Relativistic constituent-quark model of nucleon form factors

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We explore the electromagnetic properties of relativistic constituent-quark models of the proton and neutron, in particular their dependence on the constituent-quark mass and the confinement scale. Relativistic effects are never negligible in any model which fits the charge radius of the proton. For a fixed charge radius the confinement scale decreases with decreasing quark masses. Nonvanishing Pauli moments of the constituent quarks are needed to fit the magnetic moments for any value of the quark mass and confinement scale. It is possible to describe existing form-factor data at least up to momentum transfers $Q^2=6$ GeV^2 with quark masses significantly smaller than the conventional nonrelativistic choice of about one-third of the nucleon mass.

I. INTRODUCTION

The purpose of this paper is to present the results of exploratory calculations of the electromagnetic form factors of Poincaré-covariant constituent-quark models. Nonrelativistic constituent-quark models describe mesons or baryons as bound states of two or three constituents. All other degrees of freedom are absorbed into the constituent quarks. Such models have been quite successful in describing mass spectra [1–3]. Even for low values of the momentum transfer, the use of nonrelativistic quantum mechanics in extracting electromagnetic properties from such models is inconsistent in principle when the mass of the constituent is not large compared to the reciprocal confinement scale. However, the bound-state wave functions can always be interpreted as eigenfunctions of a Poincaré-invariant mass operator. For both relativistic and nonrelativistic systems the little group is SU(2), and the components of the total spin are the generators [4]. It is necessary and sufficient that the internal Hamiltonian (mass operator) be invariant under this group. The use of eigenfunctions of the spin and mass operators for the calculation of form factors requires an extension to eigenfunctions of a four-momentum operator. It is necessary that the current density operators and the eigenfunctions of the four-momentum functions transform consistently under the unitary representations of the Poincaré group. This extension of eigenfunctions of the spin and mass operators to eigenfunctions of the four-momentum implies the choice of a "form of dynamics [5]". This choice fixes the kinematic subgroup of the Poincaré group. Light-front Hamiltonian dynamics, [6–8] where the kinematic subgroup is the symmetry group of a null plane, has the unique advantage that the relevant components of one-body currents transform consistently. Relativistic constituent-quark models so formulated can account for the observed pion form factor for both low and high momentum transfer [9].

It is important not to confuse the wave functions representing state vectors in a Hilbert space of three constituent quarks with covariant functions defined as solutions of wave equations, or as matrix elements of covariant local Heisenberg fields, e.g., Bethe-Salpeter amplitudes [7,8,10]. The relationship of the Poincaré-invariant quantum mechanics of confined constituent quarks to a local Lagrangian field theory [11] is nontrivial and outside the scope of this paper.

At the outset it is not clear that a simple constituent-quark model can or should account for the electromagnetic structure of nucleons. For low momentum transfer we might expect the meson cloud of the nucleons to play an essential role [12] and for sufficiently high momentum transfer the elastic form factors are determined by the valence amplitude of the current-quark Fock-space wave function. In that limit perturbative QCD predicts the Q^2 dependence of the form factors [13,14]. A successful parametrization of existing data has been obtained by an interpolation between these extremes [15]. There is, however, some reason to doubt the applicability of perturbative QCD to exclusive processes in any experimentally accessible region [16].

The purpose of the present study is not to advocate the best nucleon model, but to explore the quantitative features of exactly Poincaré-covariant constituent-quark models. In this exploration we will use drastic simplifications in the model specifications while insisting on exact Poincaré covariance which implies that the nucleon states must be represented by eigenfunctions of the spin operator $\vec{j}^{\,2}$. In previous work [17,18] this requirement was realized only in a "weak-binding approximation" of questionable validity.

The main parameters of the models are a confinement scale $1/\alpha$ and the mass m_q of the constituent quarks. The nonrelativistic limit obtains for $\alpha/m_q \to 0$. The requirement of Galilean invariance and nonrelativistic expressions for the magnetic moments determine the conventional choice of about one-third of the nucleon mass for m_q. These constraints are absent in relativistic models. The main purpose of our investigation is to determine whether exactly Poincaré-invariant constituent-quark models can reproduce realistic nucleon form factors for Q^2 of several GeV^2. Three-quark nucleon wave functions must be antisymmetric in color and have isospin equal to $\frac{1}{2}$. It follows that the dependence on the internal momenta and spin must be of mixed symmetry. We restrict our calculations to wave functions which are of mixed symmetry in the canonical spin variables, and symmetric in the internal momenta. The dependence of the wave function on the spin and isospin variables is then unambiguously determined by the symmetry character. This restriction to functions symmetric in the internal momenta is an oversimplification which implies the absence of tensor forces. It should be removed in more realistic models. We further restrict the dependence on the internal momenta to a set of representative functions depending on only one parameter which fixes the confinement scale.

The assumption of a constituent-quark model implies that the effects of other degrees of freedom are represented by properties of the constituent quarks, in particular nontrivial form factors [19]. Consideration of meson theories would lead to the expectation that reduction of the electromagnetic structure of the nucleon to three constituent quarks is either not possible or would lead to constituent quarks of rather large size. Here we explore the consequences of the extreme assumption that the constituent quarks are sufficiently small, so that their form factors are effectively constant for $0 \le Q^2 \le 6$ GeV^2. Attempts to extend the model to higher Q^2 would involve more sophisticated wave functions and appropriate assumptions for the Q^2 dependence of quark form factors. Such attempts are beyond the scope of this paper.

This paper is organized as follows. The general form of the matrix elements of currents and their relation to form factors is presented in Sec. II. In Sec. III Poincaré-covariant three-quark wave functions of the proton and neutron are constructed explicitly. The formal basis for the computation of form factors can be found in Sec. IV. The numerical results are presented in Sec. V. A summary of the results and outlook can be found in Sec. VI.

II. FORM FACTORS AND CURRENT OPERATORS

Elastic form factors are related to the matrix elements of the current density operators

$$\langle \tau,\lambda',j,P'|I^{\mu}(x)|P,j,\lambda,\tau\rangle \,, \tag{2.1}$$

by the requirements of Lorentz covariance

$$U^{\dagger}(\Lambda)I^{\mu}(x)U(\Lambda)=\Lambda^{\mu}{}_{\nu}I^{\nu}(\Lambda^{-1}x) \,, \tag{2.2}$$

$$U(\Lambda)|P,j,\lambda,\tau\rangle=|\Lambda P,j,\bar{\lambda},\tau\rangle\langle\bar{\lambda}|\mathcal{R}_W[\Lambda,L(P)]|\lambda\rangle \,, \tag{2.3}$$

and current conservation

$$\begin{aligned}\partial_{\nu}I^{\nu}(x)&=[P_{\nu},I^{\nu}(x)]\\&=0\\&\rightarrow(P'_{\mu}-P_{\mu})\langle\tau,\lambda',j,P'|I^{\mu}(0)|P,j,\lambda,\tau\rangle=0 \,.\end{aligned} \tag{2.4}$$

The variables λ and τ are eigenvalues of spin and isospin components. We assume the covariant normalization of the states specified by

$$\begin{aligned}\langle\tau',\lambda',j',P'|P,j,\lambda,\tau\rangle\\=\delta^4(P'-P)2\delta(P^2+M^2)\theta(P^0)\delta_{j',j}\delta_{\lambda',\lambda}\delta_{\tau',\tau} \,.\end{aligned} \tag{2.5}$$

The states $|P,j,\lambda,\tau\rangle$ are eigenstates of the four-momentum P; that is, they are eigenstates of a mass operator and three kinematic components of P. The choice of the null-plane dynamics associated with the null vector n implies that the components $P^+\equiv n\cdot P$, and P_T are separately covariant under the kinematic Lorentz transformations. We will denote null-plane three-vectors by boldface type, e.g., $\mathbf{P}=\{P^+,P_T\}$. It will be convenient to use states

$$|\mathbf{P},j,\lambda,\tau\rangle\equiv|P,j,\lambda,\tau\rangle\sqrt{P^+} \,, \tag{2.6}$$

which are normalized according to

$$\langle\tau',\lambda',j',\mathbf{P}'|\mathbf{P},j,\lambda,\tau\rangle=\delta^3(\mathbf{P}'-\mathbf{P})\delta_{j',j}\delta_{\lambda',\lambda}\delta_{\tau',\tau} \,. \tag{2.7}$$

The boost $L(P)$ transforms the four-momentum P to rest $L(P)P=\{M,0,0,0\}$ and relates the spin operator $\vec{j}$ to the Pauli-Lubanski vector W:

$$\{0,\vec{j}\}:=L(P)W/\sqrt{-P^2} \,. \tag{2.8}$$

Under Lorentz transformations the spin operator undergoes Wigner rotations,

$$U^{\dagger}(\Lambda)\vec{j}U(\Lambda)=\mathcal{R}_W[\Lambda,L(P)]\vec{j} \,, \tag{2.9}$$

where

$$\mathcal{R}_W[\Lambda,L(P)]:=L(\Lambda P)\Lambda L^{-1}(P) \,. \tag{2.10}$$

With null-plane dynamics the boosts $L(P)$ are kinematic transformations and they form a group. If the orientation of the null vector n is chosen such that the plus component of the momentum transfer vanishes, $P'^+-P^+\equiv Q^+=0$, it follows from the covariance, Eqs. (2.2) and (2.3), current conservation, Eq. (2.4), and the normalization (2.5) that the nucleon form factors are related to current matrix elements by

$$\begin{aligned}\langle\mathbf{P}'_N|I_N^+(0)|\mathbf{P}_N\rangle&\equiv I_N^+\\&=\tfrac{1}{2}[F_{1N}^{\mathrm{IS}}(Q^2)-i\sigma_2\sqrt{\eta_N}\,F_{2N}^{\mathrm{IS}}(Q^2)]\\&\quad+\tfrac{1}{2}\tau_3[F_{1N}^{\mathrm{IV}}(Q^2)-i\sigma_2\sqrt{\eta_N}\,F_{2N}^{\mathrm{IV}}(Q^2)]\\&=\frac{1}{2}\sum_{\alpha=1}^{2}\sum_{I=0}^{1}\mathcal{S}_{\alpha N}^{I}F_{\alpha N}^{I}(Q^2) \,,\end{aligned} \tag{2.11}$$

where σ_2 and τ_3 are spin and isospin Pauli matrices, $\eta_N:=Q^2/4M_N^2$, and M_N is the nucleon mass. The coordinate axes are chosen such that $\vec{n}=\{0,0,1\}$ and $\vec{Q}_T=\{\sqrt{Q^2},0\}$. We are neglecting the proton-neutron mass difference. The superscripts IS and IV, or $I=0,1$ label isoscalar and isovector form factors, respectively. The 4×4 spin-isospin matrices $\mathcal{S}^I_{\alpha N}$ are by definition

$$\mathcal{S}^0_{1N}=1\otimes1,\quad \mathcal{S}^0_{2N}=-i\sqrt{\eta_N}\sigma_2\otimes1\ ,$$
$$\mathcal{S}^1_{1N}=1\otimes\tau_3,\quad \mathcal{S}^1_{2N}=-i\sqrt{\eta_N}\sigma_2\otimes\tau_3\ . \tag{2.12}$$

Note that the matrix element is independent of $\mathbf{P}'$ and $\mathbf{P}$ for all $\mathbf{P}'$ and $\mathbf{P}$ that satisfy $P^+=P'^+$ and $\vec{P}'_T-\vec{P}_T=\vec{Q}_T$.

Equation (2.11) can easily be solved for the form factors to give

$$F^I_\alpha(Q^2)=\tfrac{1}{2}\mathrm{Tr}[(\mathcal{S}^I_{\alpha N})^{-1}I^+_N]\ . \tag{2.13}$$

The proton and neutron form factors $F_{\alpha p}(Q^2)$ and $F_{\alpha n}(Q^2)$ are

$$F_{\alpha p}(Q^2)=\tfrac{1}{2}[F^{\mathrm{IS}}_\alpha(Q^2)+F^{\mathrm{IV}}_\alpha(Q^2)]\ ,$$
$$F_{\alpha n}(Q^2)=\tfrac{1}{2}[F^{\mathrm{IS}}_\alpha(Q^2)-F^{\mathrm{IV}}_\alpha(Q^2)]\ . \tag{2.14}$$

They are identical to the conventional Dirac and Pauli form factors [20–22]. For vanishing momentum transfer they are, respectively, equal to the charge and the anomalous magnetic moment in units e and e/M_N. The conventional Sachs form factors are

$$G_{eN}(Q^2)\equiv F_{1N}(Q^2)-\eta_N F_{2N}(Q^2)$$

and

$$G_{mN}(Q^2)\equiv F_{1N}(Q^2)+F_{2N}(Q^2)\ , \tag{2.15}$$

for both neutrons and protons.

The magnetic moments

$$\mu_N=F_{1N}(0)+F_{2N}(0) \tag{2.16}$$

and charge radii of the nucleons

$$\langle r^2_N\rangle_{\mathrm{charge}}:=\int d^3r\, r^2\langle I^0(\vec{r})\rangle_{\vec{P}_N=0}$$
$$=-6\frac{dG_{eN}(Q^2)}{dQ^2}\bigg|_{Q^2\to0}$$
$$=-6\frac{dF_{1N}(Q^2)}{dQ^2}\bigg|_{Q^2\to0}+\frac{3}{2M^2_N}F_{2N}(0)\ , \tag{2.17}$$

are Lorentz-invariant quantities related to the form factors in the limit of vanishing momentum transfer as shown in Eqs. (2.16) and (2.17).

We will assume that the nucleon is a bound state of three constituent quarks, which is antisymmetric in color and totally symmetric in the space, spin, and isospin variables. The single-quark current is of the same form as the single-nucleon current (2.11) with the subscript N replaced by q, indicating quarks. The form factors of the up and down quarks are related to the isoscalar and isovector quark form factors by

$$F_{\alpha u}=\tfrac{1}{2}[F^{\mathrm{IS}}_{\alpha q}+F^{\mathrm{IV}}_{\alpha q}]\ \text{ and }\ F_{\alpha d}=\tfrac{1}{2}[F^{\mathrm{IS}}_{\alpha q}-F^{\mathrm{IV}}_{\alpha q}]\ . \tag{2.18}$$

For point quarks these form factors are independent of Q^2, $F_{1u}=\frac{2}{3}$, and $F_{1d}=-\frac{1}{3}$. For Dirac point quarks the Pauli form factors F_{2u}, F_{2d} vanish. For constituent quarks the choice $F_{2u}=F_{2d}=0$ is just as arbitrary as any other choice.

In order to compute the matrix elements (2.11) we need representations of the states $|\mathbf{P}_N\rangle$ by symmetric functions of the null-plane momenta, the spin, and isospin components of the individual quarks. The construction of such wave functions is the subject of the next section.

III. QUARK-MODEL WAVE FUNCTIONS

The nucleon states $|\mathbf{P}_N,\lambda_N,\tau_N\rangle$ are represented by symmetric functions of the quark variables $\mathbf{p}_i$, λ_i, τ_i $(i=1,2,3)$. The λ's are eigenvalues of the longitudinal components of the null-plane spin. The Poincaré covariance of the wave functions is realized in the form

$$\Psi_{P_N,\lambda_N,\tau_N}(\mathbf{p}_1,\lambda_1,\tau_1,\mathbf{p}_2,\lambda_2,\tau_2,\mathbf{p}_3,\lambda_3,\tau_3)=\psi_{M_N,\lambda_N,\tau_N}(\xi_1,\vec{q}_{1T},\lambda_1,\tau_1,\ldots)\delta(\mathbf{P}-\mathbf{P}_N)\left[\frac{\partial(\mathbf{P},\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T})}{\partial(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}\right]^{1/2}\ , \tag{3.1}$$

where ψ is an eigenfunction of the mass, spin, and isospin operators. The momentum variables ξ_i, $\vec{q}_{iT}$, and $\mathbf{P}$ are related to $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ by

$$\mathbf{P}=\sum_i\mathbf{p}_i,\quad \xi_i=\frac{p_i^+}{P^+},\quad\text{and}\quad \vec{q}_{iT}=\vec{p}_{iT}-\xi_i\vec{P}_T\ , \tag{3.2}$$

which implies the constraints

$$\sum_{i=1}^3\vec{q}_{iT}=0\quad\text{and}\quad\sum_{i=1}^3\xi_i=1\ . \tag{3.3}$$

The Jacobian of the variable transformation $\mathbf{P},\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T}\to\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ is

$$\frac{\partial(\mathbf{P},\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T})}{\partial(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)}=\frac{1}{P^{+2}}\ . \tag{3.4}$$

The mass operator of the noninteracting three-body system is a function of the internal momentum variables $\xi_i,\vec{q}_{iT}$:

$$M_0^2=\sum_i\frac{\vec{q}^{\,2}_{iT}+m_i^2}{\xi_i}\ . \tag{3.5}$$

For the representation of the total spin operator it is convenient to define longitudinal components q_{in}, such that the relative momenta $\mathbf{q}_i$ and the spin satisfy vector commutation relations:

$$q_{in} := \frac{1}{2}\left[M_0\xi_i - \frac{m_i^2+\vec{q}_{iT}^{\,2}}{M_0\xi_i} \right] . \tag{3.6}$$

It follows from Eqs. (3.5) and (3.6) that

$$M_0^2 = \left[\sum_i \sqrt{m_i^2+\vec{q}_i^{\,2}} \right]^2 . \tag{3.7}$$

As in nonrelativistic systems the total spin operator $\vec{j}$ can be expressed as a sum of orbital and spin contributions,

$$\vec{j} = \sum_{i=1}^{3} (\vec{y}_i \times \vec{q}_i + \vec{s}_i) , \tag{3.8}$$

where the operators $\vec{\bar{s}}_i$ are related to the quark spins $\vec{s}_i$ by a Melosh rotation [20, 23]:

$$\vec{\bar{s}}_i := \mathcal{R}_M(\xi_i, \vec{q}_{iT}, m, M_0)\vec{s}_i , \tag{3.9}$$

which has the matrix representation

$$\begin{aligned}\langle \lambda' | \mathcal{R}_M(\xi,\vec{q}_T,m,M_0)|\lambda\rangle &= \mathcal{D}^{1/2}_{\lambda',\lambda}(\mathcal{R}_M) \\ &\equiv \left[\frac{m+\xi M_0 - i\vec{\sigma}\cdot(\vec{n}\times\vec{q}_T)}{[(m+\xi M_0)^2+\vec{q}_T^{\,2}]^{1/2}} \right]_{\lambda',\lambda} .\end{aligned} \tag{3.10}$$

The operators $\vec{y}_i$ by definition satisfy canonical commutation relations with the momenta $\vec{q}_i$ and commute with the operators $\vec{\bar{s}}_i$. It follows that the individual quark spins do not commute with the orbital angular momenta. This is a characteristic difference between relativistic and nonrelativistic systems.

Expressed as a function of the momenta $\vec{q}_i$ and the eigenvalues of the longitudinal components of the operators $\vec{\bar{s}}_i$ the wave function ψ has the same structure as the wave function of a nonrelativistic quark model. In this representation we can assume a product of a symmetric function of the momenta with a symmetric function of the spin-isospin variables. The spin function

$$\Phi^{S_{12}}_{\lambda_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3) := (\tfrac{1}{2},\tfrac{1}{2},\bar\lambda_1,\bar\lambda_2|S_{12},\bar\lambda_1+\bar\lambda_2) \times (S_{12},\tfrac{1}{2},\bar\lambda_1+\bar\lambda_2,\bar\lambda_3|\tfrac{1}{2},\lambda_N) \tag{3.11}$$

is symmetric or antisymmetric under the interchange $1\rightleftarrows 2$ for $S_{12}=1$ and $S_{12}=0$, respectively. It is often convenient to express these spin functions as outer products of Pauli matrices, that is,

$$\begin{aligned}\Phi^0_{\lambda_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3) &= \frac{1}{\sqrt{2}}\langle\bar\lambda_1|i\sigma_2|\bar\lambda_2\rangle\delta_{\bar\lambda_s,\lambda_N} , \\ \Phi^1_{\lambda_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3) &= -\frac{1}{\sqrt{6}}\langle\bar\lambda_1|i\vec{\sigma}\sigma_2|\bar\lambda_2\rangle\langle\bar\lambda_3|\vec{\sigma}|\lambda_N\rangle .\end{aligned} \tag{3.12}$$

The function

$$\begin{aligned}\Phi_{\lambda_N,\tau_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3,\tau_1,\tau_2,\tau_3) = \frac{1}{\sqrt{2}}[&\Phi^0_{\lambda_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3)\Phi^0_{\tau_N}(\tau_1,\tau_2,\tau_3) \\ &+\Phi^1_{\lambda_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3)\Phi^1_{\tau_N}(\tau_1,\tau_2,\tau_3)]\end{aligned} \tag{3.13}$$

is fully symmetric under all permutations, and it is the only such function with the nucleon spin and isospin equal to $\frac{1}{2}$. It can be expressed as linear combinations of outer products of spin and isospin matrices for the particle pairs $\{1,2\}$ and $\{3,N\}$, that is,

$$\Phi = \frac{1}{\sqrt{2}}[\Phi^0+\Phi^1] , \tag{3.14}$$

where

$$\begin{aligned}\Phi^0 &= \tfrac{1}{2}[i\sigma_2\otimes i\tau_2]_{1,2}\otimes[1\otimes 1]_{3,N} , \\ \Phi^1 &= -\frac{1}{6}\sum_{k,\alpha}[i\sigma_k\sigma_2\otimes i\tau_\alpha\tau_2]_{1,2}\otimes[\sigma_k\otimes\tau_\alpha]_{3,N} .\end{aligned} \tag{3.15}$$

The wave function $\psi_{M_N,\lambda_N,\tau_N}(\xi_1,\vec{q}_{1T},\lambda_1,\tau_1,\ldots)$ with the spin-isospin structure (3.13) has the form

$$\psi_{M_N,\lambda_N,\tau_N}(\xi_1,\vec{q}_{1T},\lambda_1,\tau_1,\ldots) = \phi(\vec{q}_1,\vec{q}_2,\vec{q}_3)\sum_{\bar\lambda_1,\bar\lambda_2,\bar\lambda_3}\mathcal{D}^{1/2\dagger}_{\lambda_1,\bar\lambda_1}(\mathcal{R}_{M1})\mathcal{D}^{1/2\dagger}_{\lambda_2,\bar\lambda_2}(\mathcal{R}_{M2})D^{1/2\dagger}_{\lambda_3,\bar\lambda_3}(\mathcal{R}_{M3})\Phi_{\lambda_N,\tau_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3,\tau_1,\tau_2,\tau_3) , \tag{3.16}$$

where the longitudinal components the vector $\vec{q}_i$ are given by Eq. (3.6). The requirement of rotational invariance implies a nontrivial dependence of the wave function on the constituent-quark mass. The function $\phi(\vec{q}_1,\vec{q}_2,\vec{q}_3)\equiv\phi(q_1^2,q_2^2,q_3^2)$ is a permutation symmetric, rotationally invariant function normalized according to

$$\int[d\xi]\int[d^2q_T]|\phi(\vec{q}_1,\vec{q}_2,\vec{q}_3)|^2 = \int[d^3q]|\phi(\vec{q}_1,\vec{q}_2,\vec{q}_3)|^2 = 1 , \tag{3.17}$$

where

$$\begin{aligned}[d\xi] &:= d\xi_1 d\xi_2 d\xi_3\frac{\delta(\xi_1+\xi_2+\xi_3-1)}{\xi_1\xi_2\xi_3} , \\ [d^2q_T] &:= d^2q_{1T}d^2q_{2T}d^2q_{3T}\delta(\vec{q}_{1T}+\vec{q}_{2T}+\vec{q}_{3T}) ,\end{aligned} \tag{3.18}$$

and

$$[d^3q] := d^3q_1 d^3q_2 d^3q_3\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3) \times\frac{M_0}{\sqrt{(q_1^2+m_q^2)(q_2^2+m_q^2)(q_3^2+m_q^2)}} . \tag{3.19}$$

A simple way to realize the permutation symmetry is to assume that ϕ is a function of a simple algebraic combination, for instance, $q_1^2+q_2^2+q_3^2$. For the numerical computations it is a little more convenient to choose a function of M_0^2. Specifically we will assume [24]

$$\phi(M_0)=\frac{N(m_q/\alpha)}{\alpha^2}\exp(-M_0^2/2\alpha^2)\ . \tag{3.20}$$

The dimensionless factor $N(m_q/\alpha)$ is determined by the normalization condition (3.17). This wave function depends on only two parameters: the constituent quark mass m_q and the range parameter α which specifies the confinement scale. The bound-state wave function has a well-defined nonrelativistic limit, $m_q/\alpha\to\infty$, as well as an extreme relativistic limit, $m_q/\alpha\to 0$. For the present exploratory calculations we did not vary the shape of the wave function. For low and moderate values of Q^2 we do not expect significant dependence on the detailed shape. For sufficiently high momentum transfer the wave function considered here will be manifestly inadequate. The model could be extended to higher values of Q^2 by considering wave functions with appropriate high-momentum tails and Q^2-dependent form factors of the constituent quarks.

It is possible to express the relativistic wave functions derived here in an equivalent Dirac-spinor representation, but there is no advantage in doing so. In order to compare our wave functions to those used elsewhere [17, 18] we present spinor representations in some detail in Appendix A.

IV. NUCLEON CURRENT MATRIX ELEMENTS

Having constructed nucleon wave functions we are in the position to compute the current matrix element (2.11) using the quark current operators (2.17) and the wave functions (3.16). It will be convenient to treat the wave functions ψ_{M_N} as rectangular matrices with the columns labeled by the spin-isospin variables λ_N,τ_N of the nucleon, and the rows labeled by the spin-isospin variables of the quarks:

$$\begin{aligned}I_N^+&=3\int[d\xi]\int[d^2q_T]\int[d^2q_T']\delta(\vec{q}\,'_{1T}-\vec{q}_{1T}+\xi_1\vec{Q}_T)\delta(\vec{q}\,'_{2T}-\vec{q}_{2T}+\xi_2\vec{Q}_T)\delta(\vec{q}\,'_{3T}-\vec{q}_{3T}-(1-\xi_3)\vec{Q}_T)\\&\qquad\times\psi_{M_N}^\dagger(\xi_1,\vec{q}\,'_{1T},\xi_2,\vec{q}\,'_{2T},\xi_3,\vec{q}\,'_{3T})I_{q3}^+\psi_{M_N}(\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T})\\&=\frac{1}{2}\sum_{I=0}^{1}\sum_{\beta=1}^{2}\mathcal{M}_\beta^I F_{\beta q}^I\ .\end{aligned}\tag{4.1}$$

The spin-isospin matrix $\mathcal{M}_\beta^I$ is defined by

$$\begin{aligned}\mathcal{M}_\beta^I&:=3\int[d\xi]\int[d^2q_T]\int[d^2q_T']\phi(M_0')\phi(M_0)\\&\qquad\times\delta(\vec{q}\,'_{1T}-\vec{q}_{1T}+\xi_1\vec{Q}_T)\delta(\vec{q}\,'_{2T}-\vec{q}_{2T}+\xi_2\vec{Q}_T)\delta(\vec{q}\,'_{3T}-\vec{q}_{3T}-(1-\xi_3)\vec{Q}_T)\\&\qquad\times\chi_{M_N}^\dagger(\xi_1,\vec{q}\,'_{1T},\xi_2,\vec{q}\,'_{2T},\xi_3,\vec{q}\,'_{3T})(\mathcal{S}_{\beta q}^I)_3\chi_{M_N}(\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T})\ ,\end{aligned}\tag{4.2}$$

where

$$\chi_{M_N,\lambda_N,\tau_N}(\xi_1,\vec{q}_{1T},\lambda_1,\tau_1,\ldots)=\sum_{\bar\lambda_1,\bar\lambda_2,\bar\lambda_3}\mathcal{D}^{1/2\dagger}_{\lambda_1,\bar\lambda_1}(\mathcal{R}_{M1})\mathcal{D}^{1/2\dagger}_{\lambda_2,\bar\lambda_2}(\mathcal{R}_{M2})\mathcal{D}^{1/2\dagger}_{\lambda_3,\bar\lambda_3}(\mathcal{R}_{M3})\Phi_{\lambda_N,\tau_N}(\bar\lambda_1,\bar\lambda_2,\bar\lambda_3,\tau_1,\tau_2,\tau_3)\ . \tag{4.3}$$

It follows from Eqs. (4.1) and (2.13) that the nucleon form factors are linear combinations of the quark form factors:

$$F_1^I(Q^2)=\sum_\beta C_{1,\beta}^I(Q^2)F_{\beta q}^I$$

and (4.4)

$$F_2^I(Q^2)=\frac{M_N}{m_q}\sum_\beta C_{2,\beta}^I(Q^2)F_{\beta q}^I\ ,$$

where

$$C_{1,\beta}^I(Q^2)=\tfrac{1}{2}\mathrm{Tr}[(\mathcal{S}_{1N}^I)^{-1}\mathcal{M}_\beta^I]$$

and (4.5)

$$C_{2,\beta}^I(Q^2)=\frac{m_q}{M_N}\tfrac{1}{2}\mathrm{Tr}[(\mathcal{S}_{2N}^I)^{-1}\mathcal{M}_\beta^I]\ .$$

The matrix elements $C_{\alpha,\beta}^I$ so defined are independent of the nucleon mass M_N. They are dimensionless functions of Q^2 and α/m_q. If $B_{\alpha\beta}^0$ and $B_{\alpha\beta}^1$ are defined as the contributions of Φ^0 and Φ^1 to $C_{\alpha\beta}^0$, such that

$$C_{\alpha\beta}^0=\tfrac{3}{2}(B_{\alpha\beta}^0+B_{\alpha\beta}^1)\ , \tag{4.6}$$

it follows from the isospin structure of the wave function that

$$C_{\alpha\beta}^1=\tfrac{3}{2}(B_{\alpha\beta}^0-\tfrac{1}{3}B_{\alpha\beta}^1)\ . \tag{4.7}$$

The general expressions for the proton and neutron form factors are

$$
\begin{aligned}
F_{1p} &= \sum_\beta [(\tfrac{3}{2}B^0_{1,\beta}+\tfrac{1}{2}B^1_{1,\beta})F_{\beta u}+B^1_{1,\beta}F_{\beta d}] \\
&= B^0_{1,1}+\tfrac{3}{2}F_{2u}B^0_{1,2}+(\tfrac{1}{2}F_{2u}+F_{2d})B^1_{1,2}\ , \\
F_{1n} &= \sum_\beta [(\tfrac{3}{2}B^0_{1,\beta}+\tfrac{1}{2}B^1_{1,\beta})F_{\beta d}+B^1_{1,\beta}F_{\beta u}] \\
&= \tfrac{1}{2}(B^1_{1,1}-B^0_{1,1})+\tfrac{3}{2}F_{2d}B^0_{1,2}+(\tfrac{1}{2}F_{2d}+F_{2u})B^1_{1,2}\ ;
\end{aligned} \tag{4.8}
$$

$$
\begin{aligned}
F_{2p} &= \frac{M_N}{m_q}\sum_\beta [(\tfrac{3}{2}B^0_{2,\beta}+\tfrac{1}{2}B^1_{2,\beta})F_{\beta u}+B^1_{2,\beta}F_{\beta d}] \\
&= \frac{M_N}{m_q}[B^0_{2,1}+\tfrac{3}{2}F_{2u}B^0_{2,2}+(\tfrac{1}{2}F_{2u}+F_{2d})B^1_{2,2}]\ , \\
F_{2n} &= \frac{M_N}{m_q}\sum_\beta [(\tfrac{3}{2}B^0_{2,\beta}+\tfrac{1}{2}B^1_{2,\beta})F_{\beta d}+B^1_{2,\beta}F_{\beta u}] \\
&= \frac{M_N}{m_q}[\tfrac{1}{2}(B^1_{2,1}-B^0_{2,1})+\tfrac{3}{2}F_{2d}B^0_{2,2} \\
&\qquad +(\tfrac{1}{2}F_{2d}+F_{2u})B^1_{2,2}]\ ,
\end{aligned} \tag{4.9}
$$

where $F_{1u}=-2F_{1d}=\frac{2}{3}$. For $Q^2=0$ we have $B^0_{1,1}(0)=B^1_{1,1}(0)=1$ and $B^0_{1,2}(0)=B^1_{1,2}(0)=0$.

It follows from Eq. (2.16), which relates the magnetic moments to the form factors, and Eq. (4.9) that magnetic moments are related to the matrix elements $B^I_{\alpha,\beta}$ by

$$
\begin{aligned}
\mu_p-1 &= F_{2p}(0) \\
&= \frac{M_N}{m_q}[B^0_{2,1}(0)+\tfrac{1}{3}(4F_{2u}-F_{2d})B^0_{2,2}(0)]\ ,
\end{aligned} \tag{4.10}
$$

$$
\begin{aligned}
\mu_n &= F_{2n}(0) \\
&= \frac{M_N}{m_q}[\tfrac{1}{2}[(B^1_{2,1}(0)-B^0_{2,1}(0)] \\
&\qquad +\tfrac{1}{3}(4F_{2d}-F_{2u})B^0_{2,2}(0)]\ ,
\end{aligned}
$$

where

$$
B^0_{2,2}(0)=-3B^1_{2,2}(0)=\int[d\xi]\int[d^2q_T]|\phi(M_0)|^2\left[1-\frac{\vec{q}^{\,2}_{sT}}{(\xi_3M_0+m_q)^2+\vec{q}^{\,2}_{sT}}\right], \tag{4.11}
$$

$$
B^0_{2,1}(0)=2\int[d\xi]\int[d^2q_r]|\phi(M_0)|^2\frac{m_q}{M_0}\left[\frac{(1-\xi_3)M_0(\xi_3M_0+m_q)-\frac{1}{2}\vec{q}^{\,2}_{sT}}{(\xi_3M_0+m_q)^2+\vec{q}^{\,2}_{sT}}\right], \tag{4.12}
$$

$$
\tfrac{1}{2}[B^1_{2,1}(0)-B^0_{2,2}(0)]=-2\int[d\xi]\int[d^2q_T]|\phi(M_0)|^2\frac{2}{3(1-\xi_3)}\frac{m_q}{M_0}\left[\frac{(1-\xi_3)M_0(\xi_3M_0+m_q)-\frac{1}{2}\vec{q}^{\,2}_{sT}}{(\xi_3M_0+m_q)^2+\vec{q}^{\,2}_{sT}}\right]. \tag{4.13}
$$

The ratio g_A/g_V of the weak coupling constants is

$$
\begin{aligned}
\frac{g_A}{g_V} &:= 3\left[\frac{g_A}{g_V}\right]_q\int[d\xi]\int[d^2q_T]|\phi(M_0)|^2 \\
&\quad\times\chi^\dagger_{M_N,1/2,1/2}(\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T})[\sigma_3\otimes\tau_3]_3\chi_{M_N,1/2,1/2}(\xi_1,\vec{q}_{1T},\xi_2,\vec{q}_{2T},\xi_3,\vec{q}_{3T}) \\
&= \frac{5}{3}\left[\frac{g_A}{g_V}\right]_q\int[d\xi]\int[d^2q_T]|\phi(M_0)|^2\frac{(m_q+\xi_3M_0)^2-\vec{q}^{\,2}_{sT}}{(m_q+\xi_3M_0)^2+\vec{q}^{\,2}_{sT}}=\frac{5}{3}\left[\frac{g_A}{g_V}\right]_q[2B^0_{2,2}(0)-1]\ .
\end{aligned} \tag{4.14}
$$

The factor $(g_A/g_V)_q$ is the ratio of the weak coupling constants of the constituent quarks. For current quarks this ratio would be unity. For $1.4<\alpha/m_q<3.5$ we find that the value of $(g_A/g_V)_q$ required to give $g_A/g_V=1.25$ varies from 0.9 to 1.2. In the nonrelativistic limit, $\alpha/m_q\to 0$, we have the well-known result $g_A/g_V=\frac{5}{3}(g_A/g_V)_q$ which requires $(g_A/g_V)_q=\frac{3}{4}$ [25, 26].

The charge radii are related to the slopes of the form factors at zero-momentum transfer according to Eq. (2.17). From Eq. (4.8) it follows that

$$
\left.\frac{dF_{1p}(Q^2)}{dQ^2}\right|_{Q^2\to 0}=\lim_{Q^2\to 0}\left[\frac{B^0_{1,1}(Q^2)-1}{Q^2}+F_{2u}\left[\frac{3B^0_{1,2}(Q^2)}{2Q^2}+\frac{B^1_{1,2}(Q^2)}{2Q^2}\right]+F_{2d}\frac{B^1_{1,2}(Q^2)}{Q^2}\right] \tag{4.15}
$$

and

$$
\left.\frac{dF_{1n}(Q^2)}{dQ^2}\right|_{Q^2\to 0}=\lim_{Q^2\to 0}\left[\frac{B^1_{1,1}(Q^2)-B^2_{1,1}(Q^2)}{2Q^2}+F_{2d}\left[\frac{3B^0_{1,2}(Q^2)}{2Q^2}+\frac{B^1_{1,2}(Q^2)}{2Q^2}\right]+F_{2u}\frac{B^1_{1,2}(Q^2)}{Q^2}\right]. \tag{4.16}
$$

It is instructive to compare these results with the nonrelativistic limit, $\alpha/m_q\to 0$. Let us first remove the relativistic effects in the spin composition by taking the limit $\vec{q}^{\,2}_{iT}\to 0$ in the Melosh rotations, i.e.,

$$\cdot\frac{m_q+\xi_i M_0-i\vec{\sigma}\cdot(\vec{n}\times\vec{q}_{iT})}{[(m_q+\xi_i M_0)^2+\vec{q}_{iT}^{\,2}]^{1/2}}\to 1-\frac{i\vec{\sigma}\cdot(\vec{n}\times\vec{q}_i)}{2m_q}\ ; \tag{4.17}$$

the matrix elements $B^I_{\alpha,\beta}$ are in that limit proportional to a single form factor F_0:

$$\begin{gathered}B^0_{1,1}(Q^2)\to F_0(Q^2),\quad B^0_{1,1}(Q^2)-B^1_{1,1}(Q^2)\to 0,\quad B^0_{2,1}(Q^2z)\to\tfrac{2}{3}F_0(Q^2),\quad B^0_{2,1}(Q^2)+B^1_{2,1}(Q^2)\to 0\ ,\\ B^0_{2,2}(Q^2)\to F_0(Q^2),\quad B^0_{2,2}(Q^2)+3B^1_{2,2}(Q^2)\to 0,\quad B^0_{1,2}(Q^2)\to 0,\quad B^1_{1,2}(Q^2)\to 0\ .\end{gathered} \tag{4.18}$$

The function $F_0(Q^2)$,

$$F_0(Q^2):=\int[d\xi]\int[d^2q_T]\int[d^2q'_T]\phi(M'_0)\phi(M_0)\delta(\vec{q}\,'_{1T}-\vec{q}_{1T}+\xi_1(\vec{Q}_T)\delta(\vec{q}\,'_{2T}-\vec{q}_{2T}+\xi_2\vec{Q}_T)\delta(\vec{q}\,'_{3T}-\vec{q}_{3T}-(1-\xi_3)\vec{Q}_T)\ , \tag{4.19}$$

is equal to unity for $Q^2=0$. In this limit the four nucleon form factors are related to single form $F_0(Q^2)$ by the relations

$$F_{1p}(Q^2)\to F_0(Q^2),\quad F_{1n}(Q^2)\to 0\ , \tag{4.20}$$

and

$$\begin{aligned}F_{2p}(Q^2)&\to\frac{M_N}{3m_q}(2+4F_{2u}-F_{2d})F_0(Q^2)\ ,\\ F_{2n}(Q^2)&\to\frac{M_N}{3m_q}(-2+4F_{2d}-F_{2u})F_0(Q^2)\ .\end{aligned} \tag{4.21}$$

With the assumption, $m_q=\frac{1}{3}M_N$, a fit of the nonrelativistic formula (4.21) to the magnetic moments requires $F_{2u}=-0.05$ and $F_{2d}=0.01$. Equation (4.21) shows that the anomalous magnetic moments of the nucleons vanish in the limit $m_q\to\infty$ [21,27].

Written in terms of the Sachs form factors for $Q^2\ll 4M_N^2$, Eqs. (4.20) and (4.21) are the familiar nonrelativistic relations

$$G_{mp}=\mu_p F_0,\quad G_{mn}=\mu_n F_0,\quad \text{and } G_{ep}=F_0\ . \tag{4.22}$$

Since F_{1n} vanishes in this limit the leading term of G_{en} is

$$G_{en}=-\eta_N G_{mn}\ , \tag{4.23}$$

which gives a reasonable approximation for the slope of G_{en} at $Q^2=0$.

There are still relativistic effects in the relation between the form factor $F_0(Q^2)$ and the wave function. Unless $\alpha\ll m_q$ the charge density, which is the Fourier transform of the charge form factor, is not given by the square of the Fourier transform of the wave function $\phi(\vec{q}_1,\vec{q}_2,\vec{q}_3)$.

The nonrelativistic limit, $\alpha/m_q\to 0$, of $F_0(Q^2)$ can be obtained by taking $\xi_i\to\frac{1}{3}$ and

$$\begin{aligned}M_0&\to 3m_q+\sum_{i=1}^{3}\frac{\vec{q}_i^{\,2}}{2m_q}\ ,\\ M_0^2&\to 9m_q^2+3(\vec{q}_1^{\,2}+\vec{q}_2^{\,2}+\vec{q}_3^{\,2})\ .\end{aligned} \tag{4.24}$$

As expected the form factor $F_0(Q^2)$ becomes a Gaussian in that limit,

$$\begin{aligned}F_0(Q^2)&\to\int d^3q_1\int d^3q_2\int d^3q_3\delta(\vec{q}_1+\vec{q}_2+\vec{q}_3)\phi(\vec{q}_1+\tfrac{1}{3}\vec{Q},\vec{q}_2+\tfrac{1}{3}\vec{Q},\vec{q}_3-\tfrac{2}{3}\vec{Q})\phi(\vec{q}_1,\vec{q}_2,\vec{q}_3)\\ &=\exp\left[-\frac{Q^2}{2\alpha^2}\right]\ ,\end{aligned} \tag{4.25}$$

and the charge radius of the proton tends to $\sqrt{3}/\alpha$. A fit to the empirical charge radius [28,29] of the proton requires $\alpha_{\rm NR}\approx 0.4$ GeV, which is not small compared to the quark mass. The Galilean invariant nonrelativistic quark model is not an approximation to a relativistic model for any realistic value of α.

Numerical results for the slope of $F_0(Q^2)$ at $Q^2=0$ can be represented by the expression

TABLE I. For various quark masses m_q and size parameters α we list in columns 4–6 the values of the Pauli moments and the ratios of the weak coupling constants of the constituent quarks that are needed to fit the magnetic moments and the ratio of the weak coupling constants of the nucleons. In columns 6 and 7 we list the proton charge radii and slopes of the electric form factor of the neutron at zero momentum transfer.

α (GeV)	m_q (GeV)	α/m_q	F_{2u}	F_{2d}	$(g_A/g_V)_q$	r_p (fm)	dG_{en}/dQ^2 (GeV)$^{-2}$
0.420	$M_N/3$	0	-0.050	0.010	0.75	0.810	0.544
0.554	0.33	1.68	0.039	-0.110	0.92	0.836	0.487
0.635	0.24	2.65	-0.025	-0.047	1.07	0.827	0.451
0.640	0.21	3.05	-0.046	-0.025	1.13	0.857	0.422

$$\left[\frac{dF_0(Q^2)}{dQ^2}\right]_{Q^2=0} \approx -\frac{1}{2\alpha^2}\left[1+0.15\frac{\alpha}{m_q}+0.022\frac{\alpha^2}{m_q^2}\right] . \tag{4.26}$$

For $0<\alpha/m_q<3.5$ the numerical results for the magnetic moments (4.10) can be approximated by the expressions

$$\mu_p-1\approx\frac{2M_N}{3m_q}\left[1-0.11\frac{\alpha}{m_q}+\frac{1}{2}(4F_{2u}-F_{2d})\left[1-0.056\frac{\alpha}{m_q}\right]\right] , \tag{4.27}$$

$$\mu_n\approx-\frac{2M_N}{3m_q}\left[1-0.13\frac{\alpha}{m_q}-\frac{1}{2}(4F_{2d}-F_{2u}\left[1-0.056\frac{\alpha}{m_q}\right]\right] .$$

It follows that

$$\mu_p+\mu_n-1\approx\frac{2M_N}{3m_q}\left[0.02\frac{\alpha}{m_q}+\frac{3}{2}(F_{2d}+F_{2u})\left[1-0.056\frac{\alpha}{m_q}\right]\right] . \tag{4.28}$$

A fit to the experimental data, $\mu_p+\mu_n-1=-0.12$, cannot be achieved if the Pauli moments of the constituent quarks vanish.

The relativistic spin effects also affect the charge radius. If we determine the quark form factors F_{2u} and F_{2d} by fitting the magnetic moments of the proton and neutron, the charge radius of the proton can be represented by the expression

$$r_p\approx\frac{\sqrt{3}}{\alpha}\left[1+0.2\frac{\alpha}{m_q}\right] . \tag{4.29}$$

In the extreme relativistic limit $m_q\to 0$ the matrix elements $B^I_{\alpha\alpha}$ are functions of Q^2/α^2, $B^0_{22}(0)=\frac{1}{2}$, and the matrix elements B^I_{21} vanish linearly in m_q. The matrix elements B^I_{12} diverge as $1/m_q$, and the quark form factors F_{2q} must vanish at least linearly with m_q. We should not expect to obtain reasonable low-energy nucleon properties in the zero-mass limit of a constituent-quark model, and it does not appear to be possible to do so.

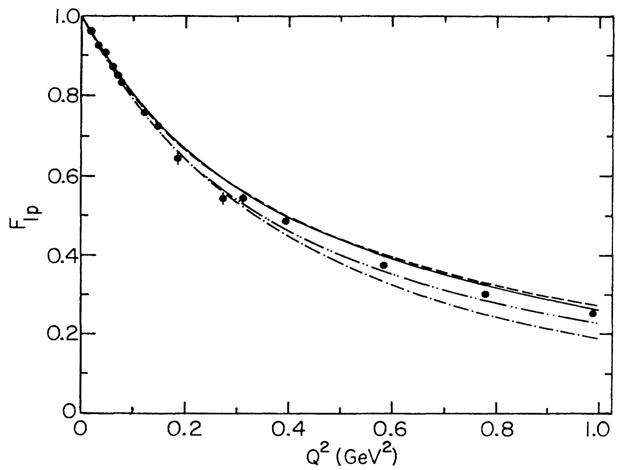

FIG. 1. The proton form factor $F_{1p}(Q^2)$ for the parameters shown in Table I. The following line codes are used: dash-dot $m_q=0.33$ GeV; solid $m_q=0.24$ GeV; dash line, $m_q=0.21$ GeV; dash-double dot for the same parameters as the solid line except that $F_{2u}=F_{2d}=0$. The experimental data are taken from the compilation in Ref. [34].

V. NUMERICAL RESULTS

We have calculated the four nucleon form factors for the range $0\le Q^2\le 6$ GeV^2 for several combinations of α and m_q shown in Table I together with the values of the quark Pauli form factors which fit the magnetic moments. The values $\alpha=0.554$ GeV, $m_q=0.33$ GeV are the parameters used in Refs. [17] and [18]. Experience with pion form factors [9] indicates that smaller constituent masses may give realistic form factors at larger values of Q^2.

Table I shows the proton charge radius and the slope of the neutron's electric form factor G_{en} at zero momentum transfer. The experimental values of the proton charge radius given in Refs. [27] and [28] are 0.81 ± 0.04

TABLE II. The effects of the Pauli form factors of the constituent quarks on the slope of the Dirac form factors of the proton and neutron at zero momentum transfer. All slopes are listed in units of GeV^{-2}. Vanishing Pauli form factors are indicated by the subscript 0.

α (GeV)	m_q (GeV)	α/m_q	$(dF_{1n}/dQ^2)_0$	(dF_{1n}/dQ^2)	$(dF_{1p}/dQ^2)_0$	(dF_{1p}/dQ^2)
0.420	$M_N/3$	0	0	0	$-1/2\alpha^2$	$-1/2\alpha^2$
0.554	0.33	1.68	-0.296	-0.057	-2.58	-2.48
0.635	0.24	2.65	-0.377	-0.092	-2.67	-2.42
0.640	0.21	3.05	-0.423	-0.124	-2.97	-2.63

(fm) and 0.862±0.012 (fm), respectively. In the conventional dipole form factor the value 0.81 (fm) is used. Experimental values for the slope of $G_{en}(Q^2)$ at zero momentum transfer are [30] 0.485±0.010 GeV^{-2} and [31] 0.511±0.008 GeV^{-2}. The main contribution to the slope of $G_{en}(Q^2)$ at zero momentum transfer comes from $F_{2n}(0)$ and is determined by the neutron's magnetic moment. Since the precisely known magnetic moment fixes the Foldy term in the slope of G_{en}, the data for dG_{en}/dQ^2 at zero momentum transfer can be interpreted as data for dF_{1n}/dQ^2. Table II shows the effect of the Pauli form factors of the constituent quarks on the slopes of the form factors F_{1p} and F_{1n} at vanishing momentum transfer. The Pauli form factors of the constituent quarks which are necessary to fit the magnetic moments also play a decisive role in obtaining reasonable values for the slope of $G_{en}(Q^2)$.

Figures 1–3 illustrate the fact that for $Q^2 < 1$ GeV^2 the data can easily be reproduced with different quark masses. Calculated values of the form factors $G_{en}(Q^2)$ and $F_{1n}(Q^2)$ are compared in Fig. 4 to the parametrizations of Höhler [32] and of Gari and Krümpelmann [15]. The main qualitative result is that the Pauli form factors of the quarks required by magnetic moments produce small neutron form factors $F_{1n}(Q^2)$ for a wide range of momentum transfers. A precise experimental measurement of $F_{1n}(Q^2)$ should provide an important constraint on these models.

At larger values of Q^2 we see important differences in the form factors for different values of the constituent quark mass and range parameter. To illustrate these effects we show in Figs. 5–7 the form factors F_{1p}, F_{2p}, and F_{2n} multiplied by Q^4 in comparison with proton [33–35] and neutron [36–39] data, and with the parametrization of Gari and Krümplemann [15]. For $m_q = 0.33$ GeV the form factors decrease too rapidly to fit the data for $Q^2 > 1$ GeV^2. This expected behavior is characteristic

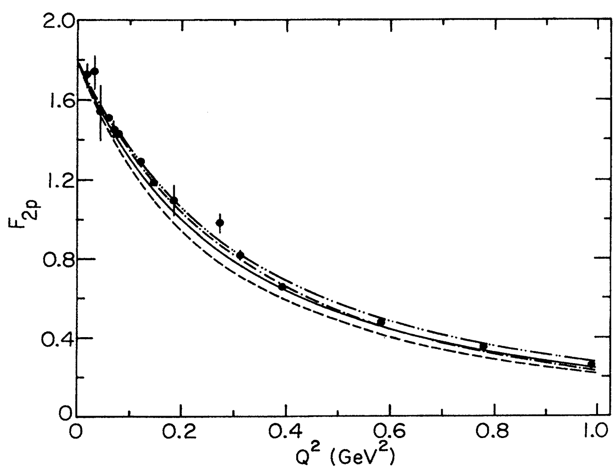

FIG. 2. The proton form factor $F_{2p}(Q^2)$ for the same parameters as in Fig. 1. The experimental data are taken from the compilation in Ref. [32].

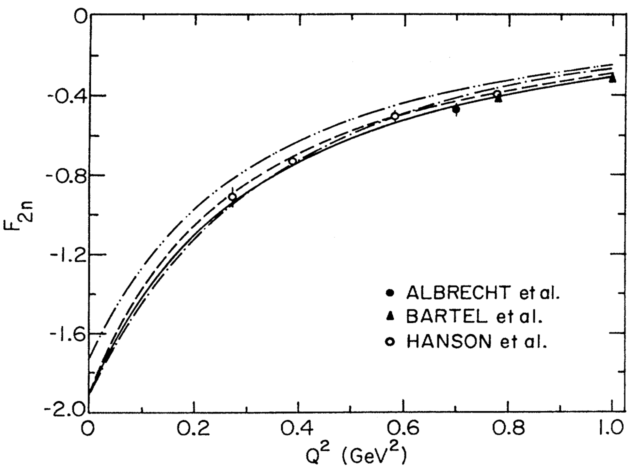

FIG. 3. The neutron form factor $F_{2n}(Q^2)$ for the same parameters as in Fig. 1. The experimental data shown are taken from Refs. [36–38].

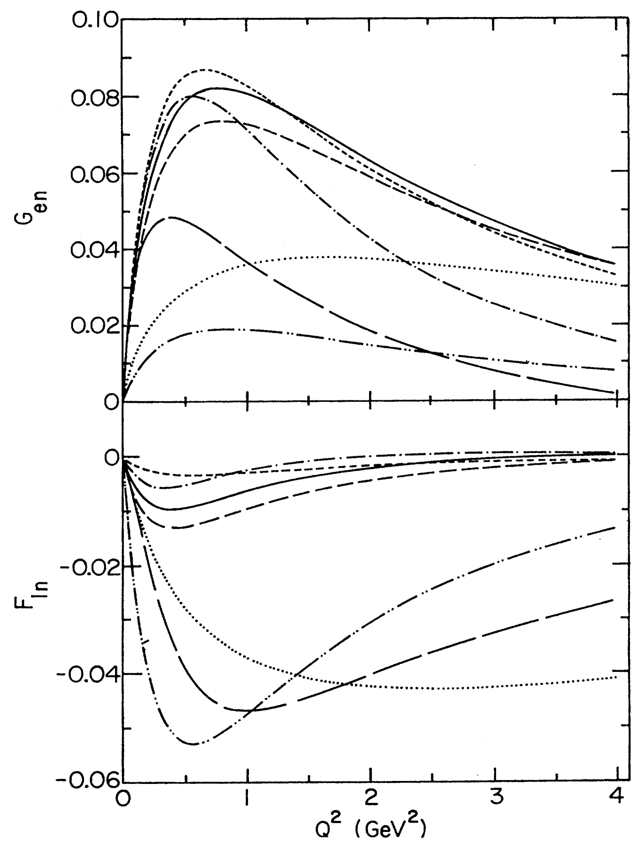

FIG. 4. The electric form factor of the neutron $G_{en}(Q^2)$ and $F_{1n}(Q^2)$ for the same parameters as in Fig. 1. The dotted line shows the form factors for $m_q = 0$, $\alpha = 0.7$ GeV, $F_{2u} = F_{2d} = 0$. These form factors are compared with the parametrizations of Höhler [32], long dashes, and Gari and Krümpelmann [15], short dashes.

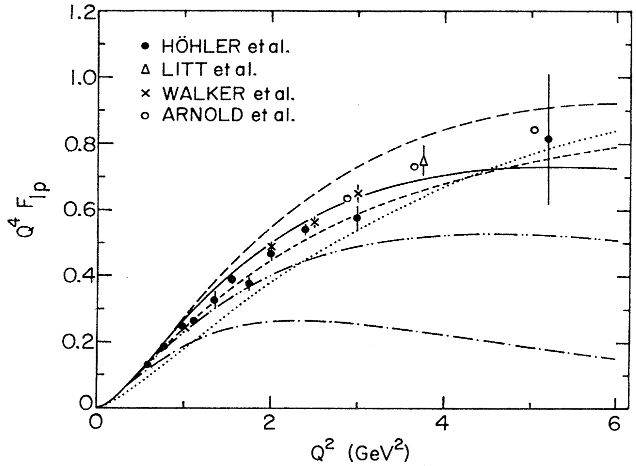

FIG. 5. The proton form factor $Q^4F_{1p}(Q^2)$ for the parameters as in Figs. 1 and 4. The short dashes show the parametrization of Gari and Krümpelmann [15]. The experimental values are taken from Refs. [33–35] as indicated in the figure. The open circles are extracted from Ref. [40] as described in the text.

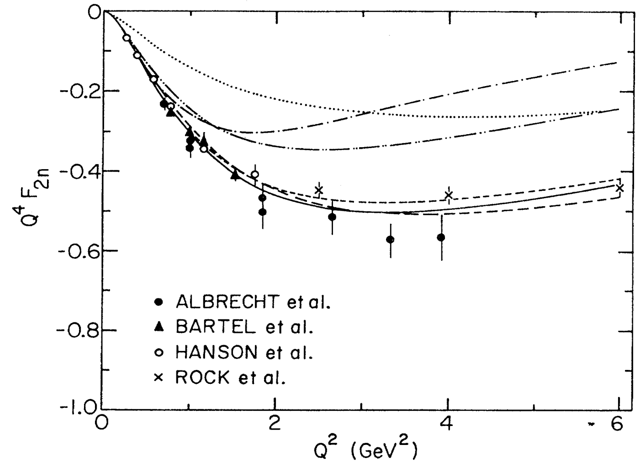

FIG. 7. The neutron form factor $Q^4F_{2n}(Q^2)$. The lines are labeled as Figs. 5 and 6. The experimental values are taken from Refs. [36–39] as indicated in the figure.

of the large mass, it does not indicate a breakdown of the constituent quark model. Reasonable agreement with the data can be achieved with smaller quark masses with slightly larger range parameters. We have not investigated the effects of a Q^2 dependence of the quark form factors, but indicate the size and sign of possible effects by showing the form factors calculated for $m_q=0.24$ GeV and $\alpha=0.635$ GeV with $F_{2u}=F_{2d}=0$. A qualitative representation of the data can also be achieved with $m_q=0$ and $\alpha=0.7$ GeV. For $\alpha=0.635$ GeV and $m_q=0.24$ GeV with $F_{2u}=-0.025$ and $F_{2d}=-0.047$ we find good agreement with recent measurement [35] of the ratio $Q^2F_{2p}/(\mu_p-1)F_{1p}$ shown in Fig. 8. Earlier measurements [34] indicated larger values. We used a constant value of 0.7 for this ratio in order to extract from Ref. [40] form factors F_{1p} and F_{2p} shown in Figs. 5 and 6.

VI. SUMMARY AND OUTLOOK

Nonrelativistic constituent-quark models of the electromagnetic structure of nucleons are always inconsistent even for small values of the momentum transfer. Our exploratory calculations have demonstrated that relativistic constituent-quark models can describe the available data for $0\le Q^2<6$ GeV^2 provided the quark mass is smaller than the conventional choice. It is inherent in the notion

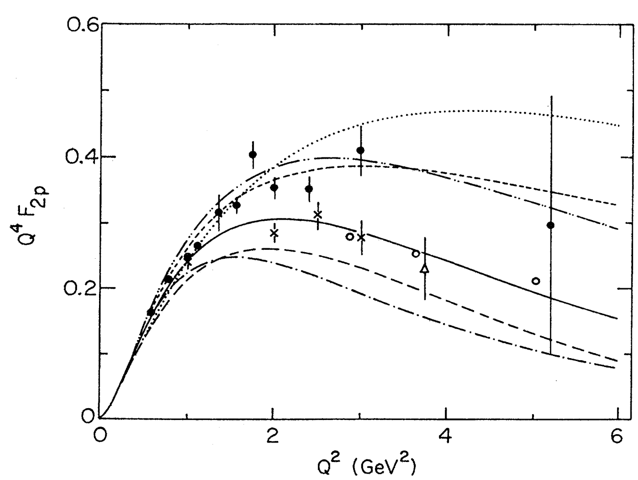

FIG. 6. The proton form factor $Q^4F_{2p}(Q^2)$. The lines and data points are labeled as in Fig. 5.

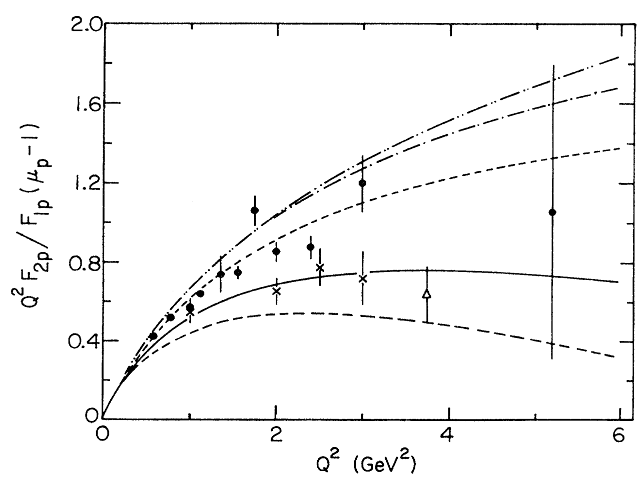

FIG. 8. The ratio of proton form factors $Q^2F_{2p}/(\mu_p-1)F_{1p}$ for various quark masses m_q and confinement parameters α. The lines and data points are labeled as in Figs. 5 and 6.

of constituent quarks that they may have form factors different from those of Dirac point particles. In the present preliminary investigation we assumed quark form factors independent of Q^2. To ascertain the limits of validity of these models for larger momentum transfer it will be necessary to consider bound-state wave functions with more realistic high-momentum features and admixtures of mixed permutation symmetry. It is also necessary investigate the role of quark form factors that have a reasonable Q^2 dependence. Such investigations must be extended to other hadrons, because success of constituent-quark models beyond hadron spectroscopy will depend on their ability to describe the properties of all hadrons with the same structure of the constituent quarks.

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APPENDIX A: Dirac-Spinor Representations

1. Spinor amplitudes of single particles

The Dirac-spinor representation S of the boost $L^{-1}(p)$ is

$$S(L^{-1}(p))=\frac{\vec{\alpha}_T\cdot\vec{\mathrm{p}}_T+p^+}{\sqrt{mp^+}}\frac{1+\alpha^3}{2}+\frac{m}{\sqrt{mp^+}}\frac{1-\alpha_3}{2} . \tag{A1}$$

Thus the null-plane spinor amplitudes

$$u(p):=S(L^{-1}(p))\frac{1+\beta}{2} \tag{A2}$$

are

$$\begin{aligned} u(p)&=\frac{\vec{\alpha}_T\cdot\vec{\mathrm{p}}_T+\beta m+p^+}{\sqrt{mp^+}}\frac{1+\alpha_3}{2}\frac{1+\beta}{2}\\ &=-\frac{\gamma\cdot p-m}{2\sqrt{mp^+}}\gamma^+\frac{1+\beta}{2} , \end{aligned} \tag{A3}$$

where $\gamma^\mu:=\beta\alpha^\mu$, $\alpha^0:=1$. For any Lorentz transformation Λ we have the covariance condition

$$S(\Lambda)u(p)=u(\Lambda p)\mathcal{R}_W[\Lambda,L(p)] , \tag{A4}$$

where $\mathcal{R}_w[\Lambda,L(p)]$ is the Wigner rotation corresponding to the Lorentz transformation Λ and the boost $L(p)$. Since the inverse of $S(\Lambda)$ is given by

$$S^{-1}(\Lambda)=\beta S(\Lambda)^\dagger\beta , \tag{A5}$$

the product $\bar{u}(p)u(p)\equiv^\dagger(p)\beta u(p)=1$ is manifestly Lorentz invariant. When Λ belongs to the subgroup of the null-plane boosts, i.e., the group generated by $\vec{\mathrm{E}}$ and K_n, the Wigner rotation $L(\Lambda^{-1}p)\Lambda L(p)$ is the identity. Thus it follows from the covariance of the spinor amplitudes that for any Lorentz transformation Λ which belongs to the group of the null-plane boosts we have

$$\begin{aligned} u(p)&=S(\Lambda)u(\Lambda^{-1}p) ,\\ \bar{u}(p)&=\bar{u}(\Lambda^{-1}p)S^{-1}(\Lambda) . \end{aligned} \tag{A6}$$

The canonical spinor amplitudes

$$u_c(p):=\frac{\vec{\alpha}\cdot\vec{\mathrm{p}}+\beta m+\omega}{\sqrt{2m(\omega+m)}}\frac{1+\beta}{2} , \tag{A7}$$

where

$$\omega:=\sqrt{\vec{\mathrm{p}}^{\,2}+m^2}=\frac{1}{2}\left[p^++\frac{m^2+\vec{\mathrm{p}}_T^{\,2}}{p^+}\right] \tag{A8}$$

and the null-plane spinor amplitudes $u(p)$ are related by the Melosh rotation $\mathcal{R}_M(p)$:

$$u(p)=u_c(p)\mathcal{R}_M(p) , \tag{A9}$$

where

$$\mathcal{R}_m(p):=\frac{p^++m-i\vec{\sigma}\cdot\vec{\mathrm{n}}\times\vec{\mathrm{p}}_T}{\sqrt{2p^+(\omega+m)}} . \tag{A10}$$

Note that

$$2p^+(m+\omega)=(p^++m)^2+\vec{\mathrm{p}}_T^{\,2} . \tag{A11}$$

The null-plane representation $\Psi(\mathbf{p})$ of any one-particle state with the norm

$$\|\Psi\|^2=\sum_\lambda\int d^3\mathbf{p}|\Psi(\mathbf{p},\lambda)|^2 \tag{A12}$$

is related to the canonical representation $\Psi_c(\vec{\mathrm{p}})$ with the norm

$$\|\Psi\|^2=\sum_\sigma\int d^3\vec{\mathrm{p}}|\Psi_c(\vec{\mathrm{p}},\sigma)|^2 , \tag{A13}$$

by

$$\sqrt{p^+}\Psi(\mathbf{p})=\mathcal{R}_M^\dagger(p)\Psi_c(\vec{\mathrm{p}})\sqrt{\omega} . \tag{A14}$$

The Dirac-spinor representation of the same state is

$$\Psi(p)=u(p)\Psi(\mathbf{p})\sqrt{p^+}=u_c(p)\Psi_c(\vec{p})\sqrt{\omega} . \tag{A15}$$

In terms of this spinor representation the norm is given by the manifestly invariant expression

$$\|\Psi\|^2=\int d^4p\, 2\delta(p^2+m^2)\bar{\Psi}(p)\Psi(p) . \tag{A16}$$

2. Spinor representations of three-quark wave functions

The spinor representation of the bound-state wave function ψ_{M_N} is given by

$$\mathbf{X}(\vec{q}_1,\vec{q}_2,\vec{q}_3):=u_1(\vec{q}_1)\otimes u_2(\vec{q}_2)\otimes u_3(\vec{q}_3)\psi_{M_N}(\vec{q}_1,\vec{q}_2,\vec{q}_3)\bar{u}_N(0)$$
$$=\phi(q_1^2,q_2^2,q_3^2)u_{c1}(\vec{q}_1)\otimes u_{c3}(\vec{q}_3)\mathbf{\Phi}u_{c2}^T(\vec{q}_2)\otimes\bar{u}_{cN}(0)$$
$$=\phi(q_1^2,q_2^2,q_3^2)\left[\frac{8m^3}{(m+\omega_1)(m+\omega_2)(m+\omega_3)}\right]^{1/2}\mathbf{\Gamma}\ , \tag{A17}$$

where

$$\mathbf{\Gamma}:=\left[\Gamma_{1,2}^0\otimes 1_{3,N}+\sum_\alpha \Gamma_{1,2}^{\mu,\alpha}\otimes\Gamma_{3,N}^{\nu,\alpha}g_{\mu\nu}\right] \tag{A18}$$

and

$$\Gamma_{1,2}^0:=-\frac{1}{\sqrt{2}}\frac{1+\beta}{2}\gamma_5 C\otimes i\tau_2,\quad \Gamma_{1,2}^{\mu,\alpha}:=\frac{1}{\sqrt{6}}\gamma^\mu C\otimes\tau_\alpha i\tau_2,\quad \Gamma_{3,N}^{\nu,\alpha}:=\frac{1+\beta}{2}\gamma^\nu\gamma_5\otimes\tau_\alpha\ . \tag{A19}$$

The charge-conjugation matrix C is $C:=i\gamma^2\gamma^0=-i\alpha_2$. To verify the result (A19) note that

$$\bar{u}_1(\vec{q}_1)\Gamma_{1,2}^0\bar{u}_2^T(\vec{q}_2)=\frac{1}{\sqrt{2}}\frac{1+\beta}{2}\frac{\vec{\alpha}\cdot\vec{q}_1+\beta m+\omega_1}{\sqrt{2m(m+\omega_1)}}\frac{1+\beta}{2}i\sigma_2\frac{\vec{\alpha}^*\cdot\vec{q}_2+\beta m+\omega_2}{\sqrt{2m(m+\omega_2)}}\frac{1+\beta}{2}i\sigma_2\otimes i\tau_2$$
$$=\frac{1}{\sqrt{2}}\left[\frac{(m+\omega_1)(m+\omega_2)}{4m^2}\right]^{1/2}\frac{1+\beta}{2}i\sigma_2\otimes i\tau_2\ , \tag{A20}$$

$$\bar{u}_3(\vec{q}_3)u_N(0)=\frac{1+\beta}{2}\frac{\vec{\alpha}\cdot\vec{q}_3+\beta m+\omega_3}{\sqrt{2m(m+\omega_3)}}\frac{1+\beta}{2}=\frac{1+\beta}{2}\left[\frac{m+\omega_3}{2m}\right]^{1/2}\ , \tag{A21}$$

and

$$\bar{u}_1(\vec{q}_1)\Gamma_{1,2}^{k,\alpha}\bar{u}_2^T(\vec{q}_2)=\frac{1}{\sqrt{6}}\frac{1+\beta}{2}\frac{\vec{\alpha}\cdot\vec{q}_1+\beta m+\omega_1}{\sqrt{2m(m+\omega_1)}}i\beta\sigma_k\sigma_2\frac{\vec{\alpha}^*\cdot\vec{q}_2+\beta m+\omega_2}{\sqrt{2m(m+\omega_2)}}\frac{1+\beta}{2}\otimes i\tau_\alpha\tau_2$$
$$=\frac{1}{\sqrt{6}}\left[\frac{(m+\omega_1)(m+\omega_2)}{4m^2}\right]^{1/2}\frac{1+\beta}{2}i\sigma_k\sigma_2\otimes i\tau_\alpha\tau_2\ , \tag{A22}$$

$$\bar{u}_3(\vec{q}_3)\frac{1+\beta}{2}\gamma^0\gamma_5u_N(0)=0\ ,\quad \bar{u}_3(\vec{q}_3)\frac{1+\beta}{2}\gamma_k\gamma_5u_N(0)=\frac{1+\beta}{2}\frac{\vec{\alpha}\cdot\vec{q}_3+\beta m+\omega_3}{\sqrt{2m(m+\omega_3)}}\gamma_k\gamma_5\frac{1+\beta}{2}=\frac{1+\beta}{2}\left[\frac{m+\omega_3}{2}\right]^{1/2}\sigma_k\ . \tag{A23}$$

It follows that the full wave function

$$\Psi_{p_N}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\psi_{M_N}(\vec{q}_1,\vec{q}_2,\vec{q}_3)\delta(\mathbf{P}-\mathbf{p}_N)\ , \tag{A24}$$

for an arbitrary nucleon momentum p_N is related to the spinor amplitudes $\mathbf{X}$ by

$$\psi_{M_N}(\vec{q}_1,\vec{q}_2,\vec{q}_3)$$
$$=\bar{u}_1(\vec{q}_1)\otimes\bar{u}_3(\vec{q}_3)\mathbf{X}(\vec{q}_1,\vec{q}_2,\vec{q}_3)\bar{u}_2^T(\vec{q}_2)\otimes u_N(0)$$
$$=\phi(q_1^2,q_2^2,q_3^2)\left[\frac{8m^3}{(m+\omega_1)(m+\omega_2)(m+\omega_3)}\right]^{1/2}$$
$$\times\bar{u}_1(\vec{q}_1)\otimes\bar{u}_3(\vec{q}_3)\mathbf{\Gamma}\bar{u}_2^T(\vec{q}_2)\otimes u_N(0)\ . \tag{A25}$$

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