega_F$, the energy and angular momentum extraction due to the MHD accretion flows can occur except the polar region of the black hole ($\lambda > 0$). However, when $\omega_H < \Omega_F$ the plasma flows carry their energy and angular momentum into the black hole.

In the final section we will use these results to give a model of the magnetosphere and to discuss the evolution due to the energy extraction from the rapidly rotating hole.

V. MAGNETOSPHERIC STRUCTURE AND ENERGY EXTRACTION

Plasma particles have the angular velocity Ω_F in the stagnation region. The acceleration of the in-going velocity will be suppressed by a repulsive force due to the orbital motion (i.e., rotation in the same direction as the hole's spin). Then the plasma inflows generated in the equatorial region far distant from the black hole cannot be accelerated to the super-Alfvénic speed and cannot fall into the horizon. This tendency should be amplified as the hole's angular velocity ω_H becomes larger. In fact, we obtain the particle flux on the horizon as

$$\dot{N} \equiv \int n u^r \sqrt{-g}\, d\theta\, d\phi = 2\int \eta\, d\Psi \sim \frac{B_0^2}{\mu \Omega_F \omega_H} \ . \qquad (5.1)$$

Because the poloidal motion of plasma particles is along the poloidal field lines, for a rapidly rotating hole ($\omega_H \gg \Omega_F$) the magnetic flux threading the horizon must be suppressed, if compared with a slowly rotating hole ($\omega_H \ll \Omega_F$). Here we considered the magnetized inflows in which the electromagnetic part of E is not dominated by the matter part. Hence our results are due to a MHD effect which was neglected in the force-free limit.

We give a schematic shape of the field lines in Fig. 3, in which the field lines obtained in the low-poloidal flow limit (Fig. 1) are deformed by the effect of plasma flows. The stagnation region may be understood as a region where a magnetically supported plasma halo is formed (the matter is in a quasiequilibrium in the poloidal direction). We can attribute the global shape of the field lines to some interaction with a thin accretion disk near the equatorial plane. In the magnetosphere there exist two separate field lines corresponding to the same value of Ψ. These field lines will be connected inside the disk where the tangential component of the magnetic field is affected by the internal current. The accretion disk will extend over the outer light surface, and the location of its inner edge will change as a function of ω_H. In the case of a rapidly rotating hole [Fig. 3(a)], the field lines threading the horizon ($\Psi \sim \Psi_0 \epsilon^{2-\gamma} \ll \Psi_0$) are connected with the far distant region, and the field lines threading the accretion disk ($\Psi \sim \Psi_0$) form a loop structure such as a disk flare (the inner edge of the disk is very close to the horizon). In the case of a slowly rotating hole, including the case $\omega_H \sim \Omega_F$ [Fig. 3(b)], the inner edge is not so close to the horizon. The polar part of the field lines threading the horizon is connected with the far distant region, but the equatorial part is connected with the accretion disk. This is a characteristic difference of the magnetospheric structure between a rapidly rotating hole ($\omega_H \gg \Omega_F$) and a slowly rotating hole ($\omega_H \sim \Omega_F$ or $\omega_H \ll \Omega_F$).

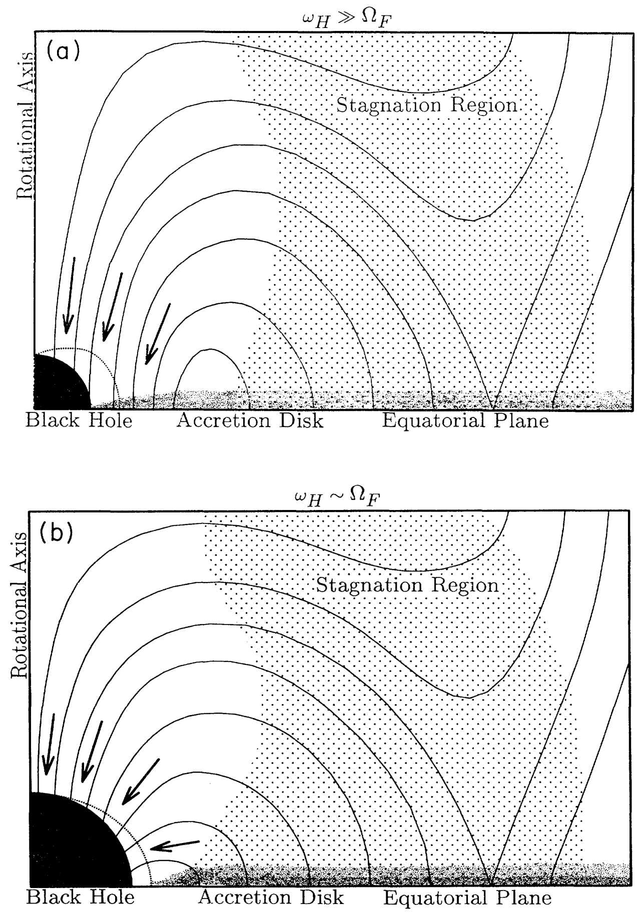

FIG. 3. Schematic figures of the magnetic-field lines to emphasize the difference between the case (a) $\omega_H \gg \Omega_F$ and the case (b) $\omega_H \sim \Omega_F$ or $\omega_H \ll \Omega_F$. Here the effect of plasma accretion flows is considered. Solid lines and the dotted line denote the magnetic-field lines and the Alfvén critical surface, respectively. The dotted region denotes the stagnation region. The shaded region near the equatorial plane denotes the thin accretion disk. The arrows denote the plasma accretion flows.

Next let us discuss the energetics of the black-hole magnetosphere. When the black hole rapidly rotates ($\omega_H \gg \Omega_F$), the energy extraction is still allowed even if some magnetohydrodynamic conditions discussed before are imposed. In this case, the angular momentum also can be extracted from the black hole. The total power of energy extraction from the black hole is given by

$$P_{\rm tot} \equiv -\dot{m} = -\int n u^r E \sqrt{-g}\, d\theta\, d\phi = -2\int E\eta\, d\Psi \ . \qquad (5.2)$$

The integration should be done on the horizon $\Delta = 0$, and the dot means the evolution rate. We can estimate $P_{\rm tot}$ by taking $E \sim -\mu$ and $\eta \sim$const in order of magnitude:

$$P_{\rm tot} \sim -E\eta \int d\Psi \sim \frac{B_0^2}{\Omega_F \omega_H} \ . \tag{5.3}$$

Remember that the total magnetic flux depends on the parameter Ω_F/ω_H. Therefore, the total power increases as the spin-down of the black hole proceeds. This increase of $P_{\rm tot}$ will stop when $\omega_H \sim \Omega_F$.

Our estimate of the power output is essentially analogous to the BZ mechanism [1], but in our discussion the poloidal magnetic flux at the horizon is obtained as a function of ω_H through the magnetohydrodynamic conditions. This clearly denotes the effect of MHD accretion flows.

We can estimate the time variation of the hole's angular momentum in the same way, and the result is, in order of magnitude,

$$\dot{J} \sim \dot{m}/\Omega_F \ . \tag{5.4}$$

This means the time variation of ω_H is

$$\dot{\omega}_H \sim -\frac{B_0^2}{m^3 \Omega_F^2} \frac{1}{\omega_H} \ . \tag{5.5}$$

Then, if initially the black hole has the mass $m = m_0$ and the angular momentum $J = J_0 \sim m_0^2$ (we consider a rapidly rotating Kerr black hole which has the spin parameter $a \sim m_0$), the time scales of the evolution of m and J are approximately given by

$$\tau_m \sim \frac{m_0}{-\dot{m}} \sim \frac{m_0 \Omega_F \omega_H}{B_0^2} \tag{5.6}$$

and

$$\tau_J \sim \frac{J_0}{-\dot{J}} \sim \epsilon \tau_m \ll \tau_m \ . \tag{5.7}$$

Hence, in our estimation the mass evolution is negligible during the extraction of the rotational energy.

In this paper we have discussed the black-hole magnetosphere by taking Ω_F as a parameter independent of ω_H. Now we also consider an evolutionary model in which Ω_F does not change so much. Then we solve Eq. (5.5) to show explicitly the time variation of ω_H:

$$\omega_H \sim \frac{1}{m} \left[1 - \frac{\zeta B_0^2}{m \Omega_F^2} t \right]^{1/2} , \tag{5.8}$$

where ζ is some undetermined constant of the order of unity and t is the observer's time. By substituting Eq. (5.8) into Eq. (5.3), we obtain the power output

$$P_{\rm tot} \sim \frac{m B_0^2}{\Omega_F} \frac{1}{[1 - (\zeta B_0^2 / m \Omega_F^2) t]^{1/2}} \ . \tag{5.9}$$

In the initial stage, the black hole is rapidly rotating ($\omega_H \gg \Omega_F$) and the MHD inflows can extract its energy and angular momentum. However, the power of the extraction is very low because the magnetic flux into the horizon is suppressed by the rapid rotation, as was shown in the previous section. The field lines threading the horizon increase as the spin down of the black hole goes on, so the power output increases. When ω_H decreases to a value somewhat larger than Ω_F, the energy extraction produces the maximum power which will be equal to $1/\epsilon$ times the initial power. This increase of $P_{\rm tot}$ occurs in time scale given by $\tau_{\rm ex} \sim m^3 \Omega_F^4 / B_0^2$, which is very short if compared with the total lifetime $\tau \sim m \Omega_F^2 / B_0^2$. We will observe that the black-hole magnetosphere becomes explosively active at this stage (explosive stage). In the later evolution when $\omega_H \to \Omega_F$, we cannot apply the estimate obtained here. The power output will decease rapidly. If we use characteristic values for AGN's, we obtain

$$P_{\rm tot} \sim \frac{G^2}{\mu_0 c^3} \frac{m^2 B_0^2}{G m \Omega_F / c^3}$$
$$\sim 10^{39} \left[\frac{0.1}{G m \Omega_F / c^3} \right] \left[\frac{m}{10^8 M_\odot} \right]^2 \left[\frac{B_0}{1\ \rm T} \right]^2 (\rm W) \tag{5.10}$$

at the initial stage where μ_0 is magnetic permeability of vacuum, and

$$P_{\rm tot} \sim \frac{G^2}{\mu_0 c^3} \frac{m^2 B_0^2}{(G m \Omega_F / c^3)^2}$$
$$\sim 10^{40} \left[\frac{0.1}{G m \Omega_F / c^3} \right]^2 \left[\frac{m}{10^8 M_\odot} \right]^2 \left[\frac{B_0}{1\ \rm T} \right]^2 (\rm W) \tag{5.11}$$

at the explosive stage. The lifetime τ ($\sim \tau_{\rm ex}/\epsilon^2$) is

$$\tau \sim \frac{\mu_0 c^5}{G^2} \frac{(G m \Omega_F / c^3)^2}{m B_0^2}$$
$$\sim 10^9 \left[\frac{G m \Omega_F / c^3}{0.1} \right]^2 \left[\frac{10^8 M_\odot}{m} \right] \left[\frac{1\ \rm T}{B_0} \right]^2 (\rm yr) \tag{5.12}$$

(evolution does not reach the stage of $\omega_H < \Omega_F$, because if $\omega_H < \Omega_F$, energy and angular momentum are injected with flows so the black hole should spin up).

We want to emphasize that the maximum activity of the black-hole magnetosphere can appear in the middle stage of the evolution if initially the black hole is rapidly rotating ($\omega_H \gg \Omega_F$). This will be clearly observed under the situation that $\epsilon \equiv m \Omega_F \ll 1$. The mechanism is essentially a result of the MHD interaction between the magnetic field and the hole's rotation. Hence we can expect that such a property of evolution will be preserved even when ϵ is not so small. This point, however, must be investigated in future work.

APPENDIX: THE GS EQUATION

We derive a useful form of the general-relativistic GS equation from the steady and axisymmetric MHD equations. (To our knowledge, the GS equation for Kerr space-time has never been given explicitly in any literature. For the Schwarzschild space-time, see Mobarry and Lovelace [13], and for the Minkowski space-time, see

Camenzind [14]). The basic equations are summarized as follows.

The particle conservation equation is

$$N^{\alpha}{}_{;\alpha}=0 , \tag{A1}$$

where

$$N^{\alpha}=nu^{\alpha} , \tag{A2}$$

with n as the proper number density.

The momentum conservation equation is

$$T^{\alpha\beta}{}_{;\beta}=0 , \tag{A3}$$

where

$$T^{\alpha\beta}=T^{\alpha\beta}_{\rm em}+T^{\alpha\beta}_{M} , \tag{A4}$$

and the electromagnetic part is

$$T^{\alpha\beta}_{\rm em}=\frac{1}{4\pi}(F^{\alpha\delta}F_{\delta}{}^{\beta}+\tfrac{1}{4}g^{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}) , \tag{A5}$$

and the matter part is

$$T^{\alpha\beta}_{M}=(\rho+P)u^{\alpha}u^{\beta}-Pg^{\alpha\beta} . \tag{A6}$$

In the following we take the cold limit ($P=0,\rho=\mu n$).

The frozen-in condition is given by

$$u^{\beta}F_{\alpha\beta}=0 , \tag{A7}$$

in addition to the Maxwell equation

$$F_{[\alpha\beta,\gamma]}=0 \tag{A8}$$

and

$$F^{\alpha\beta}{}_{;\beta}=-4\pi j^{\alpha} . \tag{A9}$$

From these equations, the momentum equation in the cold limit reduces to

$$\mu n u^{\beta}u^{\alpha}{}_{;\beta}-F^{\alpha\beta}j_{\beta}=0 . \tag{A10}$$

The GS equation describes a force balance in the transfield directions. The transfield component of the momentum equation is given by

$$\frac{F^{A}{}_{\phi}}{\tilde{B}_P^2}(\mu n u^{\beta}u_{A;\beta}-F_{A\beta}j^{\beta})=0 , \tag{A11}$$

and is rewritten in the form

$$\frac{F^{A}{}_{\phi}}{\tilde{B}_P^2}\{\mu n[-u^{B}(u_{B;A}-u_{A;B})-u^{t}\partial_A u_t-u^{\phi}\partial_A u_{\phi}] -F_{AB}j^{B}-F_{A\phi}j^{\phi}-F_{At}j^{t}\}=0 , \tag{A12}$$

where $\tilde{B}_P^2\equiv F_{B\phi}F^{B}{}_{\phi}$ ($A,B=r,\theta$). We can pick up the toroidal current density j^{ϕ} from each term in the left-hand side of Eq. (A12) as

$$-\frac{\mu n F^{A}{}_{\phi}u^{B}(u_{B;A}-u_{A;B})}{\tilde{B}_P^2} =\frac{M^2}{g_{tt}+g_{t\phi}\Omega_F}j^{\phi}+\frac{\mu\eta B_P^2}{g_{tt}+g_{t\phi}\Omega_F}\partial_{\Psi}\left[\frac{\eta}{(g_{tt}+g_{t\phi}\Omega_F)n}\right] , \tag{A13}$$

$$-\frac{\mu n}{\tilde{B}_P^2}F^{A}{}_{\phi}(u^{t}\partial_A u_t+u^{\phi}\partial_A u_{\phi})=-\mu n(u^{t}\partial_{\Psi}u_t+u^{\phi}\partial_{\Psi}u_{\phi}) , \tag{A14}$$

$$-\frac{F^{A}{}_{\phi}}{\tilde{B}_P^2}(F_{AB}j^{B}+F_{A\phi}j^{\phi})=-j^{\phi}-\frac{F_{r\theta}}{4\pi\sqrt{-g}}\partial_{\Psi}(\sqrt{-g}F^{r\theta}) , \tag{A15}$$

$$-\frac{F^{A}{}_{\phi}}{\tilde{B}_P^2}F_{At}j^{t}=-\Omega_F\frac{g_{t\phi}+g_{\phi\phi}\Omega_F}{g_{tt}+g_{t\phi}\Omega_F}j^{\phi} -\frac{\Omega_F}{4\pi}\frac{B_P^2}{g_{tt}+g_{t\phi}\Omega_F}\partial_{\Psi}\left[\frac{g_{t\phi}+g_{\phi\phi}\Omega_F}{g_{tt}+g_{t\phi}\Omega_F}\right] , \tag{A16}$$

where $\partial_{\Psi}\equiv(F^{A}{}_{\phi}/\tilde{B}_P^2)\partial_A$.

Consequently, Eq. (A12) is rewritten as the equation for j^{ϕ}:

$$\frac{k_0-M^2}{g_{tt}+g_{t\phi}\Omega_F}j^{\phi}=-\mu n(u^{t}\partial_{\Psi}u_t+u^{\phi}\partial_{\Psi}u_{\phi}) -\frac{F_{r\theta}}{4\pi\sqrt{-g}}\partial_{\Psi}(\sqrt{-g}F^{r\theta}) +\frac{\mu\eta B_P^2}{g_{tt}+g_{t\phi}\Omega_F}\partial_{\Psi}\left[\frac{\eta}{(g_{tt}+g_{t\phi}\Omega_F)n}\right] -\frac{\Omega_F}{4\pi}\frac{B_P^2}{g_{tt}+g_{t\phi}\Omega_F}\partial_{\Psi}\left[\frac{g_{t\phi}+g_{\phi\phi}\Omega_F}{g_{tt}+g_{t\phi}\Omega_F}\right] . \tag{A17}$$

By using the relations

$$\partial_{\Psi}e=\mu(\partial_{\Psi}u_t+\Omega_F\partial_{\Psi}u_{\phi}+u_{\phi}\partial\Omega_F) \tag{A18}$$

and

$$\partial_{\Psi}L=-\mu\partial_{\Psi}u_{\phi}+\frac{1}{4\pi\eta}\partial_{\Psi}\sqrt{-g}F^{r\theta}+\frac{\sqrt{-g}}{4\pi}F^{r\theta}\partial_{\Psi}\left[\frac{1}{\eta}\right] , \tag{A19}$$

we can arrive at the final form

$$\frac{k_0-M^2}{g_{tt}+g_{t\phi}\Omega_F}j^\phi=-nu^t\partial_\Psi e+\mu nu^t u_\phi\partial_\Psi\Omega_F+\frac{\eta B_\phi}{\rho_W^2}\partial_\Psi L+\frac{\eta B_\phi^2}{4\pi\rho_W^2}\partial_\Psi\left[\frac{1}{\eta}\right]+\frac{\mu\eta B_P^2}{g_{tt}+g_{t\phi}\Omega_F}\partial_\Psi\left[\frac{\eta}{(g_{tt}+g_{t\phi}\Omega_F)n}\right]-\frac{\Omega_F}{4\pi}\frac{B_P^2}{g_{tt}+g_{t\phi}\Omega_F}\partial_\Psi\left[\frac{g_{t\phi}+g_{\phi\phi}\Omega_F}{g_{tt}+g_{t\phi}\Omega_F}\right]. \tag{A20}$$

Because the Maxwell equation gives

$$j^\phi=\frac{-1}{4\pi\sqrt{-g}}\left[\partial_r\left[\frac{\sqrt{-g}}{g_{rr}}\frac{g_{tt}+g_{t\phi}\Omega_F}{\rho_W^2}(\partial_r\Psi)\right]+\partial_\theta\left[\frac{\sqrt{-g}}{g_{\theta\theta}}\frac{g_{tt}+g_{t\phi}\Omega_F}{\rho_W^2}(\partial_\theta\Psi)\right]\right], \tag{A21}$$

we can eliminate j^ϕ from Eqs. (A20) and (A21) to obtain the GS equation in the cold limit. Equation (2.24), shown in the text, is derived by using relations (2.9)–(2.11).

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