PHYSICAL REVIEW D VOLUME 44, NUMBER 1 1 JULY 1991

Masses of new particles containing b quarks

Waikwok Kwong* and Jonathan L. Rosner
Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637
(Received 21 January 1991)

Ranges of predicted masses for as yet unseen particles containing b quarks are obtained and compared with previous estimates. Nonrelativistic models are found to yield likely values in the ranges $M(B_s)=5345-5388$ MeV, $M(B_s^*)=5400-5433$ MeV, $M(B_c)=6194-6292$ MeV, $M(B_c^*)=6284-6357$ MeV, $M(\Lambda_b)=5600-5630$ MeV, with Σ_b and Σ_b^* not far above 5800 MeV.

I. INTRODUCTION

The recent observation of B mesons in hadronic collisions [1] has provided encouragement that other states containing b quarks may be visible soon. The B mesons were observed in the final states $J/\psi+K$ and $J/\psi+K^*$, indicating that the subprocess $b\to J/\psi+s$ plays an important role in b decays. If so, one might hope to detect $\bar{B}_s\equiv b\bar{s}$ via its decay to $J/\psi+(s\bar{s})$ (e.g., $J/\psi+\phi$), $\bar{B}_c\equiv b\bar{c}$ via its decay to $J/\psi+(c\bar{s})$ (e.g., $J/\psi+D_s$ or $J/\psi+D_s^*$), and $\Lambda_b=bud$ via its decay to $J/\psi+sud$ (e.g., $J/\psi+\Lambda$ or $J/\psi+K^-+p$).

A hint of B_s production in e^+e^- interactions at the energy of the $\Upsilon(5S)$ has been obtained by the CUSB group [2]. At this c.m. energy, $E_{\rm c.m.}=10.866\pm0.020$ GeV/c^2, not only B and B^* mesons but also B_s and B_s^* mesons appear to be produced. The photons emitted in $B_s^*\to B_s\gamma$ decays at this c.m. energy would be less Doppler shifted than those in $B^*\to B\gamma$ decays. By detailed study of the spectrum shape of photons in the range of $E_\gamma\approx47$ MeV, the CUSB group claims evidence for the presence of a mixture of $B^*\to B\gamma$ and $B_s^*\to B_s\gamma$ transitions. However, they do not obtain a unique solution for B_s and B_s^* masses.

In this work we summarize what can be said about the masses of $B_s^{(*)}$, $B_c^{(*)}$, Λ_b, Σ_b, and Σ_b^* on the basis of nonrelativistic models, and compare our results with those of some previous approaches [3–12]. Our approach is closest in spirit to that of Ref. [4]; in a sense we are merely updating the results of that work in the light of more recent information on particle masses. However, we also make use of the fact that enough states are already known so that the masses of mixed-flavor states like the B_s and B_c can be anticipated by interpolation, without recourse to specific models. In addition to the states B_s and B_c with $J^P=0^-$, we shall discuss the vector mesons B_s^* and B_c^* with $J^P=1^-$.

We first evaluate the masses of vector mesons in Sec. II. We then estimate hyperfine splittings in Sec. III to obtain masses for pseudoscalar mesons. A brief discussion of the Λ_b, Σ_b and Σ_b^* masses occupies Sec. IV. We summarize and compare our results with others in Sec. V.

II. VECTOR-MESON MASSES

We adopt three distinct approaches, hoping thereby to illustrate the range of uncertainties in mass predictions.

(1) A simple interpolation technique is used to fit particle masses on the basis of static quark masses and parametrizations of the binding energy in terms of the reduced mass. This approach is in the spirit of semiempirical mass formulas for nuclei.

(2) An estimate is performed using a logarithmic potential, for which the effects of changes in reduced mass are particularly easy to calculate.

(3) The general theorems and bounds of Ref. [4] are utilized. New information is available which permits somewhat more restrictive estimates than those made in Ref. [4].

A. Interpolation method

The interpolation method begins with the circumstance that numerous masses of 1^- mesons are known. We then interpolate using a formula of the form

$$M(q_1\bar{q}_2)=m_1+m_2+E(\mu)\ , \tag{1}$$

where m_i is the mass of quark i and $E(\mu)$ is some function of the reduced mass $\mu\equiv m_1m_2/(m_1+m_2)$. We use two forms of $E(\mu)$:

quadratic:

$$E(\mu)=A+B\mu+C\mu^2\ , \tag{2}$$

Padé:

$$E(\mu)=\frac{A+B\mu}{1+C\mu}\ , \tag{3}$$

both containing three parameters. We fit the masses of $\rho\approx\omega$, K^*, ϕ, D^*, D_s^*, J/ψ, B^*, and Υ (eight quantities) using the four quark masses $m_u\equiv m_d$, m_s, m_c, and m_b and three parameters in $E(\mu)$. A quantity

$$S=\sum[M(1^-)_{\rm fit}-M(1^-)_{\rm expt}]^2\ ,$$

which sums over the eight vector mesons, is minimized by allowing all seven parameters to vary freely. The

TABLE I. Results of fits to masses of 1^- particles with $E = A + B\mu + C\mu^2$ for given values of A.

A (MeV)	B	C (10^{-4} MeV^{-1})	m_u (MeV)	m_s (MeV)	m_c (MeV)	m_b (MeV)	$M(B_s^*)$ (MeV)	$M(B_c^*)$ (MeV)	$S(A)$ (MeV^2)
−1000	−0.949	1.53	1128	1282	2526	5964	5416	6288	0.20
−800	−0.854	1.43	976	1128	2352	5767	5416	6291	0.15
−600	−0.763	1.32	832	981	2187	5581	5417	6295	0.11
−400	−0.676	1.20	695	841	2030	5406	5418	6299	0.11
−200	−0.594	1.07	565	708	1880	5240	5420	6304	0.16
0	−0.516	0.93	439	581	1737	5084	5421	6310	0.34
200	−0.443	0.79	319	458	1600	4937	5423	6317	0.77
400	−0.374	0.63	203	340	1469	4797	5425	6325	1.90

minimum value $S_{\min}$ is obtained. For illustrative purpose, we show in Tables I and II the quantity $S(A)$ which is the variation of $S_{\min}$ when the parameters A in Eqs. (2) and (3) are assigned fixed values.

The ranges of $M(B_s^*)$ and $M(B_c^*)$ are remarkably small over a wide range of quark masses. This illustrates the familiar observation that a constant in the binding energy can usually be absorbed by adjusting the quark masses. To find the 1σ range of $M(B_s^*)$ and $M(B_c^*)$, we use the method outlined in the Appendix. Values of $M(B_x^*)$ or $M(B_c^*)$ were included as one of the data points. The quantity S in (A1), now summed over nine vector mesons, is minimized to obtain $S[M(B_s^*)]$ or $S[M(B_c^*)]$ as a function of $M(B_s^*)$ or $M(B_c^*)$. The results are shown in Figs. 1 and 2. Since $S_{\min}$ itself has only one degree of freedom, we have from Table VI the 1σ value of $S[M(B_s^*)]$:

$$S_{1\sigma}[M(B_s^*)] = 4.39 S_{\min} . \tag{4}$$

For Eq. (2), this is 0.448 MeV^2, which translates into $M(B_s^*) = 5418^{+3.8}_{-3.5}$ MeV; the corresponding value for B_c^* is $M(B_c^*) = 6297^{+15}_{-13}$ MeV. For Eq. (3), $S_{1\sigma} = 25.54$ MeV^2 and $M(B_s^*) = 5422^{+11}_{-16}$ MeV; the corresponding value for B_c^* is $M(B_c^*) = 6330^{+19}_{-17}$ MeV.

B. A specific potential

Potentials which satisfactorily describe the known 1^- $s\bar{s}$, $c\bar{s}$, $c\bar{c}$, and $b\bar{b}$ states include the power-law potential $V \sim r^{0.1}$ (see Refs. [8] and [13]) and the potential [14]

$$V(r) = C \ln(r/r_0) , \tag{5}$$

for which the masses of the bound states are given by simple scaling relations:

$$M_n = m_1 + m_2 + C(\epsilon_n - \ln\sqrt{2\mu C}\, r_0) . \tag{6}$$

The values of ϵ_n are (Ref. [14]) (1.044 32, 1.643, 1.8474, 2.151, 2.2897, 2.5957) for $n = (1S, 1P, 2S, 2P, 3S, 4S)$. If we fit the observed masses of $c\bar{c}$ $1S$, $1P$, $2S$ levels and $b\bar{b}$ $1S$, $1P$, $2S$, $2P$, $3S$, $4S$ levels with expression (6), we obtain an error per degree of freedom of $\sqrt{432/(9-4)} = 9.3$ MeV; we also obtain a value of $C = 722 \pm 8$ MeV.

To treat the heavy and light mesons on equal footing, we may fit (6) to the eight ground-state mesons ρ, K^*, ϕ, D^*, D_s^*, J/ψ, B^*, and Υ and obtained an average error per degree of freedom of $\sqrt{40.7/(8-6)} = 4.5$ MeV $C = 824^{+176}_{-124}$ MeV.

It is a little surprising that Eq. (6), taken as an empirical mass formula, fits ground states of both heavy and light mesons better than the excited states of just the heavy mesons alone. The constant C, however, is more

TABLE II. Results of fits to masses of 1^-particles with $E = (A + B\mu)/(1 + C\mu)$ for given values of A.

A (MeV)	B	C (10^{-4} MeV^{-1})	m_u (MeV)	m_s (MeV)	m_c (MeV)	m_b (MeV)	$M(B_s^*)$ (MeV)	$M(B_c^*)$ (MeV)	$S(A)$ (MeV^2)
−400	−2.631	9.04	966	1136	2403	5840	5416	6322	22.3
−300	−2.186	9.00	822	987	2232	5644	5415	6320	12.0
−200	−1.589	8.19	672	831	2045	5430	5415	6321	9.2
0	−0.893	6.60	477	626	1801	5156	5418	6323	7.0
200	−0.515	5.26	332	475	1625	4964	5420	6327	5.9
308	−0.379	4.58	263	404	1543	4876	5422	6330	5.7
400	−0.291	4.01	207	346	1477	4806	5423	6333	5.9
600	−0.169	2.82	92	228	1344	4668	5427	6340	9.4
700	−0.138	2.22	37	172	1281	4604	5429	6344	18.0
800	−0.130	1.53	−17	116	1220	4543	5431	6351	72.7

constrained by the level spacing with the excited states. We present, in Table III, results from the fit to the ground states of heavy and light mesons, to compare with those obtained in the previous subsection.

The errors on $M(B_s^*)$ and $M(B_c^*)$ are obtained from Fig. 3 which shows the variation of S with $M(B_s^*)$ and $M(B_c^*)$; we get $M(B_s^*)=5412\pm10$ MeV and $M(B_c^*)=6328\pm8$ MeV. In a potential $\sim r^{0.1}$, Martin [8] finds $M(B_s^*)=5408-5410$ MeV, $M(B_c^*)=6318$ MeV.

C. Model-independent method

The last method we employ is based on an attempt to estimate the change in energy $E\equiv M-M_1-M_2$ with the change in reduced mass on the basis of the Feynman-Hellmann theorem [15]:

$$\frac{dE}{d\mu}=-\frac{\langle T\rangle}{\mu}\,, \tag{7}$$

where $\langle T\rangle$ is the expectation value of the kinetic energy. This was the method employed in Ref. [4]. By the virial theorem [16]

$$\langle T\rangle=\left\langle\frac{r}{2}\frac{dV}{dr}\right\rangle\,. \tag{8}$$

We see that $\langle T\rangle=C/2$ (a constant) for the potential (5), so $\langle T\rangle$ ranges from 350 to 375 MeV for C between 0.7 and 0.75 GeV. If $\langle T\rangle$ were indeed constant, we could in-

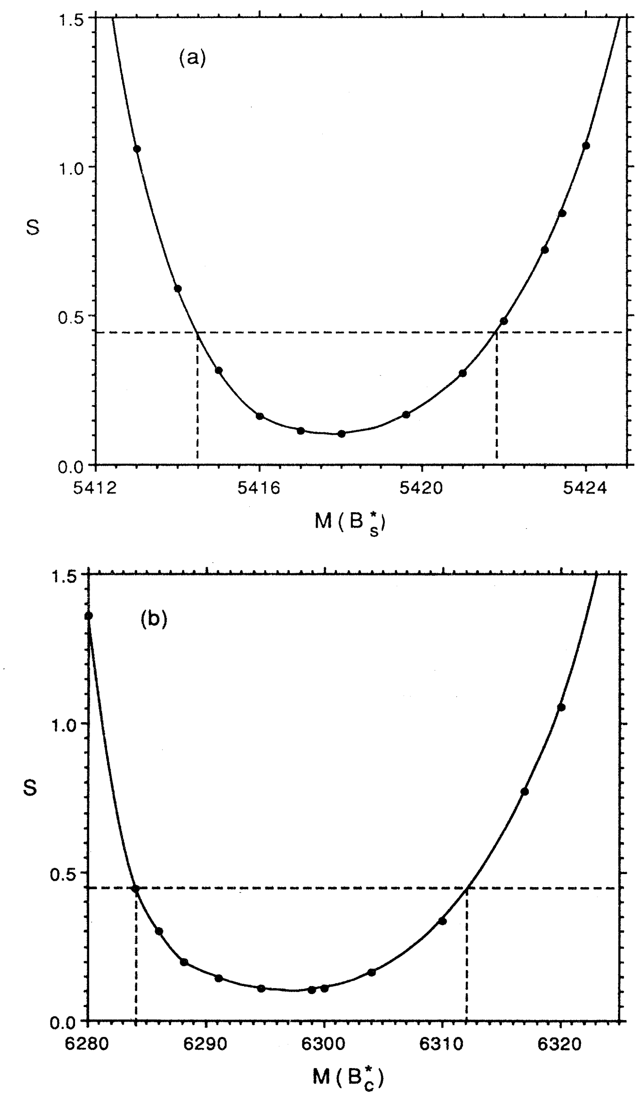

FIG. 1. Results of fits of $1S$ 1^- particle masses using an expression $E(\mu)=A+B\mu+C\mu^2$. (a) The results of minimization of $S[M(B_s^*)]=\sum(M_{\rm fit}-M_{\rm expt})^2$ with respect to all other parameters are shown, with $M(B_s^*)$ included as the ninth data point. The horizontal line shows the 1σ level. (b) Same as (a) for $S[M(B_c^*)]$.

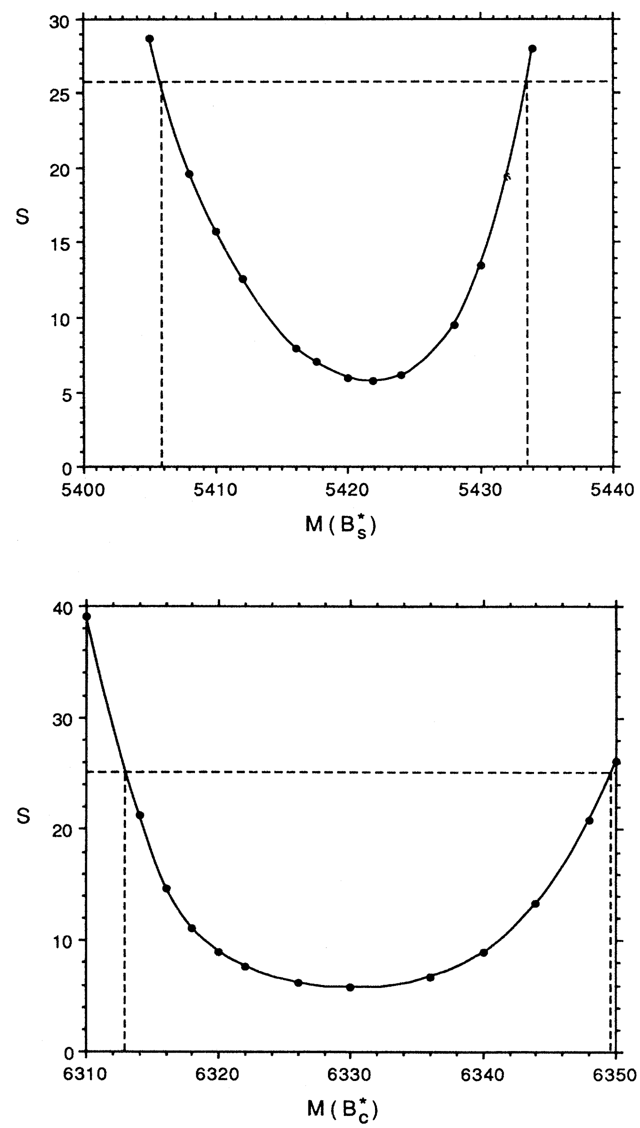

FIG. 2. Same as Fig. 1 for $E(\mu)=(A+B\mu)/(1+C\mu)$.

TABLE III. Results of fits to masses of 1^- ground-state particles with Eq. (6) for given values of C.

C (MeV)	r_0 (GeV^{-1})	m_u (MeV)	m_s (MeV)	m_c (MeV)	m_b (MeV)	$M(B_s^*)$ (MeV)	$M(B_c^*)$ (MeV)	$S(C)$ (MeV^2)
600	5.68	362	552	1764	5107	5398	6320	187.6
700	6.20	457	643	1868	5231	5406	6324	76.7
800	6.53	554	737	1973	5354	5411	6327	41.9
824	6.58	578	760	1999	5384	5412	6328	40.7
900	6.71	651	832	2078	5476	5416	6331	49.8
1000	6.80	748	928	2182	5596	5419	6334	82.3

tegrate (7), finding

$$E(\mu_2)-E(\mu_1)=-\langle T\rangle \ln\frac{\mu_2}{\mu_1}\ , \quad (9)$$

and we would simply reproduce the results based on the potential (5) as long as we took a corresponding range of quark masses. We shall use methods of Ref. [4] to estimate that, for the interpolation we wish to perform, $\langle T\rangle=300$–400 MeV, and will then use the approximate form (9) to estimate the range of possible binding effects. The use of (9) is only justified, of course, if the potential is fairly close to the form (4), which the results of Refs. [13] and [14] show it to be.

In Ref. [4] it was shown that

$$\langle T_{1S}\rangle \geq \tfrac{3}{4}(E_{1P}-E_{1S})\ .$$

For the spin-triplet $c\bar{u}$ system, $E_{1P}\simeq 2.41$ GeV (Ref. [17]), $E_{1S}=2.01$ GeV, so $\langle T_{1S}\rangle \gtrsim 300$ MeV. For the $c\bar{s}$ system, with $E_{1P}\simeq 2.54$ GeV (Ref. [18]), $E_{1S}\simeq 2.11$ GeV, $\langle T_{1S}\rangle \gtrsim 320$ MeV. We expect level spacings in the $b\bar{s}$ system to be rather similar to those in the $c\bar{s}$ system.

An empirical result for $\langle T_{1S}\rangle$ in power-law potentials $V\sim r^{\nu}$, which fits the oscillator ($\nu=2$) and Coulomb ($\nu=-1$) results exactly, is [4]

$$\langle T_{1S}\rangle \simeq \tfrac{3}{4}(E_{1P}-E_{1S})(1+\tfrac{7}{9}C_0)\ , \quad (10)$$

$$C_0\equiv\left[\frac{E_{2S}+E_{1S}-2E_{1P}}{E_{2S}-E_{1S}}\right]^2 . \quad (11)$$

The term $\frac{7}{9}C_0$ is 0 for the oscillator, 0.016 for the linear potential ($\nu=1$), and 0.187 for the potential (5) (equivalent to $\nu=0$). Equation (10) would give $\langle T\rangle=0.533C$ for the potential (5), to be compared with the exact value $0.5C$. We expect that an effective potential describing light-heavy quark systems (such as $b\bar{s}$) lies somewhere between the linear ($\nu=1$) form, appropriate for light quarks, and the logarithmic form, appropriate for interpolating between $c\bar{c}$ and $b\bar{b}$ bound states [13]. Thus, an estimate for $\langle T_{1S}\rangle$ for such systems is that it lies between $\frac{3}{4}(E_{1P}-E_{1S})\simeq 320$ MeV and

$$\tfrac{3}{4}(E_{1P}-E_{1S})\times 1.187\times(0.5)/(0.533)$$
$$=0.84(E_{1P}-E_{1S})\simeq 360 \text{ MeV}$$

if $E_{1P}-E_{1S}=430$ MeV, as we noted for the $c\bar{s}$ system.

One might argue, since the $J^P=1^-$ $1S$ levels are raised slightly by hyperfine splittings but the P-wave levels presumably are much less affected, that one should estimate $E_{1P}-E_{1S}$ using the spin-weighted average of the 1^- and 0^- $1S$ levels. For $c\bar{u}$ and $c\bar{s}$ levels, this average is about 35 MeV below the 1^- level, leading to an increase of $E_{1P}-E_{1S}$ by 35 MeV and of $\langle T_{1S}\rangle$ by up to 30 MeV. We thus feel safe in any event in estimating

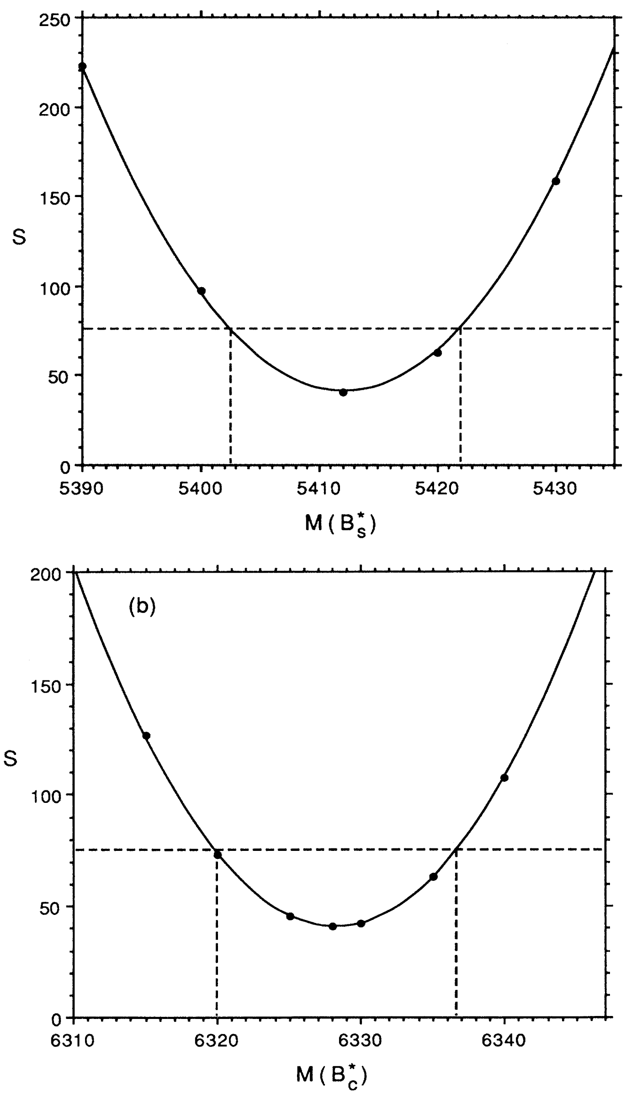

FIG. 3. Same as Fig. 1 for the potential (5).

$$300 \le \langle T \rangle \le 400 \text{ MeV} \tag{12}$$

for use in Eq. (9) when estimating the B_s^* mass. For $b\bar{c}$ we shall use a similar range, but we will use Eq. (9) in a slightly different manner.

To estimate the B_s^* mass using Eq. (9), we use the known values of $M(D^*)$, $M(B^*)$, and $M(D_s^*)$, in order to cancel out the effects of quark masses except in the reduced mass μ. We then find

$$M(B_s^*) = M(B^*) + M(D_s^*) - M(D^*) - \langle T \rangle \ln \frac{\mu(B_s^*)\mu(D^*)}{\mu(D_s^*)\mu(B^*)} . \tag{13}$$

The ratio

$$\frac{\mu(B_s^*)\mu(D^*)}{\mu(D_s^*)\mu(B^*)} = \frac{m_s + m_c}{m_s + m_b} \frac{m_u + m_b}{m_u + m_c} \tag{14}$$

ranges from 1.07 to about 1.03 as quarks range in mass from current-quark to (heavy) constituent-quark values in accord with the results of Tables I and II. The familiar result [4] that quark mass differences are more tightly constrained than the quark masses themselves is a property of the masses in Tables I and II, and has been used in obtaining this range of the ratio (14). Thus,

$$M(B_s^*) \simeq M(B^*) + M(D_s^*) - M(D^*) - (0.05 \pm 0.02)\langle T \rangle = 5.41 \pm 0.01 \text{ GeV} , \tag{15}$$

in accord with the range of estimates given above.

For the B_c^* mass, a different combination of masses can be used to eliminate effects of additive quark masses. We find

$$M(B_c^*) = \frac{M(J/\psi) + M(\Upsilon)}{2} - \langle T \rangle \ln \frac{2\sqrt{\mu(J/\psi)\mu(\Upsilon)}}{\mu(B_c^*)} = \frac{M(J/\psi) + M(\Upsilon)}{2} - \langle T \rangle \ln \frac{2\sqrt{m_c m_b}}{m_c + m_b} . \tag{16}$$

Since the argument of the logarithm is less than 1, $M(B_c^*)$ must exceed the average of the J/ψ and Υ masses, in accord with the bounds of Ref. [3]. We then seek estimates of $\langle T \rangle$ and m_c/m_b for use in Eq. (16).

The range of m_c/m_b associated with acceptable χ^2 values and u-quark masses between several MeV and about 600 MeV in Tables I and II is about 0.28–0.36. The masses of observed spin-triplet $c\bar{c}$ and $b\bar{b}$ states, when Eqs. (10) and (11) are used, yield the estimates $\langle T_{1S} \rangle_{c\bar{c}} = 367$ MeV; $\langle T_{1S} \rangle_{b\bar{b}} = 411$ MeV. Using the full range of both these quantities, we find $M(B_c^*) = 6324-6357$ MeV, or

$$M(B_c^*) = 6340 \pm 17 \text{ MeV} , \tag{17}$$

in accord with the result of Ref. [3].

D. Overall range

Combining the results from Secs. II A, II B, and II C, we find that $M(B_s^*) = 5400-5433$ MeV, $M(B_c^*) = 6284-6357$ MeV. We have quoted a range compatible with all four mass determinations order to get some idea of the systematic error inherent in nonrelativistic models.

III. PSEUDOSCALAR-MESON MASSES

A. Regularity in ΔM^2 values

An interesting regularity [19] in values of

$$\Delta M^2 \equiv M^2(1^-) - M^2(0^-)$$

occurs for systems with at least one light (u, d, or s) quark. It appears that ΔM^2 ranges between 0.5 and 0.6 GeV^2. A constant value of ΔM^2 in a system of one light quark q and one heavy quark Q corresponds to a value of $\Delta M \sim 1/M \approx 1/m_Q$. Since

$$\Delta M \sim \frac{\alpha_s |\Psi(0)|^2}{m_q m_Q} , \tag{18}$$

a universal value of $m_Q \Delta M$ implies $|\Psi(0)|^2 \sim m_q$. This is indeed the case for a linear potential when $m_q \ll m_Q$, since

$$|\Psi(0)|^2 = \frac{\mu}{4\pi} \left\langle \frac{dV}{dr} \right\rangle , \tag{19}$$

where μ is the reduced mass, and in a linear potential $\langle dV/dr \rangle$ is just the (universal) force constant.

B. B_s mass

For the 1^--0^- splitting in the $b\bar{s}$ system, let us assume that ΔM^2 still lies between 0.5 and 0.6 GeV^2. (It appears to be larger for systems of greater reduced mass, like $c\bar{c}$.) The results of Ref. [2] suggest that

$$M(1^-) - M(0^-) = 47.0 \pm 2.6 \text{ MeV} ,$$

while $\Delta M^2 = 0.6 \text{ GeV}^2$ would imply

$$M(1^-) - M(0^-) = 55 \text{ MeV} .$$

We shall thus take $M(1^-) - M(0^-)$ to range between 45 and 55 MeV. Combining this with our estimate of the overall range of $M(B_s^*)$, we find $M(B_s) = 5345-5388$ MeV.

C. B_c mass

The hyperfine splitting in the B_c system must be determined by extrapolation from the $c\bar{c}$ system. A superior method would involve interpolating between $c\bar{c}$ and $b\bar{b}$ values, but we must wait until the $\eta_b(0^-)$ is discovered for that.

We neglect variations of α_s between the $c\bar{c}$ and $b\bar{c}$ systems, and write

$$\frac{M(b\bar{c}, 1^-) - M(b\bar{c}, 0^-)}{M(J/\psi) - M(\eta_c)} = \frac{|\Psi(0)|^2_{b\bar{c}}}{|\Psi(0)|^2_{c\bar{c}}} . \tag{20}$$

In a power-law potential (Ref. [12]) $V \sim r^\nu$,

$|\Psi(0)|^2 \sim \mu^{3/(2+\nu)}$, where μ is the reduced mass. The approximate constancy of leptonic widths $\Gamma_{ll} \sim |\Psi(0)|^2/M_V^2$ for vector mesons, once quark charges have been accounted for [14], suggests $|\Psi(0)|^2 \sim \mu^2$, while in a potential (5), $|\Psi(0)|^2 \sim \mu^{3/2}$. We shall take these two dependences as upper and lower limits of sensitivity to μ with $m_c/m_b = 0.32 \pm 0.04$ as mentioned in Sec. II C, and with

$$M(J/\psi) - M(\eta_c) = 117 \pm 2 \text{ MeV} ,$$

we then find $M(b\bar{c}, 1^-) - M(b\bar{c}, 0^-)$ to range between approximately 65 and 90 MeV, and $M(B_c) = 6194-6292$ MeV.

IV. THE Λ_b, Σ_b, BARYONS

The $\Lambda = sud$, $\Lambda_c = cud$, and $\Lambda_b = bud$ all are expected to have the u and d quarks coupled to one another in a state of zero spin and isospin. Hence, their masses should differ only because of the differences between s, c, and b masses and because of different reduced-mass effects in binding.

The question is whether the ud system in each of these particles can be treated as a single entity with a well-defined effective mass. We have attempted to incorporate the observed Λ and Λ_c into models of the type described in Sec. II A, with the only new parameter being the mass of the ud diquark. The results are not self-consistent. A choice of the ud mass to fit $M(\Lambda)$ makes the Λ_c mass unacceptably low. It appears that binding effects are overestimated for the Λ_c in that case.

It appears that the spin-averaged difference between $c\bar{u}$ and $s\bar{u}$ systems is almost the same as the $\Lambda_c - \Lambda$ mass difference:

$$\frac{3M(D^*)+M(D)}{4} - \frac{3M(K^*)+M(K)}{4} \approx M(\Lambda_c) - M(\Lambda). \quad (21)$$

The left-hand side of (21) is about 1.18 GeV, while the right-hand side is about 1.17 GeV. That suggests the diquark is not behaving too differently from an ordinary nonstrange quark with regard to its effects on binding energies. In order that a baryon be heavier than a meson, a baryon must then have some added contribution to its mass, a circumstance which we will have to take as given. We would then predict, as in the naive estimate of Ref. [20], that

$$M(\Lambda_b) - M(\Lambda_c) = \frac{3M(B^*)+M(B)}{4} - \frac{3M(D^*)+M(D)}{4} \simeq 3.344 \text{ GeV} \quad (22)$$

or $M(\Lambda_b) \simeq 5.63$ GeV. Based on the small discrepancy in Eq. (21), we estimate $M(\Lambda_b)$ could be as low as 5.60 GeV. Numerous predictions of lower values have appeared in the literature quoted in Ref. [20], however. The total range estimated there is 5379–5659 MeV, with most predictions lying above 5580 MeV.

The Σ_b and Σ_b^* are expected to be heavy enough to decay to $\Lambda_b + \pi$, and may well be visible in the same experiments where Λ_b is first seen for just this reason. Estimates of $M(\Sigma_b) - M(\Lambda_b)$ typically range from 180 to 210 MeV, with

$$M(\Sigma_b) - M(\Lambda_b) > M(\Sigma_c) - M(\Lambda_c) = 168 \text{ MeV}$$

a firm prediction [12, 20] of quark models. Σ_b^* (the lowest spin-$\frac{3}{2}$ state of a b quark and two nonstrange quarks) is expected to lie only 10–40 MeV above Σ^b. The reason is that, aside from wave-function effects which might vary from the case of the strange quark to that of the b quark, one expects

$$M(\Sigma_b^*) - M(\Sigma_b) = (m_s/m_b)[M(\Sigma^*) - M(\Sigma)] \simeq (0.1)(190 \text{ MeV}) \simeq 20 \text{ MeV} . \quad (23)$$

A summary of Λ_b, Σ_b, and Σ_b^* predictions was given in Ref. [20]. In Table IV we present results quoted by those authors along with some others. Aside from the anomalously high values quoted in Ref. [22] (commented up on in Ref. [20]), we find general agreement with our estimate for $M(\Lambda_b)$, with Σ_b and Σ_b^* expected to lie not far above 5800 MeV.

V. SUMMARY AND COMPARISONS

We summarize our results for mesons and compare them with those obtained previously in Table V. There is remarkably little spread. All the results except those of Ref. [5] are based on the common assumption that a nonrelativistic potential description is valid. Some discussion in Refs. [11] and [12] addresses what would happen when this assumption is relaxed; predictive power is then apparently greatly eroded. The successes of nonrelativistic potential models for earlier predictions, even of masses of states containing strange quarks [8,13], makes us reluctant to abandon them as a phenomenological tool for interpolation among particle masses unless absolutely forced to do so.

To summarize the spread of values obtained, we find $M(B_s) = 5345-5388$ MeV $M(B_s^*) = 5400-5433$ MeV, $M(B_c) = 6194-6292$ MeV, $M(B_c^*) = 6284-6357$ MeV, $M(\Lambda_b) = 5600-5630$ MeV. The Λ_b mass is based on a guess regarding binding systematics in baryons, and is less firm than the others. Previous investigations have found $M(\Sigma_b)$ and $M(\Sigma_b^*)$ not far above 5800 MeV.

TABLE IV. Comparison of mass predictions for baryons (in MeV).

Reference	$M(\Lambda_b)$	$M(\Sigma_b)$	$M(\Sigma_b^*)$
[9]	5596	5859	5877
[10]	5640	5780	5820
[12]	$\leq 5630 \pm 30$	$\geq \Lambda_b + 168$	
[20]	≤ 5629	5670–5826	≥ 5710
[21]	5580	5800	5841
[22]	5901 ± 35	5970 ± 20	5988 ± 25
[23]	5547	5714	5766
[24]	5605	5815	5825
[25]	5620	5800	5820

TABLE V. Summary of present and previous results. Predicted masses in MeV.

1^- mesons	$M(B_s^*)$	$M(B_c^*)$	Reference
Quadratic	$5418^{+3.8}_{-3.5}$	6297^{+15}_{-13}	Present work
Padé	5422^{+11}_{-16}	6330^{+19}_{-17}	Present work
Log potential	5412±10	6328±8	Present work
Model independent	5410±10	6340±17	Present work
Overall range	5440–5433	6284–6357	Present work
Nussinov		6340±20	[3]
Regge trajectory	5420	6370	[5]
Godfrey-Isgur	5420	6310	[6][a]
Byers-Hwang	5435		[7]
Martin	5408–5410	6318	[8]
Stanley-Robson	5400	6318	[9][b]
Bag model	5420		[10]
Rel. Corrs.	5360–5440		[11]
0^- mesons	$M(B_s)$	$M(B_c)$	Reference
Overall range	5345–5388	6194–6292	Present work
Regge trajectory	5360	6320	[5]
Godfrey-Isgur	5360	6240	[6][a]
Byers-Hwang	5383		[7]
Martin	5353–5374	6250	[8]
Stanley-Robson	5360	6265	[9][b]
Bag model	5360		[10]
Rel. Corrs.	5300–5390		[11]
0^--1^- average	$M(b\bar{s})$	$M(b\bar{c})$	Reference
Various potls.	5407–5413	6308–6318	[12]

[a]Scale offset by −30 MeV to agree with $M(B)$.
[b]Scale offset by −28 MeV to agree with $M(B)$.

Discovery of mesons outside the limits quoted above, in our opinion, would cast serious doubt on the usefulness of nonrelativistic potential models for anticipating the masses of the states in questions.

In Ref. [2], two solutions were obtained for $M(B_s)$ and $M(B_s^*) \approx M(B_s)+47$ MeV, on the basis of models [26] for production of $B^{(*)}$ mesons in e^+e^- collisions above flavor threshold. In one solution, $M(B_s)-M(B) = 82.5\pm2.5$ MeV. This solution is the one favored by our result, which implies $M(B_s)-M(B)=88\pm22$ MeV. The other solution of Ref. [2], with $M(B_s)-M(B) = 121\pm9$ MeV, is disfavored by our result.

ACKNOWLEDGMENTS

We wish to thank Mel Shochet for asking the initial question which led to this investigation, and Henry Frisch and Mike Tuts for useful discussions. This work was supported in part by the United States Department of Energy under Contract No. DE-FG02-90ER 40560.

APPENDIX

In the fitting of empirical or phenomenological relations to experimental data, one often faces situations in which the experimental errors on the data points are negligibly small; any discrepancies with data can only be attributed to the crudeness or inaccuracy of the theories. In particular, when the theoretical models do not provide means of estimating "probable errors," one is left without a criterion for determining the goodness of the fits. Nevertheless, one would like to be able to compare such models and/or make predictions or extrapolations with them. In this appendix, we would like to concentrate on obtaining reasonable error estimates on the fitted parameters and any predicted quantities.

We start by assuming that a least-squares fit has been performed by minimizing the quantity

$$S = \sum_{i=1}^{n} (y_i^{\text{theory}} - y_i^{\text{data}})^2 \tag{A1}$$

and that the minimum $S_{\min}$ is obtained. Lacking a better method, we next assume the individual deviations of theory from data to be distributed randomly with common but unknown variance σ^2. It is obvious that $S_{\min}/\sigma^2$ is statistically distributed as a normalized χ^2 distribution of ν degrees of freedom, where ν is the number of data points minus the number of parameters determined from the fit. We write $S_{\min} \sim \sigma^2\chi^2(\nu)$, where $\sim$ reads *is distributed as.*

To find the error of a parameter, say, a, of the fit, we follow the usual method of giving a various fixed values and minimizing (A1) again to obtain $S(a)$ as a function of the parameter a. The difference $dS = S(a) - S_{\min}$ would be distributed as $\sigma^2\chi^2(1)$, with one degree of freedom. $\chi^2(1)$ is a well-known distribution and we only need to find an estimate for σ^2. A naive way would be to assume that we have an "average" fit. Since $\langle\chi^2(\nu)\rangle = \nu$, we have $S_{\min} \approx \sigma^2\nu$ so that $dS \sim \chi^2(1)S_{\min}/\nu$ or, approximately,

$$S(a) \sim S_{\min}\left[1 + \frac{\chi^2(1)}{\nu}\right] . \tag{A2}$$

At the 1σ level, or the 68.33% C.L., $\chi^2(1)=1$ and we have

$$S(a)_{1\sigma} \approx S_{\min}(1+1/\nu) .$$

We will see that this is not such a bad estimate for large ν.

To do a better job, we notice that σ^2 drops out of the ratio

$$dS/S_{\min} \sim \chi^2(1)/\chi^2(\nu) .$$

The distribution of this ratio of χ^2's is the well-known F distribution defined as

$$F(\nu_1,\nu_2) = \frac{\chi^2(\nu_1)/\nu_1}{\chi^2(\nu_2)/\nu_2} . \tag{A3}$$

Thus, we have $dS/S_{\min} \sim F(1,\nu)/\nu$ or

$$S(a) \sim S_{\min}\left[1 + \frac{F(1,\nu)}{\nu}\right] . \tag{A4}$$

The F distribution can be worked out from its definition (A3), and integrals for 90% confidence level upward are tabulated in all statistics textbooks. We have calculated values corresponding to 68.33% C.L. and the results are shown in Table VI. Notice that, for $\nu > 5$, (A2) is a rather good approximation. In fact, $F(1,\nu)\rightarrow\chi^2(1)$ as $\nu\rightarrow\infty$ so that (A2) reduces exactly to (A4) at large ν.

Quantities predicted by the model depend on all of the fitted parameters. Instead of finding the errors and correlations of all the parameters and then doing error propagation properly, we can treat the predicted quantity as one of the data points and do something very similar to what we have discussed. To be more specific, let us assume that, for a certain value of x our model predicts a value $y(x)=b_{\min}$. Values b different from $b_{\min}$ are assigned to $y(x)$ and added to the original data set as the $(n+1)$th data point $y_{n+1}^{\text{data}}=y(x)=b$. The quantity S in (A1) is then minimized to obtain $S(b)$ with the sum now going from 1 to $n+1$. The difference $dS=S(b)-S_{\min}$ will again have the property $dS\sim\sigma^2\chi^2(1)$ so that $S(b)\sim S_{\min}(1+F(1,\nu)/\nu)$.

TABLE VI. The 68.33% C.L. of $S(a)/S_{\min}$ for the variation of a.

ν	$1+F(1,\nu)/\nu$	$1+\chi^2(\nu)/\nu$
1	4.391	2
2	1.876	1.5
3	1.479	1.333
4	1.327	1.25
5	1.247	1.20

*Present address: Department of Physics, University of Texas, Arlington TX 76019.

[1] CDF Collaboration, F. Abe *et al.*, presented by T. Rohaly, in *Proceedings of the 25th International Conference on High Energy Physics,* Singapore, 1990, edited by K. K. Phua and Y. Yamaguchi (World Scientific, Singapore, 1991).
[2] CUSB Collaboration, J. Lee-Franzini *et al.*, Phys. Rev. Lett. **65**, 2947 (1990).
[3] S. Nussinov, Z. Phys. C **3**, 165 (1979). For more general theorems, see also S. Nussinov, Phys. Rev. Lett. **52**, 966 (1984); Phys. Lett. **136B**, 93 (1984); **164B**, 366 (1985).
[4] R. A. Bertlmann and A. Martin, Nucl. Phys. **B168**, 111 (1980).
[5] A. B. Kaidalov and A. V. Nogteva, Yad. Fiz. **47**, 505 (1988) [Sov. J. Nucl. Phys. **47**, 321 (1988)]. See also A. B. Kaidalov, Z. Phys. C. **12**, 63 (1982).
[6] Stephen Godfrey and Nathan Isgur, Phys. Rev. D **32**, 189 (1985).
[7] N. Byers and D. S. Hwang, in *Proceedings of the Workshop on High Sensitivity Beauty Physics at Fermilab,* Batavia, Illinois, 1987, edited by A. J. Slaughter, N. Lockyer, and M. Schmidt (Fermilab, Batavia, 1987), p. 199.
[8] André Martin, in *Heavy Flavors and High Energy Collisions in the 1-100 TeV Range,* edited by A. Ali and L. Cifarelli (Plenum, New York, 1989), p. 141.
[9] D. P. Stanley and D. Robson, Phys. Rev. D **21**, 3180 (1980); Phys. Rev. Lett. **45**, 235 (1980).
[10] Dale Izatt, Carleton DeTar, and Mark Stephenson, Nucl. Phys. **B199**, 269 (1982).
[11] M. R. Arafah and Fayyazuddin, King Abdulaziz University report, 1990 (unpublished).
[12] D. B. Lichtenberg, R. Roncaglia, J. G. Wills, and E. Predazzi, Z. Phys. C **47**, 83 (1990).
[13] A. Martin, Phys. Lett. **100B**, 511 (1981).
[14] C. Quigg and Jonathan L. Rosner, Phys. Lett. **71B**, 153 (1977); M. Machacek and Y. Tomozawa, Ann. Phys. (N.Y.) **110**, 407 (1978); C. Quigg and Jonathan L. Rosner, Phys. Rep. **56**, 167 (1979).
[15] R. P. Feynman, Phys. Rev. **56**, 340 (1939); H. Hellmann, *Einführung in den Quantenchemie* (Deuticke Verlag, Leipzig, 1937).
[16] See, e.g., Ref. [14].
[17] ARGUS Collaboration, H. Albrecht *et al.*, Phys. Lett. B **231**, 208 (1989); **232**, 398 (1989); CLEO Collaboration, P. Avery *et al.*, Phys. Rev. D **41**, 774 (1990).
[18] ARGUS Collaboration, H. Albrecht *et al.*, Phys. Lett. B **230**, 162 (1989); CLEO Collaboration, P. Avery *et al.*, Phys. Rev. D **41**, 774 (1990).
[19] J. Lee-Franzini, in *QCD and Beyond,* proceedings of the 20th Rencontre de Moriond, Les Arcs, France, 1985, edited by J. Tran Thanh Van (Editions Frontières, Gif-sur-Yvette, France, 1985), p. 281; M. Frank and P. J. O'Donnell, Phys. Lett. **159B**, 174 (1985); Z. Phys. C **34**, 39 (1987); K. Igi and S. Ono, Phys. Rev. D **32**, 232 (1985); H. Schnitzer, in *Hadron Spectroscopy—1985,* Proceedings of the First International Conference on Hadron Spectroscopy, College Park, Maryland, 1985, edited by S. Oneda, AIP Conf. Proc. No. 132 (AIP, New York, 1985); A. Martin (unpublished).
[20] A. Martin and J. M. Richard, Phys. Lett. B **185**, 426 (1987).
[21] J. M. Richard and P. Taxil, Phys. Lett. **128B**, 453 (1983), as extended in Ref. [20].
[22] Stuart Samuel and K. J. M. Moriarty, Phys. Lett. B **175**, 197 (1986).
[23] J. L. Basdevant and S. Boukraa, Z. Phys. C **30**, 103 (1986).
[24] S. Capstick and N. Isgur, Phys. Rev. D **34**, 2809 (1986). We have added 20 MeV to their predictions to make the predicted Λ_c mass agree with experiment.
[25] Waikwok Kwong, Jonathan L. Rosner, and Chris Quigg, Annu. Rev. Nucl. Part. Sci. **37**, 325 (1987).
[26] S. Ono, N. A. Törnqvist, J. Lee-Franzini, and A. Sanda, Phys. Rev. Lett. **55**, 2938 (1985).