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Inevitability of shell crossing in the gravitational collapse of weakly charged dust spheres

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It is shown here that shell crossing is inevitable in the gravitational collapse of weakly charged (|charge density| < mass density) dust spheres. That is, all the dust shells in the central part of the dust cloud are subject to shell crossing before they reach the stage of maximal contraction. This shell crossing, which is accompanied by a density singularity, indicates the breakdown of the dust model there. Most previous analyses of the gravitational collapse and gravitational bounce of charged dust spheres were based on the "free-surface approach"—the assumption that the surface of the dust sphere moves like a test particle in Reissner-Nordström spacetime. Because of the occurrence of shell crossing, the free-surface approach cannot be applied to the late stages of evolution. This makes it difficult to analyze, in general, what will be the final stage in the gravitational collapse of charged dust. However, for idealized models, in which the dust interior is considered to be self-similar and the shells are considered to collide inelastically, the shell crossing is shown to have a very significant effect on the causal structure: It completely prevents the gravitational bounce. This result, the inevitability of shell crossing (and its potential to affect drastically the causal structure), is in remarkable contrast to common ideas about charged-dust collapse.

I. INTRODUCTION

The idea of gravitational bounce [1–3] presents an interesting possibility for the final state in gravitational collapse. It is well known that whenever a black hole is formed in a gravitational collapse, there must be singularity inside the hole's horizon. According to the gravitational-bounce idea, deviations from spherical symmetry and/or quantum-gravity effects, which become important near the curvature singularity, may convert the contraction into expansion. The collapsing matter may thereby avoid the singularity and emerge, through a white hole, into a "new" asymptotically flat universe. To illustrate this idea, Novikov [1,2] considered the gravitational collapse of charged dust spheres. In Ref. [2] he analyzed such a homologous collapse with a uniform specific charge ε satisfying $|\varepsilon| \ll 1$. Here $\varepsilon \equiv e/\rho$, where e is the charge density, ρ is the mass density of the dust, and we set $G=C=1$. Novikov concluded that a gravitational bounce actually occurs in this case: the collapsing dust reexpands, without any divergence of density (except for the particle at the center). The matter then emerges through a white hole into a second asymptotically flat universe.

Later it was found that the innermost regions of charged black holes are unstable to electromagnetic and/or gravitational perturbations (the blue-sheet instability of the inner Cauchy horizon [4]). Because of this instability it is generally believed today that the picture of gravitational bounce, obtained from the spherical charged dust model, is oversimplified and irrelevant to realistic collapse. Nevertheless, for the issue of the final state of gravitational collapse it is important to understand the possible effects of this instability on the structure of the spacetime [5]. In order to study the effects of any instability, it is necessary to have a good understanding of the background solution. This is a sufficient reason to study gravitational-bounce solutions, even though they are unstable. Of course, another motivation comes from the special, unusual aspects of the gravitational bounce.

Until recently, the general solution of the Maxwell-Einstein equations for the evolution of charged dust spheres was not known [6,7]. (The general solution was found recently [8].) It has long been known, however, that the spherical electro-vacuum exterior of such a sphere is described by the Reissner-Nordström (RN) line element:

$$ds^2=\Delta\, dt^2-\Delta^{-1}dr^2-r^2 d\Omega^2 \;, \tag{1}$$

where $\Delta=1-2m/r+Q^2/r^2$, $d\Omega^2$ is the line element on the unit two-sphere, and m and Q are the mass and the charge of the dust spheres, respectively. In addition, there is no coupling between the motion of the various dust shells. The world line of each dust shell (including the surface of the sphere) is equivalent to that of a (charged) test particle which moves radially in a RN geometry (a geometry whose mass and charge are those produced by all shells internal to the shell being studied). This provides one with a simple approach to the evolution of charged dust spheres, which I will call the *free-surface approach.* The analysis in Ref. [2] was based on that approach.

It is well known that a test particle with $|\varepsilon|<1$ which falls radially in RN spacetime with $|Q|<m$ does not hit the central singularity. Instead, as the particle reaches some minimal r value, its motion is reversed, and it is finally ejected into an external asymptotically flat universe (see e.g., Refs. [2,3,9], and the discussion in Sec. II). Pictorially, we say that the particle moves through a "tunnel" into "another universe." In a charged dust sphere with $|\varepsilon|<1$, each shell "feels" $|Q|<m$ [10]. In

such a case one could expect, on the basis of the free-surface approach, that all the dust shells will undergo a gravitational bounce into the other universe.

As was pointed out in Ref. [2], the condition for the independent motion of each shell (and for the validity of the free-surface approach) is that there is no shell crossing, i.e., that the world lines of the various dust shells do not intersect. It is argued there, however, that for reasonable initial data (such as homologous collapse with $|\varepsilon| \ll 1$) shell crossing does not occur [11].

In this paper I analyze the relative motion of the collapsing dust shells. This analysis reveals that *shell crossing is inevitable in the gravitational collapse of any spherical charged dust configuration with* $|\varepsilon|<1$. More precisely, for every regular initial distribution of spherical charged dust with $|\varepsilon|<1$, shell crossing must occur, and all the shells in some vicinity of the center are subject to shell crossing before they reach the stage of maximum contraction. This result is proved in Sec. III, and then further explained in Sec. IV.

The occurrence of shell crossing makes it difficult to analyze the subsequent evolution of the dust sphere. One might think that the shell crossing is just a technical problem, which complicates the analysis but does not affect the qualitative structure of the spacetime: After all, if each individual shell, by its independent motion, tends to bounce, it might be expected that the mutual interaction between dust shells will somewhat modify the pattern of dust flow, but not so drastically as to prevent the gravitational bounce. To demonstrate that this is not necessarily the case, I will briefly describe the results obtained for a simple, idealized model of a collapsing charged dust cloud. The dust interior of this model is marginally bound and self-similar. It is also assumed that the dust shells collide in a completely inelastic manner [12]. The causal structure which results from this simple model is rather surprising: The Reissner-Nordström tunnel is sealed by a null spacetime singularity. This null singularity is created at the center of the dust sphere, and then evolves to intersect the external Reissner-Nordström singularity (see Fig. 2). Thus, it completely blocks the tunnel and prevents the gravitational bounce.

This paper is organized as follows. Section II outlines the basic features of the Reissner-Nordström geometry which are relevant to the analysis here. In Sec. III, I briefly discuss the principles governing the evolution of charged dust spheres, and prove that shell crossing is inevitable. In Sec. IV, I use a simple charged dust model to explain, intuitively, why shell crossing must occur. Finally, in Sec. V, I discuss the results and give some concluding remarks. In particular, I discuss the possible effects of the shell crossing on the causal structure and show, using a simplified self-similar model, that these effects may be very significant.

II. THE REISSNER-NORDSTRÖM GEOMETRY

The causal structure of (analytically extended) RN spacetime with $|Q|<m$ is shown in Fig. 1 [13]. The external and internal horizons are located at $r=r_+$ and $r=r_-$, respectively, which are the two r values where Δ vanishes:

$$r_\pm = m \pm (m^2-Q^2)^{1/2} . \tag{2}$$

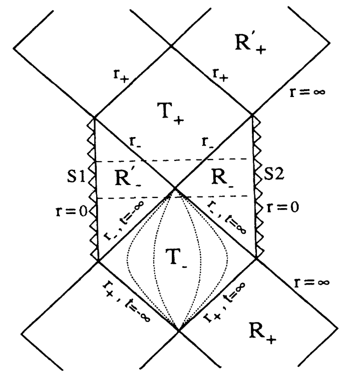

FIG. 1. Penrose diagram of the extended RN spacetime (see text).

The spacetime is composed of various R and T regions. R regions are those where $r>r_+$ or $r<r_-$. In such regions $\Delta>0$, so that the r coordinate is spacelike and the t coordinate is timelike. In the T regions, $r_- < r < r_+$, Δ is negative, so that r is timelike and t is spacelike. R_+ is the external asymptotically flat universe, where $r>r_+$. T_- is a region of trapped surfaces, and the only possible motion there is towards smaller r values. Similarly, T_+ is a region of antitrapped surfaces, where the only possible motion is towards larger r values. In the two innermost regions, labeled R_- and R'_-, $r<r_-$ and Δ is again positive. In each of these regions there is a central ($r=0$) timelike singularity ($S1$ in R'_- and $S2$ in R_-). This singularity is gravitationally repulsive, and massive test particles with $|\varepsilon|<1$ cannot hit it. The equations of motion for such a test particle in RN spacetime can be easily integrated to obtain [9]

$$u_t \equiv E - \varepsilon Q/r , \tag{3}$$

where $u_\alpha \equiv g_{\alpha\beta}u^\beta$, $u^\alpha \equiv dx^\alpha/d\tau$, and τ is the proper time. The conserved quantity E is the total mass-energy of the particles at infinite r, per unit rest mass (that is, a marginally bound particle has $E=1$). For radial motion, the normalization of the four-velocity reads $u^r u_r + u^t u_t = 1$, and Eq. (3) then yields

$$\frac{dr}{d\tau} = u^2 = \pm \left[(E^2-1) + \frac{2(m-\varepsilon EQ)}{r} - \frac{Q^2(1-\varepsilon^2)}{r^2} \right]^{1/2} . \tag{4}$$

For a test particle with $|\varepsilon|<1$ there exists some r value, $0<r_{\min}<r_-$, for which u^r vanishes. From Eq. (4), $r_{\min}$ is given by

$$(r_{\min})^{-1} = \frac{(m-\varepsilon EQ) + [(m-\varepsilon EQ)^2 + (E^2-1)Q^2(1-\varepsilon^2)]^{1/2}}{Q^2(1-\varepsilon^2)} . \tag{5}$$

For $0<r<r_{\min}$ the square root in Eq. (4) becomes imaginary, indicating that this range is forbidden. This means that at $r=r_{\min}$ u^r changes sign and becomes positive (an alternative way to show this change of sign is by analyzing the fundamental, second-order equation of motion, acceleration=Lorentz force, at $r=r_{\min}$). The particle then begins to move outward in a time-symmetric manner. It passes through T_+, and finally reaches the second asymptotically flat region R'_+.

From the light cone structure in Fig. 1 it is obvious that the regions R_- and R'_- are causally disconnected, so an infalling particle must select one of these two patches. As was shown in Ref. [9], this selection depends on the sign of u_t at $r=r_-$: If $u_t(r_-)<0$, R_- is selected and vice versa. This is also apparent from the pattern of the dotted lines $t=$const in T_- (see Fig. 1). Note that u_t is always positive in R_+ (in fact, this requirement defines the positive t direction). Also, u_t cannot change sign in any R region, because the t coordinate is timelike there [this can also be shown directly from Eq. (3)]. u_t is always positive in R'_- and negative in R_-.

The horizontal dashed lines in Fig. 1 represent spacelike slices which intersect the two $r=0$ singularities. The geometry of such a slice is semiclosed (i.e., it would be closed if the two $r=0$ singularities were absent). For each slice, let L be the proper distance, measured from the singularity $S1$. The semiclosure is demonstrated by the behavior of the function $r(L)$. Unlike the "normal" situation (i.e., in a spacetime without trapped or antitrapped surfaces), this function is not monotonic: it increases in the region R'_-, attains a maximum in T_- (or T_+), and then decreases again to zero in R_-. This semiclosure is essential for the discussion below.

III. INEVITABILITY OF SHELL CROSSING

The evolution of charged dust spheres has been discussed by several authors (see e.g., Refs. [2,6,14]). As long as shell crossing does not occur, there is no coupling between the motion of the various dust shells. Each shell moves freely as a test particle in a (hypothetical) RN spacetime with m and Q being the total mass and charge enclosed in that shell. Hereafter, we shall use the quantity m to label the dust shells. The velocity u^r of each shell is given by Eq. (4), where Q, E, and ε are regarded now as functions of m.

Accordingly, we define $r_\pm(m), r_{\min}(m)$, and $u_t^m(r)$ by substituting the corresponding m-dependent parameters in Eqs. (2), (5), and (3), respectively. The dust cloud is assumed to be initially regular. In normal situations, the existence of a regular center implies [14] $E_0\equiv E(m=0)=1$. If initially the dust cloud includes a central Minkowski bubble, $E_0>1$ is possible as well, but E_0 can never be smaller than unity without violating the regularity of the initial slice.

Let $N(m)$ denote the rest mass of the dust enclosed in the shell m. It is related to m and Q by

$$\frac{dm}{dN}=E, \quad \frac{dQ}{dN}=\varepsilon \ . \tag{6}$$

Therefore, the conserved quantities ε, E and Q are related by [14]

$$\varepsilon=E\frac{dQ}{dm} \ . \tag{7}$$

The geometry and the topology of the dust interior are different from those of RN spacetime. It is convenient, however, to extend the terminology of the RN R and T patches to the charged-dust case. The motion of each dust shell is equivalent to that of a test particles in RN, and we can use this equivalence to define the desired extension. Thus, we say that a point O on the world line of a given shell m belongs to $R_-, R'_-, R_+, R'_+, T_-$, or T_+, according to the location of O when we embedded the world line $r(\tau)$ of the shell m in an RN spacetime with the corresponding parameters m and $Q(m)$. One can easily show that, as in an RN geometry, each shell will select one of the two innermost patches, R_- or R'_-. It is also trivial to show that a shell m with $u_t^m(r=r_-(m))<0$ will pass through R_-, and one with $u_t^m(r=r_-(m))>0$ will select R'_-.

In the following discussion I assume that at the center ($m=0$) $0<|\varepsilon|<1$ (the assumption $0<|\varepsilon|$ is not necessary for the result, but it simplifies the mathematical treatment). In order to show that shell crossing is inevitable, I first prove the following three lemmas, with the assumption that no shell crossing occurs. I then show that these lemmas lead to a contradiction, and therefore this assumption must be wrong.

Lemma 1. All the dust shells within the central part of the dust sphere (defined by $m\le m_0$, for some parameter $m_0>0$) collapse into R_-.

Proof. Regularity at the center implies that in the limit $m\to 0$, Q vanishes. Eq. (7) then reads

$$\lim_{m\to 0}\frac{Q(m)}{m}=\lim_{m\to 0}\frac{dQ}{dm}\equiv Q'_0=\frac{\varepsilon_0}{E_0} \ , \tag{8}$$

where $\varepsilon_0\equiv\varepsilon(m=0)$. Here the prime denotes derivative with respect to m. By assumption, we have $0<|\varepsilon_0|<1$, and (as we discussed above) regularity at the center requires $E_0\ge 1$. Equation (3) then implies

$$\lim_{m\to 0} u_t^m(r=r_-(m))=E_0\left[1-\frac{Q_0'^2}{1-\sqrt{1-Q_0'^2}}\right] =-E_0\left[1-\frac{\varepsilon_0^2}{E_0^2}\right]^{1/2}<0 \ . \tag{9}$$

Thus, there exists $m_0>0$ such that for each shell with $m\le m_0, u_t^m(r_-(m))<0$. According to the above discussion, all these shells collapse into R_-.■

Note that this result is in a remarkable contrast to the discussion in Ref. [2] (see Sec. 3 there). In this regard, recall that if we look at the radial motion of test particles in a *fixed* RN spacetime, we find that in the limit $|\varepsilon|\ll 1$ (but with fixed $E>0$) the particles go through R'_-. This is apparent from Eq. (3) when we set $r=r_-$, because both Q and r_- are fixed. In the charged dust problem discussed here the situation is different, however, because the relevant RN parameters are determined by the dust itself. Thus, when we vary ε, the parameters Q and r_- (for a given m) are changed as well. In particular, in the limit $|\varepsilon| \ll 1$, $r_-/|Q|$ becomes $\ll 1$ too. Therefore, the term $-\varepsilon Q/r$ in Eq. (3) does not vanish in this limit, and its negative contribution overcomes the positive contribution of E.

Lemma 2. All the shells in the central region (defined by $m \le m_1$, for some $m_1 > 0$) satisfy

$$\frac{dr_{\min}(m)}{dm} > 0 \ . \tag{10}$$

Proof. It is straightforward to show, from Eqs. (5) and (8), that for reasonable regularity conditions on the behavior of $E(m)$ and $e(m)$ at $m = 0$ [15], one obtains

$$\lim_{m\to 0}\frac{dr_{\min}}{dm} = \lim_{m\to 0}\frac{r_{\min}(m)}{m} = \frac{\varepsilon_0^2}{E_0^2}\left[1+\sqrt{(1-\varepsilon_0^2/E_0^2)/(1-\varepsilon_0^2)}\,\right]^{-1} . \blacksquare \tag{11}$$

Lemma 3. All the dust shells in R_- satisfy, at any fixed moment of comoving time, $dr/dm < 0$.

Proof. Let M and M' be two neighboring shells that pass through R_-, with $dm \equiv M - M' > 0$ infinitesimally small (here the symbols M and M' denote the dust shells themselves, and at the same time they express their m values). We define $dL(\tau)$ to be the (time-dependent) infinitesimal proper distance between these two shells, as measured by a comoving observer. According to Eq. (6), dL is related to dm by $dm = E\,dN = E\rho\,dV = 4\pi E\rho r^2 dL$, where $dV = 4\pi r^2 dL$ is the three-dimensional volume element. On the other hand, dL is related to dr, the difference in the r value at a fixed comoving time, by $dr/dL = n^r$. Here n^α is the (radial) unit vector normal to the four-velocity u^α.

Imagine now that we remove all the dust shells with $m > M$ and replace them by an electrovacuum (RN) exterior. This does not influence the geometry of the dust interior; hence $dr(\tau)$ and $dL(\tau)$ are unchanged. One can easily verify that any radial world line in RN spacetime satisfies $n^r = u_t$. Clearly, this holds also for the world line of M, which is the surface of the new (hypothetical) dust sphere. Since normal derivatives are continuous on M, for its both sides we have $dr/dL = u_t = u_t^M$. This yields $dr/dm = u_t^M/4\pi E\rho r^2$. Since u_t is negative in R_- (and it is assumed that both ρ and E are positive), dr/dm is negative there as well.$\blacksquare$

In R_+, since $u_t > 0$, dr/dm is always positive. This conforms with our simple intuition about normal distributions of matter (i.e., when spacetime is not too curved). The transition from positive to negative values of dr/dm occurs at $r = \varepsilon Q/E$, which is always located in T_- [16]. Note that the negative value of dr/dm in R_- is directly connected to the semiclosure of spacelike hypersurfaces there [and to the fact that $r(L)$ is monotonically decreasing there], which we discussed is Sec. II.

We can now easily prove that all the shells in the central part of the dust sphere are subject to shell crossing. To show this, we choose $M < \min(m_0, m_1)$ and $M' \equiv M - dm$ (with $dm > 0$ and infinitesimally small). In view of lemma 1, M and M' collapse into R_-. At the (comoving) moment that M' attains its minimal r value we have, according to lemma 3, $r_{\min}(M') = r(M') > r(M) \ge r_{\min}(M)$. This, however, contradicts the statement of lemma 2. This contradiction indicates that the world lines of M and M' must intersect somewhere before $r_{\min}(M')$ is approached. The occurrence of shell crossing removes the contradiction, because the redistribution of the dust shells breaks the conservation of m, E and Q for each shell.

We conclude, therefore, that all the shells with $m < \min(m_0, m_1)$ are subject to shell crossing before they reach their minimal r value.

IV. INTUITIVE EXPLANATION FOR THE OCCURRENCE OF SHELL CROSSING

In the preceding section, the inevitability of shell crossing was proved by showing that the other possibility is mathematically impossible. It is worthwhile to complement this proof by an intuitive discussion which explains, from the positive point of view, why shell crossing must occur. For that purpose, let us focus attention on the case $|\varepsilon| = \text{const} \ll 1$. Let us also assume that the dust cloud is initially homogeneous (that is, ρ is uniform on the initial slice), and that all the shells are marginally bound ($E = 1$). This would be approximately the case if, for instance, initially the homogeneous dust configuration is at rest and is very dilute (nonrelativistic). This is basically the model that was considered in Ref. [2] (see Appendix III there).

In general relativity, by contrast with Newtonian theory, there is no exact spherically-symmetric homogeneous solution for charge dust. This is due to the gravitational effect of the energy density associated with the electric field. This energy density always vanishes at the center, but is in general nonvanishing elsewhere, which breaks the homogeneity. Nevertheless, as long as the electric field energy density is much smaller than the dust density, this effect is insignificant, and a configuration which is initially homogeneous evolves in an approximately homogeneous manner. In particular, one finds that for the case $|\varepsilon| \ll 1$ considered here, the configuration evolves approximately homogeneously as long as m/r is much smaller than ε^{-2}. On the other hand, when m/r becomes comparable to ε^{-2}, the special features of the RN geometry become important. For instance, both $m/r_{\min}$ and m/r_- are $\cong 2\varepsilon^{-2}$. The evolution of our initially homogeneous configuration is schematically divided, therefore, into two states: the *homogeneous stage,* and the *bounce stage.* This can also be seen from the equation of motion, Eq. (4), which now reads

$$\frac{dr}{d\tau} = u^2 = -(1-\varepsilon^2)^{1/2}\left[\frac{2m}{r} - \varepsilon^2\frac{m^2}{r^2}\right]^{1/2} . \tag{12}$$

In the homogeneous stage, the second term in the square root is insignificant, and the velocity is approximately the same as in the collapse of homogeneous, marginally-bound, neutral dust [the constant factor $(1-\varepsilon^2)^{1/2}$ does not break the homogeneity; its effect is equivalent to a rescaling of the gravitational constant G, due to the Lorentz repulsion].

Note that each dust shell reaches the bounce stage at a different time. To understand the evolution of the dust configuration in the bounce stage, it is important to know which shell is the first to bounce. The time slicing relevant for this discussion is the comoving time. On the other hand, Eq. (12) [or Eq. (4)] is written in term of the proper time. Fortunately, in the homogeneous stage, due to the similarity to the Robertson-Walker solution, the proper time coincides with the comoving time. Let us find, qualitatively, how the lapse of time from the initial slice to the bounce stage depends on m. Initially, due to the homogeneity, we have $m/r \propto r^2 \propto m^{2/3}$. Therefore, the contraction factor required for a shall labeled m to approach the bounce stage, $[m/r]_{\text{bounce stage}}/[m/r]_{\text{initial slice}}$, is proportional to $\varepsilon^{-2}m^{-2/3}$. The external shell (i.e., the shell with the maximal m value, which is the surface of the dust sphere) requires the smallest contraction factor to approach the bounce stage. In the homogeneous stage all the shells have the same rate of contraction. Therefore, the external shell is the first to bounce.

Under "normal" conditions [i.e., in a weakly curved spacetime, or whenever the function $r(L)$ is monotonically increasing], as the external shells bounce before the internal ones, one would not expect the shells to cross each other. In Ref. [2] it was presumed that the collapsing shells pass through R'_-, and the bounce occurs there. Since the geometry in R'_- is "normal" [the function $r(L)$ is increasing], it was concluded there that no shell crossing is expected in the $|\varepsilon| \ll 1$ homogeneous model. However, in Sec. III (lemma 1) it was shown that the dust shells collapse into R_-. (The statement there is limited to the shells in the central part of the dust sphere. Nevertheless, for the homogeneous model discussed here, one can easily extend this result to include all the dust shells.) The bounce occurs in R_-, where the function $r(L)$ is decreasing, and the shell at the surface now has an r value smaller than that of its neighboring shells (dr/dm is negative). That is, in R_- the "external" shell is actually *surrounded* by its neighboring shells (the inversion in the sign of dr/dm occurs in T_-, where the coordinate r is timelike and the statement that a shell is surrounded by another shell is meaningless; see the discussion below). This situation is directly related to the semiclosure of the spacelike hypersurfaces, the horizontal dashed lines in Fig. 1, which was discussed in Sec. II.

It is now clear why the shells must cross each other. The external shell, which has the maximal m value, is the first to bounce. At the moment that it bounces, the neighboring shells, which are now surrounding it, are still collapsing. This tendency of the external shell to reexpand (i.e., to increase its r value), while the shells surrounding it are still collapsing, leads to shell crossing.

This argument is based on the fact that in R_-, $dr/dL \neq 0$, and if at some (comoving) moment there are two neighboring spherical dust shells with the same r value, the distance between these shells must vanish. This statement is not valid everywhere: For instance, in any comoving time slice of a closed Friedmann universe, there is a spherical shell with a maximal r value, and dr/dL vanishes there. However, this statement is always valid in any R region, for the following reason. If $n^r = dr/dL$ vanishes, the (radial) normal vector n^α is tangent to dt. On the other hand, since n^α is normal to a timelike shell, it must be spacelike. In any R region, dt is timelike, hence dr/dL cannot vanish.

In the above discussion I preferred, for conceptual clarity, to discuss the occurrence of shell crossing at the external shell. It is clear, however, that the same argument is valid for all the dust shells interior to this external one. A particular shell does not "feel" the shells external to it; hence each shell can be treated as a "boundary." Therefore, in this homogeneous model all the shells are involved in shell crossing.

In the more general, inhomogeneous case, we cannot show that all the shells are subject to shell crossing. Nevertheless, for a given initial slice with a regular behavior at the center, we can restrict attention to a small, concentric ball that includes the center. For a sufficiently small ball, the effects of inhomogeneity will be insignificant, and we can use the above qualitative arguments to analyze the evolution. It is not surprising, therefore, that for any regular initial data the central shells are necessarily subject to shell crossing.

V. DISCUSSION

In this paper I have shown that in the gravitational collapse of weakly charged-dust spheres, shell crossing is inevitable. More precisely, for every initially regular spherical distribution of weakly charged ($|\varepsilon| < 1$) dust, all the shells in some vicinity of the center are subject to shell crossing prior to their bounce.

Spherical dust models, either neutral or charged, are often used to address various problems in general relativity—mainly due to their simplicity. This simplicity results from the lack of coupling between the motion of the various dust shells (so long as the shells do not cross each other). In particular, the dust sphere's surface moves as a test particle in a Schwarzschild or RN spacetime, independently of the interior geometry, and knowledge of how the world line of this surface is embedded in the external (electro-)vacuum geometry provides us with an important piece of information about the global structure. This is true even if we do not know anything about the interior geometry.

Shell crossing, however, breaks down this simple law of evolution. In particular, the motion of the surface is no longer guaranteed to be independent of the evolution of the interior geometry. For simplicity, let us consider dust configurations for which *all* the shells are subject to shell crossing. This is found to be the typical case for sufficiently homogeneous initial data (moreover, if a given configuration does not satisfy this requirement, we can always focus attention on a sufficiently small central core of it, imagining that this central core is immersed in electrovacuum. In this way, from any initially regular configuration we can obtain a smaller configuration in which all the shells are subject to shell crossing).

If we assume that the shells cross each other without interaction, then the original surface shell will penetrate into the dust cloud [12]. The shell which then becomes the new "surface" will immediately thereafter penetrate into the cloud, leaving a new shell as the surface, and so on. The only way to avoid this complication is by assuming that, as a consequence of the shell crossing process, some sort of interaction (pressure, viscosity, etc.) will appear and prevent the shells from physically crossing each other. This assumption seems reasonable, because shell crossing is always associated with a divergence of the dust density, and also a divergence of the dust's shear tensor. Even if, for some physical system, the dust approximation is valid under normal (nonsingular) conditions, the extremely high density and shear may break it just before the shell crossing.

With the presence of such an extra interaction, the original surface shell will remain at the surface, but it will no longer move freely. It is clear, therefore, that once shell crossing occurs at the external shell, the world line of the surface will no longer evolve freely, and will necessarily be influenced by the evolution of the dust interior. Indeed, this influence makes it difficult to analyze the evolution after the occurrence of shell crossing.

What can be the effect of shell crossing on the causal structure of the spacetime? In order to get insight into this question, I have analyzed the evolution of several marginally bound self-similar charged-dust solutions [8], after the occurrence of shell-crossing I selected these solutions for their especially simple form [17]. To explore the causal structure of the self-similar interior I used the method developed by Ori and Piran to analyze the causal structure of spherically symmetric self-similar solutions [18]. To model the local behavior of the colliding dust shells, I assumed that the collision is completely inelastic. According to this assumption, the colliding shells cannot cross each other; rather, they are stuck together, forming a thin massive layer (a delta function in the mass and charge distributions). The layer is massive, because in a completely inelastic collision all the matter enclosed between two neighboring shells is stuck together when these two shells collide. This massive layer forms, with zero mass and charge, at the first event of shell crossing (the event F in Fig. 2, which is located on the world line of the surface). As time passes, it collects all the dust shells that intersect its world line, and becomes massive and charged. The principles governing the evolution of such a massive thin layer were discussed in Ref. [19] (the equation of motion was explicitly derived in Refs. [3] and [20] for the case that both exterior and interior geometries are RN. The generalization to the case of dust interior is straightforward, however). I integrated the world line of the massive layer numerically, assuming that it has vanishing surface stresses.

The results obtained from this numerical integration

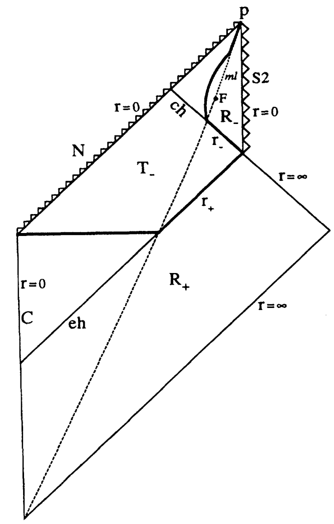

FIG. 2. A typical Penrose diagram of self-similar charged-dust collapse. The dust interior is to the left of the surface (the dashed line). This surface becomes a massive layer (the dotted line, denoted ml) at F, the first event of shell crossing. The veritcal line C is the regular center. eh and ch are the event and Cauchy horizons, respectively. N is the interior null singularity. The bold lines represent the inner and outer apparent horizons, which separate the region R_+, R_-, and T_-. The jagged lines denote spacetime singularities.

are rather surprising: For all the self-similar solutions that I have studied, it was found that the massive layer shrinks monotonically with time, and eventually crashes to the right-hand RN singularity $S2$ of the vacuum exterior (see Fig. 2). By its motion, the surface layer crosses the world lines of all the dust shells. The latter are presumed to be absorbed in the massive layer in the inelastic collision. Thus, eventually, all the collapsing matter is compressed into the $r=0$ singularity. A typical resultant Penrose diagram is shown in Fig. 2. For some sets of (marginally bound, self-similar) initial data, the Penrose diagrams are different from this one. However, in all the cases that I have studied, the resultant Penrose diagrams have a common feature: There is an "outgoing" null singularity in the dust interior (denoted N in Fig. 2). This singularity eventually intersects the right-hand RN singularity $S2$ (at the point P in Fig. 2). The massive layer, too, intersects these two singularities at P. Thus, the RN tunnel is completely sealed by the interior null singularity. Consequently, there is no gravitational bounce and no other asymptotically flat universe. Full details of my analysis of the causal structure of self-similar charged dust and the motion of the massive layer will be presented elsewhere.

This result, the collapse into a central singularity and the absence of a tunnel, is in remarkable contrast to the basic tendency of each individual dust shell to reexpand into a "new universe." It is therefore important to understand the difference in the motion of massive and nonmassive shells. For any nonmassive shell the geometry is smooth; i.e., the features of the gravitational field are the same on its two sides. By contrast, a massive shell may feel very different gravitational fields in its two sides. In particular, by its motion the massive layer penetrates into the region of trapped surfaces J_- of the interior geometry (cf. Fig. 2), while its external geometry is still that of the RN region R_-. The attractive force induced by the interior T_- region overcomes the repulsion of the exterior R_- region. The reason for this is simple. In principle, the gravitational repulsion in the exterior R_- can be overcome by attractive forces, but the gravitational attraction in the T_- interior can never be overcome by repulsive forces because the T_- region is one of trapped surfaces. The T_- interior thus wins the struggle, and the massive layer crashes to $r=0$.

It is important to note that this result, the replacement of the gravitational bounce by a complete collapse to $r=0$, has nothing to do with the density singularity associated with the shell crossing. The existence of shell crossing merely indicates that any attempt to analyze the motion of the surface of the dust cloud (and hence, the global structure) must take into account the structure of the could's interior. It is this interior structure (in particular, the trapped surfaces in the interior T_-) that prevents the reexpansion and pulls all the collapsing matter into the central singularity.

The analysis presented here does not exclude the possibility of completely regular evolution (without shell crossing) if $|\varepsilon|>1$. Normally, $|\varepsilon|>1$ everywhere will imply $|Q|>m$, which means that there is no black hole and no tunnel. However, if we give the infalling dust shells sufficiently high kinetic energy, m may become greater than $|Q|$ even though $|\varepsilon|>1$. Such gravitational bounce solutions of charged dust with $|\varepsilon|>1$ will be discussed elsewhere.

Independently of the research, by various people, on charged-dust collapse, de la Cruz and Israel [3] have studied the motion of spherical charged massive layers. This topic was further analyzed by Boulware [20]. For massive layers with a Minkowski interior and an RN exterior, it was found that a gravitational bounce inside a black hole is possible, but only if $|\varepsilon|>1$ and $|Q|<m$ (again, this requires a sufficient initial kinetic energy). Here ε is the ratio of the electric charge to the rest mass of the massive layer. This does not contradict the results presented here, which were obtained only for the case $|\varepsilon|<1$. Note, however, that the thin layers discussed there cannot be made of pure dust, for the following reason. Let us assume that initially the charged dust forms a very thin spherical configuration, with Minkowski interior and RN exterior. We keep in mind, however, that this configuration has a finite width (though very small), and it thus can be regarded as a collection of many independent, concentric dust shells. The innermost dust shell feels $m=Q=0$, and moves as a test particle in Minkowski spacetime, i.e., with a constant value of u^r, reaching $r=0$ within a finite proper time. On the other hand, the outer shell feels an RN geometry (with $|Q|<m$) and tends to bounce before $r=0$ is reached. Thus, such an initially-thin dust distribution must spread out when the outer shells begin to bounce. The rate of spreading out is virtually independent of the initial width. One thus finds (even in the limit of zero initial width) that a thin layer made of pure dust will spread out during the bounce stage. In order to preserve the layer's zero width, one has to introduce some finite, negative stress in the radial spacelike direction [21].

The analysis presented here, concerning the inevitability of shell crossing, was carried out before I found the general explicit solution for charged-dust spheres. The conclusions of this analysis can also be derived from my general solution (see Sec. 5 of Ref [8]). I choose, however, to present my original analysis instead, because of its simplicity and because it gives a better insight into the occurrence of the shell crossing.

ACKNOWLEDGMENTS

After this work was completed, I learned that J Bardeen performed a similar analysis many years ago, and arrived at similar conclusions. His analysis and results were never published. He also derived, but did not publish, the general solution for spherical charged dust [7]. I am grateful to him for an interesting conversation, in which we compared and discussed our results, and for communicating his unpublished notes to me. I would also like to thank K. Thorne, W. Israel, T. Piran, I. Novikov, and R. Nemiroff for helpful discussions. This research was supported by the Racah Institute of Physics at The Hebrew University of Jerusalem (where the research was initiated), and by NSF Grant No. AST-8817792 to Caltech, and BSF Grant No. 86-00346 to the Hebrew University.

[1] I. D. Novikov, Pis'ma Zh. Eksp. Teor. Fiz. **3**, 223 (1966) [JEPT Lett. **3**, 142 (1966)].

[2] I. D. Novikov, Astron. Zh. **43**, 911 (1967) [Sov. Astron. **10**, 731 (1967)].

[3] V. de la Cruz and W. Israel, Nuovo Cimento **51A**, 744 (1967).

[4] See, e.g., M. Simpson and R. Penrose, Int. J. Theor. Phys. **7**, 183 (1973).

[5] Recently, Poisson and Israel presented a simplified model which demonstrates the possible effects of the blue-sheet instability on the geometry (the mass-inflation model). See E. Poisson and W. Israel, Phys. Rev. Lett. **63**, 1663 (1989); Phys. Lett. B **233**, 74 (1989); Phys. Rev. D **41**, 1796 (1990).

[6] Several works supply us with important information (such as first integrals of the field equations) about charged-dust spheres. See, e.g., M. A. Markov and V. P. Frolov, Theor. Math. Phys. **3**, 301 (1970), and Vickers [14]. None of these works give explicit expressions for the evolution. One still has to solve a partial differential equation in order to complete these solutions.

[7] After this work was completed, I learned that J. Bardeen derived the general solutoin for spherical charged dust

many years ago, but never published it. See the abstract in Bull. Am. Phys. Soc. **13**, 41 (1968).

[8] A. Ori, Class. Quantum Grav. **7**, 985 (1990).

[9] F. de Felice and K. Maeda, Prog. Theor. Phys. **68**, 1967 (1982).

[10] This inequality may be violated if the parameter E is too small [see Eq. (7)]. Recall, however, that according to the discussion in Sec. III, in the central region one always has $E \geq 1$.

[11] See, in particular, Section 3 in Ref. [2].

[12] For a discussion on the physical nature of the shell-crossing process, see D. M. Eardley, in *Gravitation and Astrophysics,* edited by B. Carter and J. Hartle (Plenum, New York, 1987).

[13] For a discussion of the structure of RN spacetime, see, e.g., C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 34.

[14] P. A. Vickers, Ann. Inst. Henri Poincaré **43**, 137 (1973).

[15] A sufficient regularity condition is, for instance, that dE/dm and $d\varepsilon/dm$ have limits as $m \to 0$.

[16] By itself, the vanishing of dr/dL in T_- has nothing to do with shell crossing. Shell crossing corresponds to the vanishing of dL, rather than dr.

[17] Note that in the self-similar spherical charged-dust solution with $0<|\varepsilon|<1$, *all* the shells are subject to shell crossing.

[18] A Ori and T. Piran, Phys. Rev. Lett. **59**, 2137 (1987); Phys. Rev. D **42**, 1068 (1990).

[19] W. Israel, Nuovo Cimento **44B**, 1 (1966).

[20] D. G. Boulware, Phys. Rev. D **8**, 2363 (1973).

[21] One should not be confused by the phrase "dust" in the terminology related to massive layers (e.g., in Refs. [3] and [19]). A singular massive layer is regarded there as "dust" if it has no surface tensions. Clearly, this definition has nothing to do with the stress in the (spacelike) normal direction. In the terminology of continuous matter that we use here, dust is a perfect fluid for which the stresses vanish in all the three spacelike directions.